

Research Article

General Asymptotic Supnorm Estimates for Solutions of One-Dimensional Advection-Diffusion Equations in Heterogeneous Media

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We derive general bounds for the large time size of supnorm values $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}$ of solutions to one-dimensional advection-diffusion equations $u_t + (b(x, t)u)_x = u_{xx}$, $x \in \mathbb{R}$, $t > 0$ with initial data $u(\cdot, 0) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for some $1 \leq p_0 < \infty$ and arbitrary bounded advection speeds $b(x, t)$, introducing new techniques based on suitable energy arguments. Some open problems and related results are also given.

1. Introduction

In this work, we obtain very general large time estimates for supnorm values of solutions $u(\cdot, t)$ to parabolic initial value problems of the form

$$u_t + (b(x, t)u)_x = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad (1a)$$

$$u(\cdot, 0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 1 \leq p_0 < \infty, \quad (1b)$$

for arbitrary continuously differentiable advection fields $b \in L^\infty(\mathbb{R} \times [0, \infty[)$. Here, by *solution* to (1a) and (1b) in some time interval $[0, T_*[$, $0 < T_* \leq \infty$, we mean a function $u : \mathbb{R} \times [0, T_*[\rightarrow \mathbb{R}$ which is bounded in each strip $S_T = \mathbb{R} \times [0, T]$, $0 < T < T_*$, solves (1a) in the classical sense for $0 < t < T_*$, and satisfies $u(\cdot, t) \rightarrow u_0$ in $L^1_{\text{loc}}(\mathbb{R})$ as $t \rightarrow 0$. It follows from the a priori estimates given in Section 2 that all solutions of problem (1a), (1b) are actually globally defined ($T_* = \infty$), with $u(\cdot, t) \in C^0([0, \infty[, L^p(\mathbb{R}))$ for each $p \geq p_0$ finite. Given $b \in L^\infty(\mathbb{R} \times [0, \infty[)$, what then can be said about the size of supnorm values $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}$ for $t \gg 1$?

When $\partial b / \partial x \geq 0$ for all $x \in \mathbb{R}$, $t \geq 0$, it is well known that, for each $p_0 \leq p \leq \infty$, $\|u(\cdot, t)\|_{L^p(\mathbb{R})}$ is monotonically decreasing in t , with

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq K(p_0) \|u_0\|_{L^{p_0}(\mathbb{R})} t^{-1/2p_0} \quad \forall t > 0 \quad (b_x \geq 0) \quad (2)$$

for some constant $0 < K(p_0) < 2^{-1/p_0}$ that depends only on p_0 ; see, for example, [1–5]. For general $b(x, t)$, however, estimating $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}$ is much harder. To see why, let us illustrate with the important case $p_0 = 1$, where one has

$$\|u(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} \quad \forall t > 0, \quad (3)$$

as recalled in Theorem 1. Writing (1a) as

$$u_t + b(x, t)u_x = u_{xx} - b_x(x, t)u, \quad (4)$$

we observe on the right hand side of (4) that $|u(x, t)|$ is pushed to grow at points (x, t) where $b_x(x, t) < 0$. If this condition persists long enough, large values of $|u(x, t)|$ might be generated, particularly at sites where $-b_x(x, t) \gg 1$. Now, because of constraint (3), any persistent growth in solution size will eventually create long thin structures as shown in

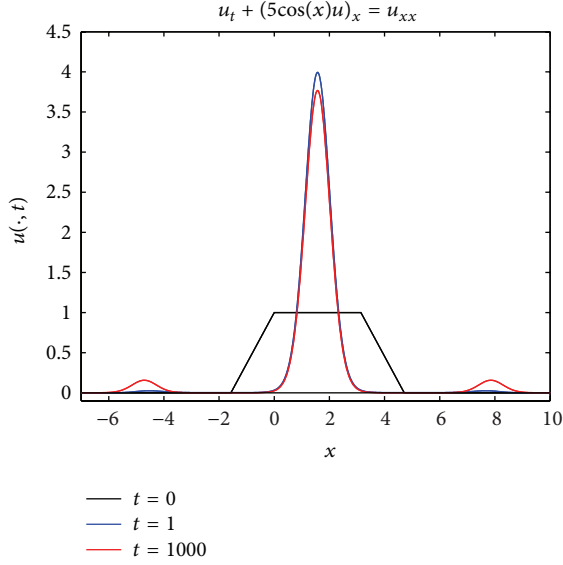


FIGURE 1: Solution profiles showing typical growth in regions with $b_x < 0$, where $b = 5 \cos x$. After reaching maximum height, solution starts decaying very slowly due to its spreading and mass conservation (decay rate is not presently known).

Figure 1, which, in turn, tend to be effectively dissipated by viscosity. The final overall behavior that ultimately results from such competition is not immediately clear, either on physical or on mathematical grounds.

As shown in (4), it is not the magnitude of $b(x, t)$ itself but instead its *oscillation* that is relevant in determining $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}$. Accordingly, we introduce the quantity $B(t)$ defined by

$$B(t) = \frac{1}{2} \left(\sup_{x \in \mathbb{R}} b(x, t) - \inf_{x \in \mathbb{R}} b(x, t) \right), \quad t \geq 0, \quad (5)$$

which plays a fundamental role in the analysis. Our main result is now easily stated.

Main Theorem. *For each $p \geq p_0$, one has¹*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ \leq \left(\frac{3\sqrt{3}}{2\pi} p \right)^{1/p} \cdot \mathcal{B}^{1/p} \cdot \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R})}, \end{aligned} \quad (6)$$

where $\mathcal{B} = \limsup_{t \rightarrow \infty} B(t)$.

