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COMPUTATION OF EXTREME CLUSTERED NATURAL FREQUENCIES OF DAMPED SECOND-ORDER LINEAR SYSTEMS

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Resumo: In recent contribution, we have proposed methods for computing the largest and the smallest isolated natural frequencies, either real or a complex pair, of a second-order mechanical linear system described by its *mass*, *damping* and *stiffness* matrices. Those methods were presented as a generalization of the well-known direct and inverse power methods for computing dominant and sub-dominant eigenvalues of a matrix; they applied results on fundamental solutions for second-order systems, as well as the widely used technique of subspace iteration. The present work addresses the case where those extreme frequencies, instead of being isolated, come in clusters. This situation is very common when system matrices come from the Finite Element Method. The strategy we propose can be applied to both sparse and dense problems, and strategies using specialized software libraries as LAPACK or SPARSEKIT can be deployed for achieving outstanding performance. Numerical examples with test matrices from real structural vibration problems are provided.

Palavras-chave: second-order, power-method, subspace-iteration

1. INTRODUCTION

Second-order mechanical linear systems can be described (see [2, 6]), using a set of ordinary second-order equations

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t), \tag{1}$$

where $M, C, K \in \mathbb{R}^{n\times n}$ are called, respectively *mass*, *damping* and *stiffness* matrices. Matrices M and K are symmetric, M is usually positive definite, while K is usually positive semi-definite. Quantities $q(t)$ and $\dot{q}(t)$ are the generalized coordinates of the system (they are also state variables), usually called generalized position and velocity, respectively. The right hand side $f(t)$ represents the external forces that act on the system for a given time t .

Mathematically [6] important properties regarding asymptotic stability and resonance frequencies of system (1) are determined by algebraic facts on the matrix pencil

$$P(\lambda) = \lambda^2 M + \lambda C + K \tag{2}$$

for $\lambda \in \mathbb{C}$ in the sense that they impose requirements on the solution (eigen) pairs (λ_i, x_i) of

$$\lambda^2 Mx + \lambda Cx + Kx = 0. \tag{3}$$

Important fact on linear systems theory is that when the right hand size of (1) matches any linear combination of the homogeneous solutions $e^{\lambda_i t}$, $i = 1, 2, \ldots, n$ then the transfer function for system (1) (see [2, 6]) achieves its maximum values, meaning that the amplitudes of the response of the system will be at their maximum values. This is a simple explanation, for linear systems, for a well-known phenomenon called resonance, which happens in non-linear systems in general. Therefore, it is necessary to avoid imposing to a system frequencies that might be close to its natural ones. For this reason it is important to know the range of natural frequencies of a system, and computation of extreme natural frequencies will establish lower and upper bounds for that range.

In recent contribution [1], we have proposed methods for computing the largest and the smallest isolated natural frequencies, either real or a complex conjugated pair, of a second-order linear system such as 1.

This paper further investigates the problem of computing accurately the smallest (sub-dominant) and the largest (dominant) natural frequencies of a system (1) in the common situation where those quantities are magnitude-clustered, that is, frequencies in dominant or sub-dominant groups are difficult to isolate according to their magnitude. Besides, the size of these groups is generally unknown, and can even be large in some cases.

2. FUNDAMENTAL MATRIX SOLUTION AND SUBSPACE ITERATION

Consider the second order eigenvalue problem (3) where matrix M is non-singular. Following [3], a *fundamental ma-*

trix solution can be defined by means of an associated second order difference problem

$$\begin{cases} MH_{k+2} + CH_{k+1} + KH_k = 0, \quad k = 0, 1, 2, 3, \ldots \\ H_0 = 0, MH_1 = I. \end{cases} \tag{4}$$

where I denotes the identity $n \times n$ matrix.

In a more general setup in [3], the fundamental matrix solution H_k defined above is shown to verify properties that allow further naming it as *discrete impulse response* or *Green function of initial state*, despite other important properties related to decomposition of forced vibrations, variation of parameter's method, generalization of Cayley-Hamilton's theorem, and others.

