

# Identification and the Information Matrix: How to Get Just Sufficiently Rich?

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**Abstract**—In prediction error identification, the information matrix plays a central role. Specifically, when the system is in the model set, the covariance matrix of the parameter estimates converges asymptotically, up to a scaling factor, to the inverse of the information matrix. The existence of a finite covariance matrix thus depends on the positive definiteness of the information matrix, and the rate of convergence of the parameter estimate depends on its “size”. The information matrix is also the key tool in the solution of optimal experiment design procedures, which have become a focus of recent attention. Introducing a geometric framework, we provide a complete analysis, for arbitrary model structures, of the minimum degree of richness required to guarantee the nonsingularity of the information matrix. We then particularize these results to all commonly used model structures, both in open loop and in closed loop. In a closed-loop setup, our results provide an unexpected and precisely quantifiable trade-off between controller degree and required degree of external excitation.

**Index Terms**—Identifiability, information matrix, input richness, transfer of excitation.

## I. INTRODUCTION

THE problem addressed in this paper is: *Given an identifiable model structure, what are the conditions required on the data collection experiment that make the information matrix positive definite?* As we shall see, for an open-loop experiment, the positive definiteness of the information matrix depends on the richness of the input signal; for a closed-loop experiment, it depends on the complexity of the controller and/or on the richness of the externally applied excitation. Our precise contribution therefore will be to determine the smallest degree of input signal richness (in open loop), and the smallest combination of controller complexity and degree of richness of the external excitation (in closed loop) that makes the information matrix non-

singular. Our results will be expressed as necessary and sufficient conditions on signal richness and/or controller complexity. Why is this problem of interest?

The information matrix plays a key role in estimation theory, and in particular in prediction error identification. When the system is in the model set, the asymptotic parameter covariance matrix is, up to a scaling factor, the inverse of the information matrix. Thus, the existence of a finite covariance matrix depends on the positive definiteness of the information matrix, and the rate of convergence towards the global minimum of the prediction error criterion depends on the “size” of this information matrix. When the system is not in the model set, it can be shown that the positive definiteness of the information matrix at some parameter vector  $\theta$  is equivalent with identifiability and informativity of the data at that  $\theta$ : see [1] for details.

Our motivation for the analysis reported in this paper, however, goes beyond this fundamental role of the information matrix in prediction error identification. It was driven by the recent surge of interest in the question of experiment design, itself triggered by the new concept of *least costly identification experiment for robust control* [2]–[5]. Briefly speaking, least costly experiment design for robust control refers to achieving a prescribed accuracy of the estimated model at the lowest possible price, which is typically measured in terms of the duration of the identification experiment, the perturbation induced by the excitation signal, or any combination of these. Central to least costly identification is the information matrix whose inverse is the covariance of the parameter estimates when the system is in the model set. It so happens that the solutions of optimal experiment design problems are most easily expressed in the form of multisines, i.e. input (or reference) signals that have a discrete spectrum. These are precisely the signals that may result in insufficiently informative data, causing the information matrix to be singular. The degree of richness of a signal is precisely connected to the number of points of support of its frequency spectrum.

In this context, questions regarding the minimum excitation, in the sense of smallest degree of richness, that is necessary for the information matrix to be positive definite or to achieve a required level of accuracy become relevant, such as the following:

- 1) what is the smallest degree of input signal richness that is required in an open-loop experiment?
- 2) assuming that the system operates in closed-loop, when is noise excitation sufficient?
- 3) if noise excitation is not sufficient in a closed-loop experiment, then how much additional degree of richness is required at the reference input?

Manuscript received May 14, 2007; revised October 06, 2008. First published November 10, 2009; current version published December 09, 2009. This work was supported by the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office and by the Brazilian Ministry of Education through CAPES. Recommended by Associate Editor B. Ninness.

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Digital Object Identifier 10.1109/TAC.2009.2034199

- 4) assuming that excitation can be applied at different entry points of a multi-input system operating in closed loop, is it necessary to excite each input?

Sufficient conditions for a successful identification using noise excitation only (question 2) have been given, under different sets of assumptions, in [3], [6], [7]. The key condition is in terms of the complexity of the feedback controller; this complexity condition relates the controllability (or observability) indices of the controller to the controllability (or observability) indices of the plant. Question 4 has been addressed in [8] where it is shown that, when identification cannot be performed using noise excitation only, this does not imply that all reference inputs must be excited. The effect on the covariance of the estimated parameters of applying excitation at one or more entry points has been examined in [5] for open loop identification and in [9] for closed-loop identification.

In attempting to address questions 1 and 3 above, we discovered to our surprise that, whereas the scientific literature abounds with sufficient conditions on input signal richness, there appear to be no result on the smallest possible degree of richness that delivers a positive definite information matrix in a given identification setup. In other words, necessary and sufficient conditions on input richness that will guarantee a full rank information matrix are strangely lacking.

The purpose of this contribution, therefore, is to address the following two questions:

- assuming open-loop identification with a given model structure, what is the smallest degree of input signal richness that is necessary to achieve a positive definite information matrix?
- assuming closed-loop identification with a given model structure and assuming that the controller is not sufficiently complex to yield a positive definite information matrix using noise excitation only, what is then the smallest degree of reference signal excitation that is necessary to achieve a positive definite information matrix?

In addressing these questions, we shall proceed through the following steps.

- 1) We observe that the information matrix is the covariance of a regression vector which is the sum of a known excitation signal ( $u$  in open loop,  $r$  in closed loop) filtered by a vector filter of rational functions, and of white noise filtered by another vector filter of rational functions.
- 2) We introduce a geometric framework that allows us to handle in the same way the kernel of the space spanned by stationary stochastic regression vectors and the kernel of the space spanned by vectors of rational transfer functions. This framework is a convenient tool to establish results on the transfer of excitation from input signals to regression vectors through linear time-invariant vector filters.
- 3) Our first main result is then to establish necessary and sufficient conditions on the richness of a scalar input signal to a vector filter of rational transfer functions so that the resulting pseudoregression vector is persistently exciting.
- 4) This main result is then applied to the classical model structures (ARX, ARMAX, BJ, OE) to establish necessary and sufficient conditions on the input signal richness or on the combination of controller complexity and reference

signal richness to guarantee a full rank information matrix. For the closed-loop case, our results establish a precise tradeoff between controller complexity and required reference signal richness.

We note that whereas most of the theory treats the ‘‘classical’’ model structures, which we also specialize to in Section VI, our geometric framework allows us to deal with arbitrary model structures. Our analysis and results will be established for single input single output systems, but the framework we develop lends itself naturally to extensions to multiple input multiple output systems.