In particular, in the important case $p_0 = 1$ considered above, we obtain, using (3),

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3\sqrt{3}}{2\pi} \right) \cdot \mathcal{B} \cdot \|u_0\|_{L^1(\mathbb{R})}, \quad (7)$$

so that $u(\cdot, t)$ stays uniformly bounded for all time in this case.² Estimates similar to (6) can also be shown to hold for the n -dimensional problem

$$u_t + \operatorname{div}(\mathbf{b}(x, t)u) = \Delta u, \quad u(\cdot, 0) \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad (8)$$

but to simplify our discussion we consider here the case $n = 1$ only. Our derivation of (6), which improves some unpublished results by the third author, uses the 1D inequality

$$\|v\|_{L^\infty(\mathbb{R})} \leq C_\infty \|v\|_{L^1(\mathbb{R})}^{1/3} \|v_x\|_{L^2(\mathbb{R})}^{2/3}, \quad v \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}), \quad (9)$$

where $C_\infty = (3/4)^{2/3}$, and can be readily extended to other problems of interest like 1D systems of viscous conservation laws [6, Ch. 9] or the more general equation

$$\begin{aligned} u_t + (b(x, t, u)u)_x &= (a(x, t, u)u_x)_x, \\ a(x, t, u) &\geq \mu(t) > 0, \end{aligned} \quad (10)$$

with bounded values $b(x, t, u)$; provided that we assume $\int_0^\infty \mu(t)dt = \infty$: using a similar argument, we get the estimate³ [7, Ch. 2]

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ \leq \left(\frac{3\sqrt{3}}{2\pi} p \right)^{1/p} \cdot \mathcal{B}_\mu^{1/p} \cdot \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R})}, \end{aligned} \quad (11)$$

for each $p \geq p_0$, where

$$\mathcal{B}_\mu = \limsup_{t \rightarrow \infty} \frac{B(t)}{\mu(t)}, \quad (12a)$$

$$B(t) = \frac{1}{2} \left(\sup_{x \in \mathbb{R}} b(x, t, u(x, t)) - \inf_{x \in \mathbb{R}} b(x, t, u(x, t)) \right). \quad (12b)$$

More involving applications, such as problems with superlinear advection or degenerate diffusion, which require considerable extra work, will be studied in the future.

2. A Priori Estimates

This section contains some preliminary results on the solutions of problem (1a) and (1b) needed later for our derivation of estimate (6), which is completed in Section 3. (Recall that a solution on some given time interval $[0, T_*[$, $0 < T_* \leq \infty$, is a function $u(\cdot, t) \in L_{\text{loc}}^\infty([0, T_*[, L^\infty(\mathbb{R}))$ which is smooth (C^2 in x , C^1 in t) in $\mathbb{R} \times]0, T_*[$ and solves (1a) there, verifying the initial condition in the sense of $L_{\text{loc}}^1(\mathbb{R})$, i.e., $\|u(\cdot, t) - u_0\|_{L^1(\mathbb{K})} \rightarrow 0$ as $t \rightarrow 0$ for each compact $\mathbb{K} \subset \mathbb{R}$. Local existence theory can be found in, e.g., [8, Ch. 6].) We start with a simple Gronwall-type estimate for $\|u(\cdot, t)\|_{L^q(\mathbb{R})}$, $p_0 \leq q < \infty$. The corresponding result for the supnorm ($q = \infty$) is more difficult to obtain and will be given at the end of Section 2; see Theorem 4.

Theorem 1. *If $u(\cdot, t) \in L_{\text{loc}}^\infty([0, T_*[, L^\infty(\mathbb{R}))$ solves problem (1a), (1b), then $u(\cdot, t) \in C^0([0, T_*[, L^q(\mathbb{R}))$ for each $p_0 \leq q < \infty$, and*

$$\begin{aligned} \|u(\cdot, t)\|_{L^q(\mathbb{R})} \\ \leq \|u(\cdot, 0)\|_{L^q(\mathbb{R})} \cdot \exp \left\{ \frac{1}{2} (q-1) \int_0^t B(\tau)^2 d\tau \right\} \end{aligned} \quad (13)$$

for all $0 < t < T_*$.

Proof. The proof is standard, so we will only sketch the basic steps. Taking $S \in C^1(\mathbb{R})$ such that $S'(v) \geq 0$ for all v , $S(0) = 0$, $S(v) = \text{sgn}(v)$ for $|v| \geq 1$, let (given $\delta > 0$) $L_\delta(u) = \int_0^u S(v/\delta) dv$, so that $L_\delta(u) \rightarrow |u|$ as $\delta \rightarrow 0$, uniformly in u . Let $\Phi_\delta(u) = L_\delta(u)^q$. Given $R > 0$, $0 < \epsilon \leq 1$, let $\zeta_R(\cdot)$ be the cut-off function $\zeta_R(x) = 0$ for $|x| \geq R$, $\zeta_R(x) = \exp\{-\epsilon\sqrt{1+x^2}\} - \exp\{-\epsilon\sqrt{1+R^2}\}$ for $|x| < R$. Multiplying (1a) by $\Phi'_\delta(u(x, t)) \cdot \zeta_R(x)$ if $q \neq 2$, or $u(x, t) \cdot \zeta_R(x)$ if $q = 2$, and integrating the result on $\mathbb{R} \times [0, t]$, we obtain, letting $\delta \rightarrow 0$ and then $R \rightarrow \infty$, since $u \in L^\infty(\mathbb{R} \times [0, t])$,