2.1. Subspace iteration for a dominant cluster of size p

Suppose that M is non-singular and let the solutions of (3) be ordered like

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > |\lambda_{p+1}| \geq \cdots \geq |\lambda_{2n}|. \tag{5}$$

For any given non-singular matrix $V_0 \in \mathbb{R}^{n \times p}$, we define

$$Y_k = H_k V_0 \tag{6}$$

where H_k satisfies (4). It is straightforward to verify that Y_k satisfies the recursive formula

$$\begin{cases} MY_{k+2} + CY_{k+1} + KY_k = 0, \quad k = 0, 1, 2, 3, \ldots \\ Y_0 = 0, MY_1 = V_0. \end{cases} \tag{7}$$

To avoid numerical overflow, as well as to bring orthogonality, for every $k \geq 0$, once Y_{k+2} is computed, we normalize (orthogonalise) the computed quantities taking the economic (skinny) QR factorization :

$$Y_{k+1} = Q_{k+1} R_{k+1} \tag{8}$$

and normalize them through

$$Y_{k+2} \leftarrow Y_{k+2} R_{k+1}^{-1} \tag{9}$$

$$Y_{k+1} \leftarrow Q_{k+1} \tag{10}$$

$$Y_k \leftarrow Y_k R_{k+1}^{-1} \tag{11}$$

where, in fact, the respective linear systems are solved.

Now, writing

$$Y_{k+2} = Y_{k+1} \Theta_{k+1} \tag{12}$$

gives

$$\Theta_{k+1} = Y_{k+1}^T Y_{k+2} \tag{13}$$

by the orthogonality of the normalized Y_{k+1}.

If the p dominant eigenvalues have a set of p linearly independent eigenvectors, it can be shown, like in the first or generalized first order case of the Subspace Iteration technique (see [4]), that the sequence Y_k will converge to a basis of that space. In particular, our construction provides that (Θ_{k+1}, Y_{k+1}) will converge to (Θ_d, Y_d) such that

$$MY_d \Theta_d^2 + CY_d \Theta_d + KY_d = 0 \tag{14}$$

and

- $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ are the eigenvalues of Θ_d;
- Y_d is an orthonormal basis of the respective eigenspace.

Algorithm 2.1: Computing p dominant eigenspace
Input: n by n matrices M, C and K; tolerance parameter TOL for stopping criteria; integer p and non-singular matrix $V \in \mathbb{R}^{n \times p}$
Output: approximation Θ_d to a real p by p matrix whose eigenvalues are the p dominant eigenvalues of $\lambda^2 Mx + \lambda Cx + Kx = 0$; orthogonal matrix Y_d with the basis vectors for this eigenspace.
Assumptions: matrix M is non-singular, C is non-null, there exist p linearly independent eigenvectors associated to the p dominant eigenvalues.
Step 1: Let $Y_0 = 0$, let $s_0 = 0 \in \mathbb{R}^p$;
Step 2: Solve $MZ = V$ and let $Y_1 = Z$.
For $k = 0, 1, 2, 3, \ldots$, until tolerance condition is reached
Step 3: Solve $MY_{k+2} = -CY_{k+1} - KY_k$ for Y_{k+2}
Step 4: Skinny QR factorization: $Q_{k+1} R_{k+1} = Y_{k+1}$
Step 5: Solve $R_{k+1}^T Z = Y_{k+2}^T$ and update $Y_{k+2} \leftarrow Z^T$
Step 6: Update $Y_{k+1} \leftarrow Q_{k+1}$
Step 7: Compute $\Theta_{k+1} = Q_{k+1}^T Y_{k+2}$
Step 8: Compute v_{k+1}, the vector of eigenvalues of Θ_{k+1} ordered by magnitude and real part.
Step 9: Compare s_k and s_{k+1} and check on tolerance condition
End For
Return $\Theta_d = \Theta_{k+1}$ and $Y_d = Y_{k+1}$.

Remarks:

- Since $Y_0 = 0$, if $C = 0$ (undamped case) then Algorithm 2.1 breaks with $Y_1 = 0$, and that is the reason why we impose $C \neq 0$ as assumption for Algorithm 2.1. Moreover, for a singular matrix C, if $CY_1 = 0$ the algorithm will also breaks in first step; in this case it suffices to re-start the recursion with another V_0.
- In Step 3, system properties that bring structure for matrix M can be applied in the solution of this linear system for better performance. Such properties include symmetry, positiveness and bandwidth.
- In Step 5, computation yields to linear algebraic triangular p by p system with multiple right hand sides.
- Algorithm 2.1 is heavily composed of solution of algebraic linear systems, and high-performance strategies [5] can widely be applied, for instance, through specialized software as LAPACK [7] or SPARSEKIT [9].
- If we cannot guarantee the existence of a p-dominant set of eigenvectors with a linearly independent set of p-eigenvectors, then Algorithm 2.1 may not converge. Algorithm 2.2, presented below, can be applied to apply Algorithm 2.1 for increasing trial values of p, until convergence on its main iteration is detected.