The effect of the design parameters of system identification, such as input signal, reference signal or feedback controller, on the information matrix has been much analyzed in recent years in the context of experiment design for system identification [3], [10]–[12]. The geometric approach developed in this paper, which is based on kernel spaces of vector-valued stationary stochastic processes analyzed in the time-domain, is closely related to that developed in [13], [14] for the computation of the variance of a range of quantities that depend on the parameter estimates, which is based on inner products of vector-valued random processes analyzed in the frequency domain.

The paper is organized as follows. In Section II we set up the notations and the key tools of the prediction error identification framework. In Section III, we recall the concepts of identifiability, informative experiments, and the information matrix. The body of our results is in Section IV where we derive necessary and sufficient conditions for the positive definiteness of the information matrix for arbitrary model structures, in open and closed loop. In Section V we present necessary and sufficient conditions on the input signal that make a regressor persistently exciting. In Section VI we particularize the results of Sections IV and V to the most commonly utilized model structures, in open loop and in closed loop; in closed-loop identification, this leads to a remarkable and somewhat unexpected trade-off between controller complexity and required degree of richness of the external excitation. Our theoretical results are illustrated and confirmed by simulations in Section VII. In line with common practice, we conclude with conclusions.

## II. THE PREDICTION ERROR IDENTIFICATION SETUP

Consider the identification of a linear time-invariant discrete-time single-input single-output system  $\mathcal{S}$

$$y(t) = G_0(q)u(t) + H_0(q)e(t). \quad (1)$$

In (1)  $q$  is the forward-shift operator,  $G_0(q)$  and  $H_0(q)$  are the process transfer functions,  $u(t)$  is the control input, and  $e(t)$  is a zero mean white noise sequence with variance  $\sigma_e^2$ . Both transfer functions,  $G_0(q)$  and  $H_0(q)$ , are rational and proper; furthermore,  $H_0(\infty) = 1$ , that is the impulse response  $h(t)$  of the filter  $H_0(q)$  satisfies  $h(0) = 1$ . To be precise, we shall define  $\mathcal{S} \triangleq [G_0(q) H_0(q)]$ .

This true system may be under feedback control with a proper rational stabilizing controller  $K(q)$

$$u(t) = K(q)[r(t) - y(t)]. \quad (2)$$

The system (1) is identified using a model structure parameterized by a vector  $\theta \in \mathfrak{R}^d$ :

$$y(t) = G(q, \theta)u(t) + H(q, \theta)\varepsilon(t). \quad (3)$$

For a given  $\theta \in \mathfrak{R}^d$ ,  $M(\theta) \triangleq [G(q, \theta)H(q, \theta)]$  is called a *model*, while the *model structure*  $\mathcal{M}$  is defined as a differentiable mapping from a connected open subset  $D_\theta \in \mathfrak{R}^d$  to a *model set*  $\mathcal{M}^*$

$$\mathcal{M} : \theta \in D_\theta \longrightarrow M(\theta) = [G(q, \theta) \ H(q, \theta)] \in \mathcal{M}^*. \quad (4)$$

It is assumed that the loop transfer function  $G_0(q)K(q)$  has a non-zero delay, as well as  $G(q, \theta)K(q) \forall \theta \in D_\theta$ . The true system is said to belong to this model set,  $\mathcal{S} \in \mathcal{M}^*$ , if there is a  $\theta_0 \in D_\theta$  such that  $M(\theta_0) = \mathcal{S}$ . In a prediction error identification framework, a model  $M(\theta)$  uniquely defines the one-step-ahead predictor of  $y(t)$  given all input/output data up to time  $t$

$$\hat{y}(t|t-1, \theta) = W_u(q, \theta)u(t) + W_y(q, \theta)y(t) \quad (5)$$

where  $W_u(q, \theta)$  and  $W_y(q, \theta)$  are stable filters obtained from the model  $M(\theta)$  as follows:

$$W_u(q, \theta) = H^{-1}(q, \theta)G(q, \theta), \quad W_y(q, \theta) = [1 - H^{-1}(q, \theta)]. \quad (6)$$

Since there is a 1-1 correspondence between  $[G(q, \theta), H(q, \theta)]$  and  $[W_u(q, \theta), W_y(q, \theta)]$ , the model  $M(\theta)$  will in the future refer indistinctly to either one of these equivalent descriptions. For later use, we introduce the following vector notations:

$$W(q, \theta) \triangleq [W_u(q, \theta) \ W_y(q, \theta)], \quad z(t) \triangleq \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}. \quad (7)$$

We shall also consider throughout this paper that the vector process  $z(t)$  is quasistationary (see Definition 2.1 in [15]), so that the spectral density matrix  $\Phi_z(\omega)$  is well defined. Accordingly, we shall also use the notation

$$\bar{E}[f(t)] \triangleq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^{\infty} E[f(t)]$$

where  $E[\cdot]$  denotes expectation. The one-step-ahead prediction error is defined as

$$\begin{aligned} \varepsilon(t, \theta) &\triangleq y(t) - \hat{y}(t|t-1, \theta) \\ &= y(t) - W(q, \theta)z(t) \\ &= H^{-1}(q, \theta)[y(t) - G(q, \theta)u(t)]. \end{aligned} \quad (8)$$

Using a set of  $N$  input-output data and a least squares prediction error criterion yields the estimate  $\hat{\theta}_N$  [15]

$$\hat{\theta}_N = \arg \min_{\theta \in D_\theta} \frac{1}{N} \sum_{t=1}^N \varepsilon^2(t, \theta). \quad (9)$$

Under mild technical conditions on the data set [15],  $\hat{\theta}_N \xrightarrow{N \rightarrow \infty} \theta^* \triangleq \arg \min_{\theta \in D_\theta} \bar{V}(\theta)$ , with

$$\bar{V}(\theta) \triangleq \bar{E}[\varepsilon^2(t, \theta)]. \quad (10)$$

If  $\mathcal{S} \in \mathcal{M}^*$  and if  $\hat{\theta}_N \xrightarrow{N \rightarrow \infty} \theta_0$ , then the parameter error converges to a Gaussian random variable

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{N \rightarrow \infty} N(0, P_\theta) \quad (11)$$

where

$$P_\theta = [I(\theta)]^{-1} |_{\theta=\theta_0} \quad (12)$$

$$I(\theta) = \frac{1}{\sigma_\varepsilon^2} \bar{E}[\psi(t, \theta)\psi(t, \theta)^T] \quad (13)$$

$$\psi(t, \theta) = \frac{\partial \hat{y}(t|t-1, \theta)}{\partial \theta} = \nabla_\theta W(q, \theta)z(t) \quad (14)$$

and where  $\nabla_\theta W(e^{j\omega}) \triangleq \partial W(e^{j\omega}) / \partial \theta$ . We shall refer to the matrix  $I(\theta)$  as the *information matrix* at  $\theta$ , although in the literature this term usually refers only to its value at  $\theta = \theta_0$ . The matrix  $I(\theta)$  is positive semi-definite by construction and will play the central role in this paper. In fact the paper is dedicated to determining necessary and sufficient conditions on the data set to make  $I(\theta)$  positive definite.