$$\begin{aligned} U_\epsilon(t) + V_\epsilon(t) &\leq U_\epsilon(0) + \int_0^t G_\epsilon(\tau) U_\epsilon(\tau) d\tau, \\ U_\epsilon(t) &= \int_{\mathbb{R}} |u(x, t)|^q w_\epsilon(x) dx, \end{aligned} \quad (14a)$$

where $w_\epsilon(x) = \exp\{-\epsilon\sqrt{1+x^2}\}$, $G_\epsilon(t) = (1/2)q(q-1)B(t)^2 + \epsilon 2q \cdot \sup_{0 \leq \tau \leq t} \|u(\cdot, \tau)\|_{L^\infty(\mathbb{R})} + \epsilon$, and

$$\begin{aligned} V_\epsilon(t) &= \begin{cases} \frac{1}{2}q(q-1) \\ \times \int_0^t \int_{\mathbb{R}} |u(x, \tau)|^{q-2} |u_x(x, \tau)|^2 w_\epsilon(x) dx d\tau, & \text{if } q \neq 2, \\ \int_0^t \int_{\mathbb{R}} |u_x(x, \tau)|^2 w_\epsilon(x) dx d\tau, & \text{if } q = 2. \end{cases} \end{aligned} \quad (14b)$$

By Gronwall's lemma, (14a) and (14b) give $U_\epsilon(t) \leq U_\epsilon(0) \cdot \exp\{\int_0^t G_\epsilon(\tau) d\tau\}$, from which we obtain (13) by simply letting $\epsilon \rightarrow 0$. This shows, in particular, that $u(\cdot, t) \in L^\infty_{\text{loc}}([0, T_*], L^q(\mathbb{R}))$ if $p_0 \leq q < \infty$. Now, to get $u(\cdot, t) \in C^0([0, T_*], L^q(\mathbb{R}))$, it is sufficient to show that, given $\epsilon > 0$ and $0 < T < T_*$ arbitrary, we can find $R = R(\epsilon, T) \gg 1$ large enough so that we have $\|u(\cdot, t)\|_{L^q(|x| > R)} < \epsilon$ for any $0 \leq t \leq T$. Taking $\psi \in C^2(\mathbb{R})$ with $0 \leq \psi \leq 1$ and $\psi(x) = 0$ for all $x \leq 0$, $\psi(x) = 1$ for all $x \geq 1$, let $\Psi_{R,M} \in C^2(\mathbb{R})$ be the cut-off function given by $\Psi_{R,M}(x) = 0$ if $|x| \leq R-1$, $\Psi_{R,M}(x) = \psi(|x| - R + 1)$ if $R-1 < |x| < R$, and $\Psi_{R,M}(x) = 1$ if $R \leq |x| \leq R+M$, $\Psi_{R,M}(x) = \psi(R+M+1-|x|)$ if $R+M < |x| < R+M+1$, $\Psi_{R,M}(x) = 0$ if $|x| \geq R+M+1$, where $R > 1$, $M > 0$ are given. Multiplying (1a) by $\Phi'_\delta(u(x, t)) \cdot \Psi_{R,M}(x)$ if $q \neq 2$, or $u(x, t) \cdot \Psi_{R,M}(x)$ if $q = 2$, and integrating the result on $\mathbb{R} \times [0, t]$, $0 < t \leq T$, we obtain, as in (14a) and (14b), by letting $\delta \rightarrow 0$, $M \rightarrow \infty$, that $\|u(\cdot, t)\|_{L^q(|x| > R)} < \epsilon/2 + \|u(\cdot, 0)\|_{L^q(|x| > R-1)}$ for all $0 \leq t \leq T$, provided that we take $R > 1$ sufficiently large. This gives the continuity result, and the proof is complete. \square

An important by-product of the proof above is that we have (letting $\epsilon \rightarrow 0$ in (14a) and (14b), and using (13)), for each $0 < T < T_*$ and $q \geq \max\{p_0, 2\}$,

$$\int_0^T \int_{\mathbb{R}} |u(x, \tau)|^{q-2} |u_x(x, \tau)|^2 dx d\tau < \infty. \quad (15)$$

Therefore, if we repeat the steps above leading to (14a) and (14b), we obtain (letting $\delta \rightarrow 0$, $R \rightarrow \infty$, $\epsilon \rightarrow 0$, in this order, taking (13) and (15) into account) the identity

$$\begin{aligned} \|u(\cdot, t)\|_{L^q(\mathbb{R})}^q + q(q-1) \int_0^t \int_{\mathbb{R}} |u(x, \tau)|^{q-2} |u_x(x, \tau)|^2 dx d\tau \\ = \|u(\cdot, 0)\|_{L^q(\mathbb{R})}^q \\ + q(q-1) \int_0^t \int_{\mathbb{R}} (b(x, \tau) - \beta(\tau)) |u(x, \tau)|^{q-2} \\ \times u(x, \tau) u_x(x, \tau) dx d\tau \end{aligned} \quad (16)$$

for every $0 < t < T_*$ and $\max\{p_0, 2\} \leq q < \infty$, where

$$\beta(t) = \frac{1}{2} \left(\sup_{x \in \mathbb{R}} b(x, t) + \inf_{x \in \mathbb{R}} b(x, t) \right), \quad t \geq 0. \quad (17)$$

The core of the difficulty in the analysis of (1a) and (1b) is apparent here: under the sole assumption that b is bounded, it is not much clear how one should go about the last term in (16) in order to get more than (13) above. Actually, it will be convenient to consider (16) in the (equivalent) differential form, that is,

$$\begin{aligned} \frac{d}{dt} \|u(\cdot, t)\|_{L^q(\mathbb{R})}^q \\ + q(q-1) \int_{\mathbb{R}} |u(x, t)|^{q-2} |u_x(x, t)|^2 dx \\ = q(q-1) \int_{\mathbb{R}} (b(x, t) - \beta(t)) |u(x, t)|^{q-2} \\ \times u(x, t) u_x(x, t) dx \end{aligned} \quad (18)$$

for all $t \in [0, T_* \setminus E_q]$, where $E_q \subset [0, T_*]$ has zero measure. We then readily obtain, using (9) and the one-dimensional Nash inequality [9]

$$\|v\|_{L^2(\mathbb{R})} \leq C_2 \|v\|_{L^1(\mathbb{R})}^{2/3} \|v_x\|_{L^2(\mathbb{R})}^{1/3}, \quad C_2 = \left(\frac{3\sqrt{3}}{4\pi} \right)^{1/3}, \quad (19)$$

where the value given above for C_2 is optimal [10], the following result.