Algorithm 2.2: Computing a dominant eigenspace

Input: n by n matrices M, C and K; tolerance parameter TOL for stopping criteria; starting vector $V_1 \in \mathbb{R}^n$ not null.
Output: approximation Θ_d to a real p by p matrix whose eigenvalues are the p dominant eigenvalues of $\lambda^2 Mx + \lambda Cx + Kx = 0$; orthogonal matrix Y_d with the basis vectors for this eigenspace.
Assumptions: matrix M is non-singular, C is non-null, there exist p linearly independent eigenvectors associated to p dominant eigenvalues, for some integer p.
Step 1: Let $p \leftarrow 1$, $go \leftarrow 1$;
While $go = 1$

Step 2: Apply Algorithm 2.1 to (M, C, K) and V_p. If convergence is detected:

Wait to the tolerance TOL the user expects; set $go \leftarrow 0$;

Otherwise update $p \leftarrow p + 1$, set V_p using the columns of last Y_{k+1} produced by Algorithm 2.1, padded with an additional column, to have a linearly independent set.

End-While
Return $\Theta_d = \Theta_{k+1}$ and $Y_d = Y_{k+1}$.

2.2. Computation for a sub-dominant cluster of size p

Suppose that K is non-singular and let the solutions of (3) be ordered like

$$|\lambda_1| \geq \cdots \geq |\lambda_{2n-p}| > |\lambda_{2n-p+1}| \geq \cdots \geq |\lambda_{2n}|. \quad (15)$$

Since the eigenvalues of (K, C, M), which satisfy

$$\rho^2 Kx + \rho Cx + Mx = 0 \quad (16)$$

are $\rho = 1/\lambda$, where (λ, x) satisfies (3) , we just have to apply Algorithm 2.1 to (K, C, M) (iteration will be called "inverted"), and take the reciprocal of the eigenvalues afterwards, to accomplish the desired computation. Again, Algorithm 2.2 can be applied to (K, C, M) in the common situation where such p is unknown; on success, we take the reciprocal of every eigenvalue we found.

2.3. Shifting and convergence

Another simple result states that the solutions (γ, x) of

$$\gamma^2 Mx + \gamma D_\sigma x + K_\sigma x = 0, \quad (17)$$

where

$$D_\sigma = D + 2\sigma M \quad (18)$$

$$K_\sigma = K + \sigma D + \sigma^2 M, \quad (19)$$

are just $\gamma = \lambda - \sigma$, where (λ, x) is a solution of (3). Therefore, every time an inverted iteration seams to be converging to a group of p real quantities that are closed to a real number β, we can define $\sigma = 1/\beta$ and change D and K to the matrices above to speed-up the convergence (shifted inverted strategy) and shift back by adding σ to the sequence.

Let $\rho_1, \rho_2, \ldots, \rho_{2n}$ be the solutions of (16), ordered like

$$|\rho_1| \geq |\rho_2| \geq |\rho_p| > |\rho_{p+1}| \geq \cdots \geq |\rho_{2n}|. \quad (20)$$

Adapting results obtained in [1], for a shift $\alpha = 1/\beta$ and matrices D_σ and K_σ as in (18-19), it can be shown that we can build a tuned-up sequence converging linearly, with a rate given by

$$\delta = -\log_{10}\left(\left|\frac{\frac{1}{\rho_p} - \sigma}{\frac{1}{\rho_{p+1}} - \sigma}\right|\right) \quad (21)$$

additional figures per iteration, asymptotically.

3. NUMERICAL EXAMPLES

3.1. Extreme frequencies of a Transmission Tower

Consider the 153 by 153 matrices M and K from *Harwell-Boeing* BCSSCTRUC05 matrix set (available in [8]). For simplicity, take $C = \alpha I$, that is, the structure is assumed to have some diagonal damping, where $\alpha = 8.186 \times 10^2$.

Table 1 summarizes the successful application of Algorithm 2.2 for computation of a dominant cluster of frequencies. This cluster has size 4, matrices $\Lambda_d \in \mathbb{R}^{4\times 4}$ and orthogonal $Y_d \in \mathbb{R}^{153\times 4}$ were computed (but not shown here because of space limitation) such that

$$\|MY_d\Lambda_d^2 + CY_d\Lambda_d + KY_d\|_F = 1.47 \times 10^{-2} \quad (22)$$

and the computed frequencies were

$$\{\lambda_4; \lambda_3; \lambda_2; \lambda_1\} = 10^4 \begin{Bmatrix} -1.11656894040187 \\ -1.11656891648987 \\ -1.11266724654359 \\ -1.11266921656658 \end{Bmatrix} \quad (23)$$

Table 1 – Dominant cluster for Transmission Tower.