### III. IDENTIFIABILITY, INFORMATIVE DATA, AND THE INFORMATION MATRIX

Several formal concepts of identifiability have been proposed in the scientific literature, and these definitions have evolved over the years. Here we adopt the uniqueness-oriented definition proposed in [15], which refers to the injectivity of the mapping from parameter space to the space of transfer function models.

*Definition 3.1: (Identifiability):* A parametric model structure  $M(\theta)$  is locally identifiable at a value  $\theta_1$  if  $\exists \delta > 0$  such that, for all  $\theta$  in  $\|\theta - \theta_1\| \leq \delta$

$$W(e^{j\omega}, \theta) = W(e^{j\omega}, \theta_1) \text{ at almost all } \omega \Rightarrow \theta = \theta_1.$$

The model structure is globally identifiable at  $\theta_1$  if the same holds for  $\delta \rightarrow \infty$ . Finally, a model structure is globally identifiable if it is globally identifiable at almost all  $\theta_1$ . ■

Most commonly used model structures (except ARX) are not globally identifiable, but they are globally identifiable at all values  $\theta$  that do not cause pole-zero cancellations: see Chapter 4 in [15]. We introduce the *identifiability Gramian*  $\Gamma(\theta) \in \mathfrak{R}^{d \times d}$

$$\Gamma(\theta) \triangleq \int_{-\pi}^{\pi} \nabla_\theta W(e^{j\omega}, \theta) \nabla_\theta W^H(e^{j\omega}, \theta) d\omega \quad (15)$$

where for any  $M(e^{j\omega})$ , the notation  $M^H(e^{j\omega})$  denotes  $M^T(e^{-j\omega})$ . The relevance of this matrix (and the name “identifiability Gramian”) stems from the fact that the positive definiteness of  $\Gamma(\theta_1)$  is a sufficient condition for local identifiability at  $\theta_1$ ; see problem 4G.4 in [15].

*Proposition 3.1:* A parametric model structure  $M(\theta)$  is locally identifiable at  $\theta_1$  if  $\Gamma(\theta_1)$  is nonsingular. ■

*Proof:* For  $\theta$  close to  $\theta_1$  we can write

$$W(e^{j\omega}, \theta) = W(e^{j\omega}, \theta_1) + (\theta - \theta_1)^T \nabla_\theta W(e^{j\omega}, \theta) + \sigma(|\theta - \theta_1|^2)$$

where  $\lim_{\theta \rightarrow \theta_1} (\sigma(|\theta - \theta_1|^2)/|\theta - \theta_1|) = 0$ . Therefore

$$\int_{-\pi}^{\pi} \|W(e^{j\omega}, \theta) - W(e^{j\omega}, \theta_1)\|^2 d\omega = (\theta - \theta_1)^T \Gamma(\theta_1) (\theta - \theta_1) + \rho (|\theta - \theta_1|^2) \quad (16)$$

where  $\lim_{\theta \rightarrow \theta_1} (\rho(|\theta - \theta_1|^2)/|\theta - \theta_1|^2) = 0$ . The result then follows from the definition of local identifiability. ■

Identifiability (local, or global) is a property of the parametrization of the model  $M(\theta)$ . It tells us that if the model structure is globally identifiable at some  $\theta_1$ , then there is no other parameter value  $\theta \neq \theta_1$  that yields the exact same predictor as  $M(\theta_1)$ . However, it does not guarantee that two different models in the model set  $\mathcal{M}^*$  cannot produce the same prediction errors when driven by the same data, thus yielding the same value for the prediction error criterion. This requires, additionally, that the data set is informative enough to distinguish between different predictors, which leads us to the definition of informative data with respect to a model structure.

*Definition 3.2: (Informative Data) [15]:* A quasistationary data set  $z(t)$  is called informative with respect to a parametric model set  $\{M(\theta), \theta \in \mathcal{D}_\theta\}$  if, for any two models  $W(z, \theta_1)$  and  $W(z, \theta_2)$  in that set

$$\begin{aligned} \bar{E}\{[W(z, \theta_1) - W(z, \theta_2)]z(t)\}^2 &= 0 \\ \implies W(e^{j\omega}, \theta_1) &= W(e^{j\omega}, \theta_2) \text{ at almost all } \omega. \end{aligned} \quad (17)$$

■

We observe that the definition of informative data with respect to a parametric model structure is a global one: (17) must hold at all pairs of parameter vectors  $(\theta_1, \theta_2) \in \mathcal{D}_\theta$ . If in addition, the model structure is globally identifiable at  $\theta_1$ , say, then the condition on the left hand side of (17) implies that  $\theta_2 = \theta_1$ , i.e. there can be no  $\theta_2 \neq \theta_1$  for which  $\bar{E}\{[W(z, \theta_1) - W(z, \theta_2)]z(t)\}^2 = 0$ .

The definition of informative data is with respect to a given model set, not with respect to the true system, which may or may not belong to the model set. In an identification experiment, one typically first selects a globally identifiable model structure; this is a user's choice. Experimental conditions must then be selected that make the data informative with respect to that structure; this is again a user's choice. However, the data are generated by the true system, in open or in closed loop. Thus, the conditions that make a data set  $z(t)$  informative with respect to some model structure depend on the true system and on the possible feedback configuration.

We now turn to the information matrix defined in (13) and (14). Combining these expressions and using Parseval's relationship yields

$$I(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \nabla_\theta W(e^{j\omega}, \theta) \Phi_z(\omega) \nabla_\theta W^H(e^{j\omega}, \theta) d\omega \quad (18)$$

where  $\Phi_z(\omega)$  is the power spectrum of the data  $z(t)$  generated by an identification experiment. The expression (18) shows how

the information matrix combines information about the identifiability of the model structure and about the richness<sup>1</sup> of the data (through  $\Phi_z(\omega)$ ). We note that  $I(\theta) > 0$  only if  $\Gamma(\theta) > 0$ , but we shall show that the rank of  $I(\theta)$  can be lower than the rank of  $\Gamma(\theta)$  if the data  $z(t)$  with spectrum  $\Phi_z(\omega)$  are not "rich enough". The main contribution of this paper will be to establish, for a given parametric model structure, the weakest possible richness conditions on the input signal  $u(t)$  (in open-loop identification) or  $r(t)$  (in closed-loop identification) that make the information matrix full rank at all  $\theta$  where the identifiability Gramian  $\Gamma(\theta)$  has full rank.