Theorem 2. Let $q \geq 2p_0$. If $\hat{t} \in [0, T_* \setminus E_q]$ is such that $(d/dt)\|u(\cdot, t)\|_{L^q(\mathbb{R})}^q|_{t=\hat{t}} \geq 0$, then

$$\|u(\cdot, \hat{t})\|_{L^q(\mathbb{R})} \leq \left(\frac{q}{2} C_2^3 \right)^{1/q} B(\hat{t})^{1/q} \|u(\cdot, \hat{t})\|_{L^{q/2}(\mathbb{R})}, \quad (20a)$$

$$\|u(\cdot, \hat{t})\|_{L^\infty(\mathbb{R})} \leq \left(\frac{q}{2} C_2 C_\infty \right)^{2/q} B(\hat{t})^{2/q} \|u(\cdot, \hat{t})\|_{L^{q/2}(\mathbb{R})}. \quad (20b)$$

Proof. Consider (20a) first. From (5), (17), and (18), we have

$$\begin{aligned} \int_{\mathbb{R}} |u(x, \hat{t})|^{q-2} |u_x(x, \hat{t})|^2 dx \\ \leq B(\hat{t}) \int_{\mathbb{R}} |u(x, \hat{t})|^{q-1} |u_x(x, \hat{t})| dx. \end{aligned} \quad (21)$$

This gives

$$\int_{\mathbb{R}} |u(x, \hat{t})|^{q-2} |u_x(x, \hat{t})|^2 dx \leq B(\hat{t})^2 \|u(\cdot, \hat{t})\|_{L^q(\mathbb{R})}^q, \quad (22)$$

or, in terms of $\hat{v} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ defined by $\hat{v}(x) = |u(x, \hat{t})|^{q/2}$ if $q > 2$, $\hat{v}(x) = u(x, \hat{t})$ if $q = 2$,

$$\|\hat{v}_x\|_{L^2(\mathbb{R})} \leq \frac{q}{2} B(\hat{t}) \|\hat{v}\|_{L^2(\mathbb{R})}. \quad (23)$$

Using (19), we then get $\|\hat{v}\|_{L^2(\mathbb{R})}^2 \leq (q/2) C_2^3 B(\hat{t}) \|\hat{v}\|_{L^1(\mathbb{R})}^2$, which is equivalent to (20a). Similarly, (20b) can be obtained, using (9). \square

Thus, we can use (20a) and (20b) when $\|u(\cdot, t)\|_{L^q(\mathbb{R})}$ is not decreasing. If it is decreasing, (18) becomes useless but at least we know in such case that $\|u(\cdot, t)\|_{L^q(\mathbb{R})}$ is not increasing, which should be useful too. Different values of q have different scenarios, which we will have to piece together in some way. The next result shows us just how. To this end, it is convenient to introduce the quantities $\mathbb{B}(t_0; t)$ and $\mathbb{U}_p(t_0; t)$ defined by

$$\mathbb{B}(t_0; t) = \sup \{B(\tau) : t_0 \leq \tau \leq t\}, \quad (24)$$

$$\mathbb{U}_p(t_0; t) = \sup \{\|u(\cdot, \tau)\|_{L^p(\mathbb{R})} : t_0 \leq \tau \leq t\}, \quad (25)$$

given $p \geq p_0$, $0 \leq t_0 \leq t < T_*$ arbitrary.

Theorem 3. *Let $q \geq 2p_0$. For each $0 \leq t_0 < T_*$, we have*

$$\begin{aligned} & \mathbb{U}_q(t_0; t) \\ & \leq \max \left\{ \|u(\cdot, t_0)\|_{L^q(\mathbb{R})}; \left(\frac{q}{2} C_2^3 \right)^{1/q} \mathbb{B}(t_0; t)^{1/q} \mathbb{U}_{q/2}(t_0; t) \right\} \end{aligned} \quad (26)$$

for all $t_0 \leq t < T_*$.

Proof. Set $\lambda_q(t) = ((q/2) C_2^3)^{1/q} \mathbb{B}(t_0; t)^{1/q} \mathbb{U}_{q/2}(t_0; t)$. There are three cases to consider.

Case I. $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})} > \lambda_q(t)$ for all $t_0 \leq \tau \leq t$. By (20a), Theorem 2, we must then have $(d/d\tau)\|u(\cdot, \tau)\|_{L^q(\mathbb{R})}^q < 0$ for all $\tau \in [t_0, t] \setminus E_q$, so that $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})}$ is monotonically decreasing in $[t_0, t]$. In particular, $\mathbb{U}_q(t_0; t) = \|u(\cdot, t_0)\|_{L^q(\mathbb{R})}$ in this case, and (26) holds.