p	iter	figs	figs/iter	fast enough
1	10	2.14	0.2135	maybe
1	36	4.05	0.1125	maybe
1	836	6.69	0.0080	no
2	10	2.23	0.2229	maybe
2	65	4.05	0.0623	maybe
2	865	5.56	0.0064	no
3	10	2.25	0.2249	maybe
3	38	4.03	0.0839	maybe
3	838	8.86	0.0106	no
4	10	2.18	0.2184	maybe
4	47	4.05	0.0861	maybe
4	149	10.04	0.0674	yes
4	279	15.11	0.0541	yes

Table 2 summarizes the successful application of Algorithm 2.2 to (K, C, M) for computation of a sub-dominant cluster of frequencies. Sub-dominant frequency happens to be sufficiently isolated, that is, the computation is good enough for $p = 1$ and a very simple choice of shifts α. Matrices $\Lambda_s \in \mathbb{R}^{4\times 1}$ and unitary $Y_s \in \mathbb{R}^{153}$ were computed, and are such that

$$\|MY_s\Lambda_s^2 + CY_s\Lambda_s + KY_s\|_F = 7.6833 \times 10^{-11} \quad (24)$$

$$\lambda_{306} = -0.54131285542577 \quad (25)$$

Table 2 – Sub-dominant cluster for Trans. Tower, with shifts.

p	iter	figs	σ	figs/iter	fast enough
1	10	3.37	0	0.3373	yes
1	47	8.003	-0.5365	0.2163	yes
1	57	15.31	-0.5413	0.2687	yes

A shifting strategy that can be deployed to tune-up the computation of real dominant clusters is to apply Algorithm 2.1 to (K, C, M) and generate a real sequence that can be speed-up using shifts σ. Only at the end of computation the quantities are taken to their reciprocals. Table 3 shows the application of this idea to computation of the a dominant cluster, contrasting to Table 1. We were able to compute $\lambda_1 = -1.112669216566579 \times 10^4$ very faster and even apart from the other 3 quantities in the cluster.

Table 3 – Dominant cluster for Trans. Tower, with shifts.

p	iter	figs	β	figs/iter	fast enough
1	10	2.14	0	0.2135	yes
1	29	4.11	-9.1216 $\times 10^{-5}$	0.1419	yes
1	51	8.39	-8.9874 $\times 10^{-5}$	0.1646	yes
1	61	15.03	-8.9874 $\times 10^{-5}$	0.2464	yes

3.2. Extreme frequencies of an Ore Car

Consider the 1473 by 1473 matrices M and K from *Harwell-Boeing* BCSSCTRUC12 matrix set (available in [8]), for the model of an Ore (Mine) Car. For simplicity, take $C = \alpha I$, that is, the structure is assumed to have a very light diagonal damping, where $\alpha = 5.0 \times 10^{-2}$. This mass matrix is known to be positive semi-definite; in particular, it is singular, and therefore no dominant cluster of frequencies are sought.

Table 4 summarizes the successful application of Algorithm 2.2 to (K, C, M) for computation of a sub-dominant cluster of frequencies. Matrices $\Lambda_s \in \mathbb{R}^{2\times 2}$ and unitary $Y_s \in \mathbb{R}^{1473}$ were computed, and are such that

$$\|MY_s\Lambda_s^2 + CY_s\Lambda_s + KY_s\|_F = 4.8505 \times 10^{-8} \quad (26)$$

$$\lambda_{1472,1473} = \left\{ \begin{array}{l} -10.179889412 + 64.80802264 18\mathbf{i} \\ -10.179889412 - 64.80802264 18\mathbf{i} \end{array} \right\} \quad (27)$$

Table 4 – Sub-dominant cluster for lightly damped Ore Car.

p	iter	figs	figs/iter	fast enough
1	30	0.1144	0.0038	maybe
1	130	1.2281	0.0094	maybe
1	180	0.2096	0.0012	no
2	60	1.2832	0.0214	yes
2	210	5.4276	0.0258	yes
2	499	12.015	0.0241	yes

4. CONCLUSION

A method for obtaining clustered dominant and sub-dominant natural frequencies in second-order linear systems has been proposed. The method is a generalization of Power and Inverse Power methods for computing dominant and sub-dominant eigenvalues of a given matrix through subspace iteration, and is also an application of the recently established concept of fundamental solution for second (and higher) order linear systems. Convergence to clusters of real frequencies can be speed-up with shifting strategies and very fast convergence can be obtained in some cases. The method is capable of taking advantage of natural properties of the system matrices (like symmetry and bandwidth - sparsity) in order to achieve better computational performance. Algorithms and numerical examples on test matrices have been provided.

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