#### IV. ANALYSIS OF THE INFORMATION MATRIX

The information matrix  $I(\theta)$  can be positive definite only at values of  $\theta$  that are (at least) locally identifiable, i.e. where  $\Gamma(\theta) > 0$ . At those values, the positive definiteness of  $I(\theta)$  depends additionally on the data set through  $\Phi_z(\omega)$ . The focus of the paper, from now on, will be to seek conditions on the data set under which the information matrix  $I(\theta) > 0$  at all values of  $\theta$  at which  $\Gamma(\theta) > 0$ . We will proceed with a geometric approach to this problem, which allows the derivation of generic results valid for any model structure.

To simplify all expressions, we assume that  $\sigma_e = 1$ . The information matrix is then defined as  $I(\theta) = \bar{E}[\psi(t, \theta)\psi^T(t, \theta)]$  where  $\psi(t, \theta) = \nabla_\theta W(q, \theta)z(t)$  is the gradient of the predictor, which we call the *pseudoregression vector*. We first examine the expressions of this gradient, in open loop and in closed loop. To improve readability, we delete the explicit dependence on the variables  $q$  and  $\theta$  whenever it creates no confusion.

##### A. Expressions of the Pseudoregression Vector

In open-loop identification, the data are generated as

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ G_0 & H_0 \end{bmatrix} \begin{bmatrix} u(t) \\ e(t) \end{bmatrix}.$$

Substituting in (5) yields

$$\hat{y}(t|t-1, \theta) = [W_u + W_y G_0]u(t) + W_y H_0 e(t). \quad (19)$$

The pseudoregressor is then expressed in terms of the external signals as

$$\begin{aligned} \psi(t, \theta) &= [\nabla_\theta W_u + \nabla_\theta W_y G_0]u(t) + \nabla_\theta W_y H_0 e(t) \\ &= \frac{1}{H^2(\theta)} [H(\theta) \nabla_\theta G(\theta) + (G_0 - G(\theta)) \nabla_\theta H(\theta)] u(t) \\ &\quad + \frac{H_0}{H^2(\theta)} \nabla_\theta H(\theta) e(t) \\ &\triangleq V_{uol}(q, \theta) u(t) + V_{eol}(q, \theta) e(t). \end{aligned} \quad (20)$$

In closed-loop identification, the data are generated as

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} K & -KH_0 \\ KG_0 & H_0 \end{bmatrix} \begin{bmatrix} r(t) \\ e(t) \end{bmatrix}$$

<sup>1</sup>The concept of richness of a data set will be precisely defined in Section V.

where  $K = K(q)$  is the controller, and  $S = S(q) = 1/(1 + K(q)G_0(q))$  is the sensitivity function. Substituting in (5) yields

$$\hat{y}(t|t-1, \theta) = KS[W_u + W_y G_0]r(t) + H_0 S[W_y - KW_u]e(t). \quad (21)$$

The pseudoregressor is then expressed in terms of the external signals as

$$\begin{aligned} \psi(t, \theta) &= KS[\nabla_\theta W_u + \nabla_\theta W_y G_0]r(t) \\ &\quad + H_0 S[\nabla_\theta W_y - K\nabla_\theta W_u]e(t) \\ &= KS \left\{ \frac{1}{H^2(\theta)} [H(\theta)\nabla_\theta G(\theta) \right. \\ &\quad \left. + (G_0 - G(\theta))\nabla_\theta H(\theta)] \right\} r(t) \\ &\quad + \frac{H_0 S}{H^2(\theta)} \{ \nabla_\theta H(\theta) \\ &\quad - K [H(\theta)\nabla_\theta G(\theta) - G(\theta)\nabla_\theta H(\theta)] \} e(t) \\ &\triangleq V_{rc}(q, \theta)r(t) + V_{ec}(q, \theta)e(t). \quad (22) \end{aligned}$$

We observe that, for both open loop and closed loop identification, the pseudoregressor  $\psi(t, \theta)$  that “feeds” the information matrix can generically be written as

$$\psi(t, \theta) = V_w(q, \theta)w(t) + V_e(q, \theta)e(t) \quad (23)$$

where  $w(t)$  is a known excitation signal ( $u(t)$  or  $r(t)$ ),  $e(t)$  is white noise independent of  $w(\cdot)$ , while  $V_w(q, \theta)$  and  $V_e(q, \theta)$  are  $d$ -vectors of stable rational transfer functions.

### B. Range and Kernel of Quasistationary Vector Processes

In order to study the rank of the information matrix  $I(\theta)$  defined in (13) where  $\psi(t, \theta)$  takes the form (23), we introduce the following definitions.

**Definition 4.1:** Let  $V(q) : \mathcal{C} \mapsto \mathcal{K}^d(q)$  be a  $d$ -vector of proper stable rational transfer functions. The left-kernel of  $V(q)$ , denoted  $\text{Ker}\{V(q)\}$ , is the set spanned by all real-valued vectors  $\alpha \in \mathfrak{R}^d$  such that  $\alpha^T V(q) = 0 \forall q \in \mathcal{C}$ . Its dimension is called the nullity and annotated  $\nu_V$ . The rank of  $V(q)$  is defined as  $\rho_V = d - \nu_V$ , and  $V(q)$  is said to have full rank if  $\rho_V = d$ . ■

**Definition 4.2:** Let  $\psi(t) : \mathfrak{R} \mapsto \mathfrak{R}^d$  be a  $d$ -vector of quasistationary processes. The left-kernel of  $\psi(t)$ , denoted  $\text{Ker}\{\psi(t)\}$ , is the set spanned by all real-valued vectors  $\alpha \in \mathfrak{R}^d$  such that  $\bar{E}[\alpha^T \psi(t)]^2 = 0$ , or alternatively  $\alpha^T \Phi_\psi(\omega)\alpha = 0 \forall \omega$  where  $\Phi_\psi(\omega)$  is the spectral density matrix of  $\psi(t)$ . Its dimension is called the nullity and annotated  $\nu_\psi$ . The rank of  $\psi(t)$  is defined as  $\rho_\psi = d - \nu_\psi$ , and  $\psi(t)$  is said to have full rank if  $\rho_\psi = d$ . ■

The analysis of the rank of  $I(\theta)$  thus reduces to the analysis of the rank of  $\psi(t, \theta)$  since, with Definition 4.2, we have  $\text{rank}\{I(\theta)\} = \text{rank}\{\psi(t, \theta)\}$ . To study the rank of  $\psi(t, \theta)$ , we need to study

- i) how the rank of the sum of two vector stochastic processes relates to the rank of each term of the sum;
- ii) how the rank of a pseudoregressor vector  $\psi(t)$  generated by  $\psi(t) = V(q)u(t)$  relates to the rank of  $V(q)$  and the

properties of the scalar signal  $u(t)$  that generates it—we call this problem “transfer of excitation”.

The solution to these two problems starts with the following result.