Case II. $\|u(\cdot, t_0)\|_{L^q(\mathbb{R})} > \lambda_q(t)$ and $\|u(\cdot, t_1)\|_{L^q(\mathbb{R})} \leq \lambda_q(t)$ for some $t_1 \in]t_0, t]$. In this case, let $t_2 \in]t_0, t]$ be such that we have $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})} > \lambda_q(t)$ for all $t_0 \leq \tau < t_2$, while $\|u(\cdot, t_2)\|_{L^q(\mathbb{R})} = \lambda_q(t)$. We claim that $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})} \leq \lambda_q(t)$ for every $t_2 \leq \tau \leq t$: in fact, if this were not true, we could then find t_3, t_4 with $t_2 \leq t_3 < t_4 \leq t$ such that $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})} > \lambda_q(t)$ for all $t_3 < \tau \leq t_4$, $\|u(\cdot, t_3)\|_{L^q(\mathbb{R})} = \lambda_q(t)$. By (20a), Theorem 2, this would require $(d/d\tau)\|u(\cdot, \tau)\|_{L^q(\mathbb{R})}^q < 0$ for all $\tau \in]t_3, t_4] \setminus E_q$, so that $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})}$ could not increase anywhere on $[t_3, t_4]$. This contradicts $\|u(\cdot, t_3)\|_{L^q(\mathbb{R})} < \|u(\cdot, t_4)\|_{L^q(\mathbb{R})}$, and so we have $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})} \leq \lambda_q(t)$ for every $t_2 \leq \tau \leq t$,

as claimed. On the other hand, by (20a), $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})}$ has to be monotonically decreasing on $[t_0, t_2]$, just as in Case I. Therefore, we have $\mathbb{U}_q(t_0; t) = \|u(\cdot, t_0)\|_{L^q(\mathbb{R})}$ in this case again, which shows (26).

Case III. Consider $\|u(\cdot, t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q(t)$. This gives $\|u(\cdot, \tau)\|_{L^q(\mathbb{R})} \leq \lambda_q(t)$ for every $t_0 \leq \tau \leq t$, by repeating the argument used on the interval $[t_2, t]$ in Case II. It follows that we must have $\mathbb{U}_q(t_0; t) \leq \lambda_q(t)$ in this case, and the proof of Theorem 3 is complete. \square

An important application of Theorem 3 is the following result.

Theorem 4. *Let $p_0 \leq p < \infty$, $0 \leq t_0 < T_*$. Then*

$$\begin{aligned} & \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ & \leq (2p)^{1/p} \cdot \max \left\{ \|u(\cdot, t_0)\|_{L^\infty(\mathbb{R})}; \mathbb{B}(t_0; t)^{1/p} \mathbb{U}_p(t_0; t) \right\} \end{aligned} \quad (27)$$

for any $t_0 \leq t < T_*$, where $\mathbb{B}(t_0; t)$ and $\mathbb{U}_p(t_0; t)$ are given in (24) and (25) above.

Proof. Let $k \in \mathbb{Z}$, $k \geq 2$. Applying (26) successively with $q = 2p, 4p, \dots, 2^k p$, we obtain

$$\begin{aligned} & \|u(\cdot, t)\|_{L^{2^k p}(\mathbb{R})} \\ & \leq \max \left\{ \|u(\cdot, t_0)\|_{L^{2^k p}(\mathbb{R})}; K(k, \ell)^{1/p} \cdot \mathbb{B}(t_0; t)^{(1/p)(2^{-\ell} - 2^{-k})} \right. \\ & \quad \cdot \|u(\cdot, t_0)\|_{L^{2^\ell p}(\mathbb{R})}, 1 \leq \ell \leq k-1; \\ & \quad \left. K(k, 0)^{1/p} \cdot \mathbb{B}(t_0; t)^{(1/p)(1 - 2^{-k})} \cdot \mathbb{U}_p(t_0; t) \right\}, \end{aligned} \quad (28a)$$

where

$$K(k, \ell) = \prod_{j=\ell+1}^k \left(2^{j-1} p C_2^3 \right)^{2^{-j}}, \quad 0 \leq \ell \leq k-1. \quad (28b)$$

Now, for $1 \leq \ell \leq k-1$,

$$\begin{aligned} & \mathbb{B}(t_0; t)^{(1/p)(2^{-\ell} - 2^{-k})} \cdot \|u(\cdot, t_0)\|_{L^{2^\ell p}(\mathbb{R})} \\ & \leq \mathbb{B}(t_0; t)^{(1/p)(2^{-\ell} - 2^{-k})} \cdot \|u(\cdot, t_0)\|_{L^p(\mathbb{R})}^{(2^{-\ell} - 2^{-k})/(1 - 2^{-k})} \\ & \quad \cdot \|u(\cdot, t_0)\|_{L^{2^k p}(\mathbb{R})}^{(1 - 2^{-\ell})/(1 - 2^{-k})} \\ & \leq \max \left\{ \|u(\cdot, t_0)\|_{L^{2^k p}(\mathbb{R})}; \right. \\ & \quad \left. \mathbb{B}(t_0; t)^{(1/p)(1 - 2^{-k})} \cdot \|u(\cdot, t_0)\|_{L^p(\mathbb{R})} \right\} \end{aligned} \quad (29)$$

by Young's inequality (see, e.g., [11, page 622]); in particular, we get, from (28a) and (28b),

$$\begin{aligned} & \|u(\cdot, t)\|_{L^{2^k p}(\mathbb{R})} \\ & \leq (2p)^{1/p} \\ & \quad \cdot \max \left\{ \|u(\cdot, t_0)\|_{L^{2^k p}(\mathbb{R})}; \mathbb{B}(t_0; t)^{(1/p)(1-2^{-k})} \cdot \mathbb{U}_p(t_0; t) \right\}, \end{aligned} \quad (30)$$

since $K(k, \ell) \leq 2p$ for all $0 \leq \ell \leq k-1$. Letting $k \rightarrow \infty$, (27) is obtained. \square