**Theorem 4.1:** Let  $\psi(t, \theta) \in \mathcal{R}^d$  be given as in (23) where  $V_w(q, \theta)$  and  $V_e(q, \theta)$  are  $d$ -vectors of stable proper rational functions,  $w(t)$  is quasistationary and  $e(t)$  is white noise. Then

1)

$$\text{Ker}\{V_w(q, \theta)\} \subseteq \text{Ker}\{V_w(q, \theta)w(t)\} \quad (24)$$

2)

$$\text{Ker}\{V_e(q, \theta)\} = \text{Ker}\{V_e(q, \theta)e(t)\} \quad (25)$$

3)  $\psi(t, \theta)$  has full rank, or equivalently  $I(\theta) > 0$ , if and only if

$$\begin{aligned} &\text{Ker}\{V_w(q, \theta)w(t) + V_e(q, \theta)e(t)\} \\ &= \text{Ker}\{V_w(q, \theta)w(t)\} \cap \text{Ker}\{V_e(q, \theta)\} = \{\mathbf{0}\}. \quad (26) \end{aligned}$$

■

*Proof:* Denote  $\psi_1(t, \theta) \triangleq V_w(q, \theta)w(t)$ . Result (1) follows by observing that, for any  $\alpha \in \mathfrak{R}^d$ :

$$\begin{aligned} &\alpha^T \bar{E} [\psi_1(t, \theta)\psi_1^T(t, \theta)] \alpha \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha^T V_w(e^{j\omega}, \theta) \Phi_w(\omega) V_w^H(e^{j\omega}, \theta) \alpha \quad d\omega. \end{aligned}$$

Result (2) follows similarly by noting that  $\Phi_e(\omega)$  is a constant. To prove result (3), note that the independence of the signals  $w$  and  $e$  implies that

$$\begin{aligned} \alpha^T \bar{E} [\psi(t, \theta)\psi^T(t, \theta)] \alpha &= \bar{E} [\alpha^T V_w(q, \theta)w(t)]^2 \\ &\quad + \bar{E} [\alpha^T V_e(q, \theta)e(t)]^2. \quad (27) \end{aligned}$$

Therefore  $\alpha \in \text{Ker}\{\psi(t, \theta)\}$  if and only if  $\alpha$  belongs to the left-kernels of both  $V_w(q, \theta)w(t)$  and  $V_e(q, \theta)e(t)$ , and hence to their intersection. Next, it follows from result (2) that  $\text{Ker}\{V_e(q, \theta)e(t)\} = \text{Ker}\{V_e(q, \theta)\}$ . ■

Result (2) of Theorem 4.1 states that white noise causes no drop of rank. The same is actually true for any input signal that has a continuous spectrum. For the part of  $\psi(t, \theta)$  driven by the controlled signals  $u(t)$  or  $r(t)$  (see (20) and (22)), we want to consider input signals ( $u(t)$  or  $r(t)$ ) that have discrete spectra, such as multisines; this problem is treated in Section V.

Observe that the condition (26) uses the two distinct but compatible notions of kernel, defined in Definitions 4.1 and 4.2 respectively, in the same statement. This condition shows how the positive definiteness of  $I(\theta)$  depends on both the model structure through  $V_{uol}(q, \theta)$  and  $V_{eol}(q, \theta)$  (respectively,  $V_{rc}(q, \theta)$  and  $V_{ec}(q, \theta)$ ) and the excitation signal  $u(t)$  (respectively  $r(t)$ ): see (20) and (22). We now elaborate on these conditions, separately for the open-loop and for the closed-loop identification setup.

*C. Positive Definiteness Conditions for  $I(\theta)$  in Open and Closed-Loop Identification*

*Open-Loop Identification:* In open-loop identification, the pseudoregression vector is given by (20). We then have the following result.

*Theorem 4.2:* Let  $\mathcal{N}_H$  denote the left-kernel of  $\nabla_\theta H(q, \theta)$ . Then  $I(\theta)$  is positive definite if and only if either  $\mathcal{N}_H = \{\mathbf{0}\}$  or, for each non-zero  $d$ -vector  $\alpha \in \mathcal{N}_H$  we have

$$\bar{E} [\alpha^T \nabla_\theta G(q, \theta) u(t)]^2 \neq 0. \tag{28}$$

*Proof:* First note that the set of vectors  $\{\alpha \in \mathcal{N}_H \subseteq \mathcal{R}^d\}$  spans  $\text{Ker}\{\nabla_\theta W_y(q, \theta)\} = \text{Ker}\{V_{eol}(q, \theta)\}$ . Therefore, by Theorem 4.1,  $I(\theta) > 0$  if and only if either  $\mathcal{N}_H = \{\mathbf{0}\}$  or if, for each nonzero  $\alpha \in \mathcal{N}_H$ , we have  $\bar{E}[\alpha^T(\nabla_\theta W_u + G_0 \nabla_\theta W_y)u(t)]^2 \neq 0$ . Since  $\alpha^T \nabla_\theta H(q, \theta) = \mathbf{0}$ , this is equivalent with  $\bar{E}[\alpha^T \nabla_\theta G(q, \theta) u(t)]^2 \neq 0$ . ■

*Closed-Loop Identification:* In closed-loop identification, the pseudoregression vector is given by (22). We then have the following result.

*Theorem 4.3:* Let  $\mathcal{N}_{V_{ecl}}$  denote the left-kernel of  $V_{ecl}(q, \theta)$ . Then  $I(\theta)$  is positive definite if and only if either  $\mathcal{N}_{V_{ecl}} = \{\mathbf{0}\}$  or, for each non-zero  $d$ -vector  $\alpha \in \mathcal{N}_{V_{ecl}}$  we have

$$\bar{E} [\alpha^T \nabla_\theta W_y(q, \theta) r(t)]^2 = \bar{E} [\alpha^T K(q) \nabla_\theta W_u(q, \theta) r(t)]^2 \neq 0. \tag{29}$$

*Proof:* First note that for each  $\alpha \in \mathcal{N}_{V_{ecl}} \subseteq \mathcal{R}^d$  we have  $\alpha^T \nabla_\theta W_y(q, \theta) = \alpha^T K(q) \nabla_\theta W_u(q, \theta)$ . By Theorem 4.1,  $I(\theta) > 0$  if and only if either  $\mathcal{N}_{V_{ecl}} = \{\mathbf{0}\}$  or if, for each non-zero  $\alpha \in \mathcal{N}_{V_{ecl}}$ , we have  $\bar{E} [\alpha^T K S(\nabla_\theta W_u + G_0 \nabla_\theta W_y) r(t)]^2 \neq 0$ . Now observe that  $\alpha^T K S(\nabla_\theta W_u + G_0 \nabla_\theta W_y) = \alpha^T S(1 + K G_0) \nabla_\theta W_y = \alpha^T \nabla_\theta W_y = \alpha^T K \nabla_\theta W_u$ . This proves the result. ■

V. TRANSFER OF EXCITATION

This section is entirely devoted to the relationship between the rank of  $V(q)$ , the rank of  $\psi(t)$  and the properties of the scalar signal  $u(t)$  when  $\psi(t) = V(q)u(t)$ . It contains the main technical result of this paper, in the form of necessary and sufficient conditions on the richness of  $u(t)$  such that  $\text{rank}\{\psi(t)\} = \text{rank}\{V(q)\}$ . In order to analyze the rank properties of regressors obtained by filtering scalar signals with discrete spectra, we introduce the concept of degree of richness of a signal and of a persistently exciting regression vector.

*Definition 5.1:* A quasistationary vector signal  $\psi(t)$  is called *persistently exciting (PE)* if  $\bar{E}[\psi(t)\psi^T(t)] > 0$ . ■

Whether a quasistationary vector signal  $\psi(t)$  obtained as a filtered version (by a vector  $V(q)$  of transfer functions) of a quasistationary scalar signal  $u(t)$  is PE or not depends on whether  $\text{Ker}\{V(q)\} = \{\mathbf{0}\}$  but also on the degree of richness of the input  $u(t)$ . The richness of a scalar signal is defined as follows.

*Definition 5.2:* A quasistationary scalar signal  $u(t)$  is *sufficiently rich of order  $n$  (denoted SRn)* if the following regressor is PE

$$\phi_{1,n}(t) \triangleq \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n) \end{bmatrix} = \begin{bmatrix} q^{-1} \\ q^{-2} \\ \vdots \\ q^{-n} \end{bmatrix} u(t). \tag{30}$$

It is *sufficiently rich of order exactly  $n$  (denoted SREn)* if it is SRn but not SRn+1. ■

Definition 5.2 is equivalent with many other classically used definitions, except that nowadays the most common terminology is to say that a signal is PE of order  $n$  rather than SR of order  $n$ . At the risk of being considered old-fashioned, we prefer the term *sufficiently rich* because sufficient reflects the notion of *degree of richness* while persistent does not. The vector  $\phi_{1,n}(t)$  serves as a basis for all regression vectors that are obtained as (vector)-filtered versions of a scalar signal  $u(t)$ . For future use, we introduce the notation

$$\mathcal{B}_{k,n}(q) \triangleq [q^{-k} \quad q^{-k-1} \quad \dots \quad q^{-n}]^T, \text{ for } k \leq n. \tag{31}$$

By our assumption of quasistationarity,  $u(t)$  is SRn if  $\mathcal{B}_{k+1,k+n}(q)u(t)$  is PE for any  $k$ . Thus, we could just as well have used  $\phi_{0,n-1}(t)$  in lieu of  $\phi_{1,n}(t)$  in Definition 5.2: the definition is shift-invariant. **We denote by  $\mathcal{U}_n$  the set of all SRn signals.** We now address the following problem.

What are the necessary and sufficient conditions on the richness of  $u(t)$  such that  $\text{Ker}\{\psi(t)\} = \text{Ker}\{V(q)\}$  when  $\psi(t) = V(q)u(t)$ ?

To help us solve this problem, we have . . . . . not much. As it happens, the only available results, as far as we know, are sufficiency results [7], [16], [17]. The vector filter  $V(q)$  can be uniquely decomposed as

$$V(q) = \frac{N(q^{-1})}{D(q^{-1})} = \frac{q^{-m}}{D(q^{-1})} R \mathcal{B}_{0,k-1}(q) \tag{32}$$

where  $D(q^{-1}) = 1 + d_1 q^{-1} + \dots + d_p q^{-p}$ , with  $d_p \neq 0$ ,  $R \in \mathcal{R}^{d \times k}$  is the matrix of real coefficients of the expansion of the numerator matrix  $N(q^{-1})$  into powers of  $q^{-1}$ , and  $m$  is a possible common delay in all elements of  $N(q^{-1})$ .

*Lemma 5.1:* Let  $\psi(t) = V(q)u(t)$  with  $\psi(t) \in \mathcal{R}^d$ ,  $u(t)$  quasistationary,  $V(q)$  proper and stable, and let  $V(q)$  be decomposed as in (32) with  $\rho_V = c$ . Let the rows of  $Q \in \mathcal{R}^{c \times k}$  be a basis for the rowspace of  $R$ , and define the  $c$ -vectors  $U(q) = (q^{-m}/D(q^{-1}))Q \mathcal{B}_{0,k-1}(q)$  and  $\phi(t) = U(q)u(t)$ . Then, for any  $u(t)$ ,  $\text{Ker}\{\psi(t)\} = \text{Ker}\{V(q)\}$  if and only if  $\text{Ker}\{\phi(t)\} = \text{Ker}\{U(q)\} = \mathbf{0}$ .

*Proof:* Since the rows of  $Q$  form a basis for the rowspace of  $R$  we can write

$$R = T \begin{bmatrix} Q \\ 0 \end{bmatrix} \tag{33}$$

for some nonsingular matrix  $T \in \mathcal{R}^{d \times d}$ . Then, for any  $\alpha \in \mathcal{R}^d$  we have

$$\alpha^T R = \alpha^T T \begin{bmatrix} Q \\ 0 \end{bmatrix} = \beta^T Q \tag{34}$$

where  $\beta$  is uniquely defined by  $\alpha^T T \triangleq (\beta^T \gamma^T)$  with  $\beta \in \mathcal{R}^c$  and  $\gamma \in \mathcal{R}^{d-c}$ . It follows from (34) that

$$\begin{aligned} \alpha^T \psi(t) &= \frac{q^{-m}}{D(q^{-1})} \alpha^T R \mathcal{B}_{0,k-1}(q) u(t) \\ &= \frac{q^{-m}}{D(q^{-1})} \beta^T Q \mathcal{B}_{0,k-1}(q) u(t) = \beta^T \phi(t). \end{aligned}$$

Therefore the following four statements are equivalent:

- $\text{Ker}\{\psi(t)\} = \text{Ker}\{V(q)\}$ ;
- $\bar{E}[\alpha^T \psi(t)]^2 = 0$  iff  $\alpha^T R \in \text{Ker}\{\mathcal{B}_{0,k-1}(q)u(t)\}$ ;
- $\bar{E}[\beta^T \phi(t)]^2 = 0$  iff  $\beta^T Q \in \text{Ker}\{\mathcal{B}_{0,k-1}(q)u(t)\}$ ;
- $\text{Ker}\{\phi(t)\} = \text{Ker}\{U(q)\}$ .

Finally, since  $Q$  has full rank,  $\text{Ker}\{U(q)\} = \mathbf{0}$ . ■

We can now state our main result on transfer of excitation.