It follows from Theorems 1 and 4 that $u(\cdot, t)$ is globally defined ($T_* = \infty$). Now, from (27), we immediately obtain, letting $t \rightarrow \infty$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ & \leq (2p)^{1/p} \cdot \max \left\{ \|u(\cdot, t_0)\|_{L^\infty(\mathbb{R})}; \mathbb{B}(t_0)^{1/p} \mathbb{U}_p(t_0) \right\} \end{aligned} \quad (31)$$

for any $t_0 \geq 0$, where $\mathbb{B}(t_0)$ and $\mathbb{U}_p(t_0)$ are given by

$$\mathbb{B}(t_0) = \sup \{B(t) : t \geq t_0\}, \quad (32)$$

$$\mathbb{U}_p(t_0) = \sup \{\|u(\cdot, t)\|_{L^p(\mathbb{R})} : t \geq t_0\}. \quad (33)$$

Taking $(t_0^{(n)})_n$ such that $t_0^{(n)} \rightarrow \infty$ and $\|u(\cdot, t_0^{(n)})\|_{L^\infty(\mathbb{R})} \rightarrow \liminf_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})}$, and applying (31) with $t_0 = t_0^{(n)}$ for each n , we then obtain, letting $n \rightarrow \infty$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ & \leq (2p)^{1/p} \cdot \max \left\{ \liminf_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})}; \mathcal{B}^{1/p} \cdot \mathcal{U}_p \right\}, \end{aligned} \quad (34)$$

where \mathcal{B} and \mathcal{U}_p are given by

$$\mathcal{B} = \limsup_{t \rightarrow \infty} B(t), \quad \mathcal{U}_p = \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R})}. \quad (35)$$

3. Large Time Estimates

In this section, we use the results obtained above to derive two basic large time estimates (given in Theorems 5 and 6) for solutions $u(\cdot, t)$ of problem (1a), (1b), which represent important intermediate steps that will ultimately lead to the main result stated in Theorem 7.

Theorem 5. Let $q \geq 2p_0$, and $\mathcal{B} \geq 0$ be as defined in (35). Then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^q(\mathbb{R})}, \\ & \leq \left(\frac{q}{2} C_2^3 \right)^{1/q} \cdot \mathcal{B}^{1/q} \cdot \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^{q/2}(\mathbb{R})}, \end{aligned} \quad (36)$$

where $C_2 = (3\sqrt{3}/(4\pi))^{1/3}$ is the constant in the Nash inequality (19).

Proof. We set $p = q/2$ and assume that \mathcal{U}_p is finite. As in the proof of Theorem 2, we take $v \in L^\infty(\mathbb{R} \times [0, \infty[)$ given by $v(x, t) = |u(x, t)|^p$ if $p > 1$, $v(x, t) = u(x, t)$ if $p = 1$. It follows that

$$\|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 = \|u(\cdot, t)\|_{L^{2p}(\mathbb{R})}^{2p}, \quad (37)$$

$$\|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 = p^2 \int_{\mathbb{R}} |u(x, t)|^{2p-2} |u_x(x, t)|^2 dx.$$

Therefore, from (18), we have, for some null set $E_{2p} \subset [0, \infty[$,

$$\begin{aligned} & \frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 + 4 \left(1 - \frac{1}{2p} \right) \|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \\ & \leq 4p \left(1 - \frac{1}{2p} \right) B(t) \|v(\cdot, t)\|_{L^2(\mathbb{R})} \|v_x(\cdot, t)\|_{L^2(\mathbb{R})} \end{aligned} \quad (38)$$

for all $t \in [0, \infty[\setminus E_{2p}$, and so, by (19),

$$\begin{aligned} & \frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 + 4 \left(1 - \frac{1}{2p} \right) \|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \\ & \leq 4pC_2 \left(1 - \frac{1}{2p} \right) B(t) \|v(\cdot, t)\|_{L^1(\mathbb{R})}^{2/3} \|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^{4/3}. \end{aligned} \quad (39)$$

This gives, by Young's inequality ([11, page 622]), for all $t \in [0, \infty[\setminus E_{2p}$,

$$\begin{aligned} & \frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 + \frac{4}{3} \left(1 - \frac{1}{2p} \right) \|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \\ & \leq \frac{4}{3} \left(1 - \frac{1}{2p} \right) (pC_2)^3 B(t)^3 \|v(\cdot, t)\|_{L^1(\mathbb{R})}^2. \end{aligned} \quad (40)$$

Setting

$$\lambda_p = \limsup_{t \rightarrow \infty} g(t), \quad (41)$$

$$g(t) = (pC_2^3)^{1/2} B(t)^{1/2} \|v(\cdot, t)\|_{L^1(\mathbb{R})},$$

we claim that

$$\limsup_{t \rightarrow \infty} \|v(\cdot, t)\|_{L^2(\mathbb{R})} \leq \lambda_p. \quad (42)$$

In fact, let us argue by contradiction. If (42) is false, we can pick $0 < \eta \ll 1$ and a sequence $(t_j)_{j \geq 0}$, $t_j \rightarrow \infty$, such that $\|v(\cdot, t_j)\|_{L^2(\mathbb{R})} > \lambda_p + \eta$ (for all $j \geq 0$) and $g(t) \leq \lambda_p + \eta/2$ for all $t \geq t_0$. From (20a), Theorem 2, it will then follow that

$$\|v(\cdot, t)\|_{L^2(\mathbb{R})} > \lambda_p + \eta, \quad \forall t \geq t_0. \quad (43)$$