**Theorem 5.1:** Let  $\psi(t) \triangleq V(q)u(t)$  with  $\psi(t) \in \mathcal{R}^d$ ,  $u(t)$  quasistationary,  $V(q)$  proper and stable, and let  $V(q)$  be decomposed as in (32). Then  $\text{Ker}\{\psi(t)\} = \text{Ker}\{V(q)\}$  for almost all  $u(t) \in \mathcal{U}_n$  if and only if  $n \geq \rho_V$ .

*Proof:* If  $\rho_V < d$ , we can replace  $\psi(t) = V(q)u(t)$  by  $\phi(t) = U(q)u(t)$  with  $U(q)$  defined from  $V(q)$  as in Lemma 5.1 above, where  $U(q)$  has full rank. Thus, using Lemma 5.1, we can assume without loss of generality that  $\rho_V = d$ . Using Parseval's Theorem and (32) we can write

$$\begin{aligned} \bar{E}[\alpha^T \psi(t)]^2 &= \bar{E} \left[ \alpha^T \frac{q^{-m}}{D(q^{-1})} R \mathcal{B}_{0,k-1}(q) u(t) \right]^2 \\ &= \alpha^T R \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) \frac{1}{|D(e^{-j\omega})|^2} \mathcal{B}_{0,k-1}(e^{-j\omega}) \right. \\ &\quad \left. \mathcal{B}_{0,k-1}^H(e^{-j\omega}) d\omega \right) R^T \alpha. \end{aligned}$$

Let  $u(t)$  be SREn with finite  $n$ . Its spectrum can then be written as  $\Phi_u(\omega) = \sum_{i=1}^n \lambda_i \delta(\omega - \omega_i)$  where  $\delta(\cdot)$  is the Dirac impulse function,  $\lambda_i \in \mathfrak{R}$ , and the  $\omega_i$  are  $n$  distinct frequencies in the interval  $(-\pi, \pi]$ . Define its support as the vector  $\mathbf{f} = [e^{j\omega_1} e^{j\omega_2} \dots e^{j\omega_n}] \in \Omega_n$ , where  $\Omega_n \subset \mathcal{C}^n$  is the set of all supports  $\mathbf{f}$  which result in an SREn signal, that is, those  $\mathbf{f}$  such that the  $e^{j\omega_i}$  are distinct.  $\Omega_n$  is an  $n$ -dimensional subset of  $\mathcal{C}^n$  which defines the class of signals  $u(t)$  that we consider. Then we can write

$$\begin{aligned} \bar{E}[\alpha^T \psi(t)]^2 &= \alpha^T R \left( \frac{1}{2\pi} \sum_{i=1}^n \lambda'_i \mathcal{B}_{0,k-1}(e^{-j\omega_i}) \right. \\ &\quad \left. \times \mathcal{B}_{0,k-1}^H(e^{-j\omega_i}) \right) R^T \alpha \end{aligned}$$

where  $\lambda'_i = \lambda_i / |D(e^{-j\omega_i})|^2$ . Hence

$$\bar{E}[\alpha^T \psi(t)]^2 = \alpha^T R F(\mathbf{f}) \Lambda F^H(\mathbf{f}) R^T \alpha \quad (35)$$

with

$$F(\mathbf{f}) = [\mathcal{B}_{0,k-1}(e^{-j\omega_1}) \quad \mathcal{B}_{0,k-1}(e^{-j\omega_2}) \quad \dots \quad \mathcal{B}_{0,k-1}(e^{-j\omega_n})] \quad (36)$$

and  $\Lambda = \text{diag}\{\lambda'_1, \lambda'_2, \dots, \lambda'_n\}$ . Note that for  $k \geq n$ , the  $n$  first rows of  $F(\mathbf{f})$  form a Vandermonde matrix; since all  $e^{j\omega_i}$  are

distinct, it follows that this Vandermonde matrix has full rank. Therefore  $\rho(F(\mathbf{f})) = n$  whenever  $k \geq n$ . But  $\psi(t)$  is full rank if and only if  $P(\mathbf{f}) \triangleq R F(\mathbf{f}) \Lambda F^H(\mathbf{f}) R^T$  has rank equal to  $d$ , which is equivalent to  $\det(P(\mathbf{f})) \neq 0$ . Suppose that  $n < d$ ; then  $\rho(F(\mathbf{f})) = n < d$  which, noting that  $\rho(P(\mathbf{f})) \leq \rho(F(\mathbf{f}))$ , implies  $\det(P(\mathbf{f})) = 0$ , thus proving the necessity of  $n \geq d$ .

For  $n \geq d$ , the determinant  $\det(P(\mathbf{f}))$  is a nontrivial polynomial in the vector variable  $\mathbf{f}$  and  $\psi(t)$  loses rank exactly at the roots of this polynomial. Since the roots of a polynomial define a set of measure zero in the space of its variable,  $\psi(t)$  is full-rank for almost all  $\mathbf{f} \in \Omega_n$ . ■

Theorem 5.1 completely characterizes the required signal richness of  $u(t)$  that keeps the kernel of the regressor vector  $\psi(t) \triangleq V(q)u(t)$  identical to the kernel of  $V(q)$ , thus causing no drop of rank, i.e.  $\rho_\psi = \rho_V$ . It tells us that signals that do not result in full transfer of excitation form a set of measure zero in the set of signals. The next theorem explains that there is indeed a gap between a necessary condition and a sufficient condition on the signal richness for the preservation of the rank. This gap explains why the rank preservation property holds for almost all  $u(t) \in \mathcal{U}_n$ .

**Theorem 5.2:** Let  $\psi(t) \triangleq V(q)u(t)$  with  $\psi(t) \in \mathcal{R}^d$ ,  $u(t)$  quasistationary,  $V(q)$  proper and stable, and let  $V(q)$  be decomposed as in (32) with  $\text{rank}(V) = \rho_V$ .

- If  $u(t)$  is **not** SR of order  $\rho_V$ , then  $\text{Ker}(\psi(t)) \supset \text{Ker}(V(q))$ .
- If  $u(t)$  is SRk then  $\text{Ker}(\psi(t)) = \text{Ker}(V(q))$ .

*Proof:* With  $R$  and  $F$  defined as in (32) and (36), this result follows immediately from Sylvester's inequality:

$$\rho(R) + \rho(F) - k \leq \rho(RF) \leq \min(\rho(R), \rho(F))$$

which yields  $\rho(RF) < \rho(R)$  for  $n < \rho_V$  and  $\rho(RF) \geq \rho(R)$  for  $n \geq k$ . ■

Theorem 5.2 provides a necessary richness condition - SR of order  $\rho_V$  - and a sufficient condition - SR of order  $k$ . Hence, any signal that is SRk guarantees that  $\text{Ker}(\psi(t)) = \text{Ker}(V(q))$  and any signal that is not SR of order  $\rho_V$  guarantees that  $\text{Ker}(\psi(t)) \neq \text{Ker}(V(q))$ . For signals with richness in between  $\rho_V$  and  $k$ , some will result in full transfer of excitation - that is,  $\text{Ker}(\psi(t)) = \text{Ker}(V(q))$  - and some will not. That is, for such signals, only a few frequencies - a set of measure zero on the real axis - will cause  $\text{Ker}(\psi(t)) \neq \text{Ker}(V(q))$ . The following example illustrates our results.