In fact, suppose that (43) were false, so that we had $\|v(\cdot, \tilde{t})\|_{L^2(\mathbb{R})} \leq \lambda_p + \eta$ for some $\tilde{t} > t_0$. Taking $j \gg 1$ with $t_j > \tilde{t}$, we could then find $\hat{t} \in [\tilde{t}, t_j]$ such that $\|v(\cdot, \hat{t})\|_{L^2(\mathbb{R})} > \lambda_p + \eta$ for all $t \in [\hat{t}, t_j]$, while $\|v(\cdot, \hat{t})\|_{L^2(\mathbb{R})} = \lambda_p + \eta$, and so there would exist $t_* \in [\hat{t}, t_j] \setminus E_{2p}$ with $(d/dt)\|v(\cdot, t)\|_{L^2(\mathbb{R})}^2$ positive at $t = t_*$. By (20a), we would have $\|v(\cdot, t_*)\|_{L^2(\mathbb{R})} \leq \lambda_p$, but this would contradict the fact that $\|v(\cdot, t)\|_{L^2(\mathbb{R})} \geq \lambda_p + \eta$

everywhere on $[\hat{t}, t_j]$. Thus, we conclude that (43) cannot be false, as claimed. We then obtain, from (19), (40), and (43),

$$\begin{aligned} & \|v(\cdot, t)\|_{L^2(\mathbb{R})}^6 \\ & \leq C_2^6 \|v(\cdot, t)\|_{L^1(\mathbb{R})}^4 \|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \\ & \leq g(t)^6 + \frac{2p}{2p-1} \|v(\cdot, t)\|_{L^1(\mathbb{R})}^4 \left(-\frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 \right) \end{aligned} \quad (44)$$

for all $t \in [t_0, \infty[\setminus E_{2p}$. Recalling that $\|v(\cdot, t)\|_{L^2(\mathbb{R})} > \lambda_p + \eta$, $g(t) \leq \lambda_p + \eta/2$, for all $t \geq t_0$, this gives

$$-\frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 \geq K(\eta), \quad \forall t \in [t_0, \infty[\setminus E_{2p} \quad (45)$$

for some constant $K(\eta) > 0$ independent of t , which cannot be, since this implies

$$\|v(\cdot, t_0)\|_{L^2(\mathbb{R})}^2 \geq K(\eta) \cdot (t - t_0) \quad \forall t > t_0. \quad (46)$$

This contradiction shows (42), which is equivalent to (36), and the proof is complete. \square

Applying (36) successively with $q = 2p, 4p, \dots, 2^k p$, we get

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^{2^k p}(\mathbb{R})} \\ & \leq \left[\prod_{j=1}^k (2^{j-1} p C_2^3)^{2^{-j}} \right]^{1/p} \cdot \mathcal{B}^{(1/p)(1-2^{-k})} \cdot \mathcal{U}_p \end{aligned} \quad (47)$$

for $k \geq 1$ arbitrary, where $\mathcal{U}_p = \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R})}$. Letting $k \rightarrow \infty$, this suggests

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq K(p) \cdot \mathcal{B}^{1/p} \cdot \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R})}, \quad (48a)$$

where

$$K(p) = \left[\prod_{j=1}^{\infty} (2^{j-1} p C_2^3)^{2^{-j}} \right]^{1/p} = \left(\frac{3\sqrt{3}}{2\pi} p \right)^{1/p} \quad (48b)$$

(cf. (6) above), as long as the limit processes $k \rightarrow \infty$, $t \rightarrow \infty$ can be interchanged. That this is indeed the case is a consequence of (34) and the following result.

Theorem 6. *Let $p \geq p_0$. Then*

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ & \leq (p C_2 C_\infty)^{1/p} \cdot \mathcal{B}^{1/p} \cdot \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R})}, \end{aligned} \quad (49)$$

where C_2, C_∞ are the constants given in (19) and (9).

Proof. Again, assuming \mathcal{U}_p finite (otherwise, (49) is obvious; cf. endnote⁴), we introduce, as in the previous proof, $v \in L^\infty(\mathbb{R} \times [0, \infty[)$ given by $v(x, t) = |u(x, t)|^p$ if $p > 1$, and $v(x, t) = u(x, t)$ if $p = 1$. Thus, (40) is valid, and setting $\lambda_p \in \mathbb{R}$, $g \in L^\infty([0, \infty[)$ by

$$\lambda_p = \limsup_{t \rightarrow \infty} g(t), \quad g(t) = p C_2 B(t) \|v(\cdot, t)\|_{L^1(\mathbb{R})}, \quad (50)$$

we have that (49) is obtained if we show that

$$\liminf_{t \rightarrow \infty} \|v(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C_\infty \cdot \lambda_p. \quad (51)$$

We argue by contradiction and assume that (51) is false. Taking then $0 < \eta \ll 1$, $t_0 \gg 1$ so that $\|v(\cdot, t)\|_{L^\infty(\mathbb{R})} \geq C_\infty \cdot (\lambda_p + \eta)$ and $g(t) \leq \lambda_p + \eta/2$ hold for all $t \geq t_0$, we get, by (9) and (40),

$$\begin{aligned} & \|v(\cdot, t)\|_{L^\infty(\mathbb{R})}^3 \leq C_\infty^3 \|v(\cdot, t)\|_{L^1(\mathbb{R})} \|v_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \\ & \leq C_\infty^3 g(t)^3 + C_\infty^3 \frac{2p}{2p-1} \|v(\cdot, t)\|_{L^1(\mathbb{R})} \\ & \quad \times \left(-\frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 \right) \end{aligned} \quad (52)$$

for all $t \in [t_0, \infty[\setminus E_{2p}$. Since $\|v(\cdot, t)\|_{L^\infty(\mathbb{R})} \geq C_\infty \cdot (\lambda_p + \eta)$, $g(t) \leq \lambda_p + \eta/2$, this gives

$$-\frac{d}{dt} \|v(\cdot, t)\|_{L^2(\mathbb{R})}^2 \geq K(\eta), \quad \forall t \in [t_0, \infty[\setminus E_{2p} \quad (53)$$

for some constant $K(\eta) > 0$ independent of t . As before, this implies that $\|v(\cdot, t_0)\|_{L^2(\mathbb{R})}^2 \geq K(\eta) \cdot (t - t_0)$ for all $t \geq t_0$, which is impossible because $\|v(\cdot, t_0)\|_{L^2(\mathbb{R})}$ is finite. This contradiction establishes (51) above, completing the proof of Theorem 6. \square

We are finally in good position to derive (6), (48a), and (48b). Combining (34) and (49) above, we obtain

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq (2p^2)^{1/p} \cdot \mathcal{B}^{1/p} \cdot \mathcal{U}_p \quad (54)$$

for each $p \geq p_0$, so that we have, in particular,

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq (2^{2k+1} p^2)^{1/2^k p} \cdot \mathcal{B}^{1/2^k p} \cdot \mathcal{U}_{2^k p} \quad (55)$$

for each $k \geq 0$. By (47), we then get

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \\ & \leq \left\{ (2^{2k+1} p^2)^{2^{-k}} \cdot \prod_{j=1}^k (2^{j-1} p C_2^3)^{2^{-j}} \right\}^{1/p} \cdot \mathcal{B}^{1/p} \cdot \mathcal{U}_p \end{aligned} \quad (56)$$

for all k . Letting $k \rightarrow \infty$, Theorem 7 is obtained, and our argument is complete.

Theorem 7. Let $p \geq p_0$. Assuming $b \in L^\infty(\mathbb{R} \times [0, \infty[)$, then (6), (48a), and (48b) hold.

It is worth noticing that the corresponding estimate for the n -dimensional problem (8), namely,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \\ & \leq K(n, p) \cdot \mathcal{B}^{n/p} \cdot \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)}, \end{aligned} \quad (57)$$

where $\mathcal{B} \geq 0$ is similarly defined, can be also derived in arbitrary dimension $n > 1$.

4. Concluding Remarks

We close our discussion of the problem (1a), (1b), given $b \in L^\infty(\mathbb{R} \times [0, \infty[)$, $1 \leq p_0 < \infty$, indicating a few questions which were not answered by our analysis:

- characterize all $b \in L^\infty(\mathbb{R} \times [0, \infty[)$ for which it is true that $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \rightarrow 0$ (as $t \rightarrow \infty$) for every solution $u(\cdot, t)$ of problem (1a) and (1b);
- same question as (a) above, but requiring only that $\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} < \infty$ (as $t \rightarrow \infty$) for every solution $u(\cdot, t)$ of problem (1a) and (1b), in case $p_0 > 1$;⁵
- given $p_0 > 1$, characterize all $b \in L^\infty(\mathbb{R} \times [0, \infty[)$ such that $\|u(\cdot, t)\|_{L^{p_0}(\mathbb{R})} \rightarrow 0$ (as $t \rightarrow \infty$) for every solution $u(\cdot, t)$ of problem (1a) and (1b);
- same question as (c) above, but requiring only that $\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^{p_0}(\mathbb{R})} < \infty$ (as $t \rightarrow \infty$) for every solution $u(\cdot, t)$ of problem (1a) and (1b);
- for $p_0 = 1$, characterize all $b \in L^\infty(\mathbb{R} \times [0, \infty[)$ such that $\|u(\cdot, t)\|_{L^1(\mathbb{R})} \rightarrow |m|$ (as $t \rightarrow \infty$) for every solution $u(\cdot, t)$, where $m = \int_{\mathbb{R}} u_0(x) dx$ is the solution mass;
- for $p_0 = 1$, and $b \in L^\infty(\mathbb{R} \times [0, \infty[)$ not satisfying property (e), what are the values of $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^1(\mathbb{R})}$ in case of initial states that change sign?

These questions can be similarly posed for solutions $u(\cdot, t)$ of autonomous problems

$$u_t + (b(x)u)_x = u_{xx}, \quad u(\cdot, 0) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad (58)$$

where $b \in L^\infty(\mathbb{R})$ does not depend on the time variable. For (58), question (e) has been answered in [12] (see also [13]). Another interesting question is the following:

- when (58) admits no stationary solutions other than the trivial solution $u = 0$, is it true that $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} = 0$ for every solution $u(\cdot, t)$?

Moreover, for solutions $u(\cdot, t)$ of (1a) and (1b) or (58) with $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \rightarrow 0$ as $t \rightarrow \infty$, there is the question of determining the proper decay rate.⁶ As suggested by Figure 1, solution decay may sometimes happen at remarkably slow rates.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Endnotes

- In (6), (11), and other similar expressions in the text, it is assumed that $0 \cdot \infty = \infty$.
- The constants $(3\sqrt{3}p/(2\pi))^{1/p}$ in (6) and (7) are not optimal; minimal values are not known.
- In (6), (11), and other similar expressions in the text, it is assumed that $0 \cdot \infty = \infty$.
- In (6), (11), and other similar expressions in the text, it is assumed that $0 \cdot \infty = \infty$.
- For $p_0 = 1$, any $b \in L^\infty(\mathbb{R} \times [0, \infty[)$ satisfies property (b); compare (7) in Section 1.
- In case we have $b_x \geq 0$ for all x, t , the answer is given in (2) above.

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