**Example 5.1:** Consider the regressor  $\psi(t) = V(q)u(t)$ , with  $V(q) \triangleq R[1 \ q^{-1} \ q^{-2} \ q^{-3} \ q^{-4}]^T$  where

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider first  $u(t) = \lambda_1 + \lambda_2 \sin(\omega_1 t)$ , which is SRE3. For such signal,  $RF$  is a  $3 \times 3$  matrix, whose determinant is  $\det(RF) = -2j[3 \sin(\omega_1) - 2 \sin(2\omega_1) - \sin(3\omega_1) + \sin(4\omega_1)]$ . Its roots in  $(-\pi, \pi]$  are at  $(-\pi/3), 0, \pi/3$  and  $\pi$ , but  $\omega_1 = 0$  and  $\omega_1 = \pi$  do not keep  $u(t) \in \mathcal{U}_3$ . Thus,  $\psi(t)$  will have rank 3 for all  $u(t) \in \mathcal{U}_3$  except for  $u(t) = \lambda_1 + \lambda_2 \sin((\pi/3)t)$ .

Now let  $u(t) = \lambda_1 \sin(\omega_1 t) + \lambda_2 \sin(\omega_2 t)$  which is SRE4 provided that  $\omega_1 \neq \omega_2$  and that neither  $\omega_1$  nor  $\omega_2$  equal 0 or  $\pi$ . We have

$$RF = \begin{bmatrix} e^{j\omega_1} + e^{j4\omega_1} & e^{j\omega_2} + e^{j4\omega_2} & e^{-j\omega_1} + e^{-j4\omega_1} & e^{-j\omega_2} + e^{-j4\omega_2} \\ e^{j2\omega_1} & e^{j2\omega_2} & e^{-j2\omega_1} & e^{-j2\omega_2} \\ 1 + e^{j4\omega_1} & 1 + e^{j4\omega_2} & 1 + e^{-j4\omega_1} & 1 + e^{-j4\omega_2} \end{bmatrix}.$$

It is rather easy to see that  $RF$  will have full rank for all values of  $\omega_1$  and  $\omega_2$  for which  $u(t)$  is SRE4, except those for which  $\omega_1 + \omega_2 = \pi$ . We observe, using Theorem 5.2, that the richness of the two considered signals are in between the necessary richness ( $u(t)$  is SR3) and the sufficient richness ( $u(t)$  is SR5).

## VI. POSITIVE DEFINITENESS OF $I(\theta)$ FOR ARMAX AND BJ MODEL STRUCTURES

The combined results of Theorems 4.2, 4.3 and 5.1 produce necessary and sufficient richness conditions on the excitation signal that guarantee positive definiteness of the information matrix at all  $\theta$  at which the model structure is identifiable, i.e. at which  $\Gamma(\theta) > 0$ . In order to show how these results materialize into explicit richness conditions on the external signals as a function of the polynomial degrees of the model structure and the controller (in a closed-loop setup), we derive these explicit conditions for the ‘‘classical’’ model structures: ARMAX, ARX, BJ and OE. We do this for both open-loop and closed-loop identification.

### A. Open-Loop Identification

**ARMAX Model Structure:** Consider the ARMAX model structure

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})e(t), \text{ where} \quad (37)$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a} \quad (38)$$

$$B(q^{-1}) = b_1 q^{-1} + \dots + b_{n_b} q^{-n_b} \quad (39)$$

$$C(q^{-1}) = 1 + c_1 q^{-1} + \dots + c_{n_c} q^{-n_c}. \quad (40)$$

We comment that for ARMAX model structures, one must consider as generic the possible existence of common roots between the polynomials  $A$  and  $B$ , as well as between  $A$  and  $C$ . However, the three polynomials  $A$ ,  $B$  and  $C$  must be coprime at any identifiable  $\theta$ ; at a  $\theta$  at which  $A$ ,  $B$  and  $C$  have a common root,  $\Gamma(\theta)$  would be singular.

In open-loop identification it is well known (see Theorem 13.1 in [15], for instance) that an excitation that is SR of order  $n_b + n_a$  is sufficient for an experiment to be informative both for ARMAX and Box-Jenkins model structures. Using the results of Sections IV and V we derive similar results, in the form of necessary and sufficient conditions.

**Theorem 6.1:** For the ARMAX model structure (37), the information matrix  $I(\theta)$  is positive definite at a  $\theta$  at which the model structure is identifiable if and only if  $u(t)$  is SRk, where  $k = n_b + n_u(\theta)$  and  $n_u(\theta)$  is the number of common roots of the polynomials  $A(q^{-1})$  and  $C(q^{-1})$  at that  $\theta$ .  $I(\theta)$  is positive

definite at all  $\theta$  at which the model structure is identifiable if and only if  $u(t)$  is SRk with  $k = n_b + \min\{n_a, n_c\}$ .

*Proof:* For the ARMAX model structure, we have

$$\nabla_{\theta} G = \frac{1}{A^2} \begin{pmatrix} -Bq^{-1} \\ \vdots \\ -Bq^{-n_a} \\ \hline Aq^{-1} \\ \vdots \\ Aq^{-n_b} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \nabla_{\theta} H = \frac{1}{A^2} \begin{pmatrix} -Cq^{-1} \\ \vdots \\ -Cq^{-n_a} \\ \hline 0 \\ \vdots \\ 0 \\ \hline Aq^{-1} \\ \vdots \\ Aq^{-n_c} \end{pmatrix}. \quad (41)$$

Let  $\alpha^T = (\alpha_A^T | \alpha_B^T | \alpha_C^T)$  denote any vector in the left-kernel of  $\nabla_{\theta} H(q, \theta)$ , and let  $\gamma_A(q^{-1}) \triangleq \alpha_A^T \mathcal{B}_{1, n_a}$ ,  $\gamma_B(q^{-1}) \triangleq \alpha_B^T \mathcal{B}_{1, n_b}$ , and  $\gamma_C(q^{-1}) \triangleq \alpha_C^T \mathcal{B}_{1, n_c}$ . Then

$$\begin{aligned} \alpha^T \nabla_{\theta} H(q, \theta) = 0 &\Leftrightarrow \gamma_A(q^{-1})C(q^{-1}) = \gamma_C(q^{-1})A(q^{-1}) \\ &\Leftrightarrow q\gamma_A(q^{-1})C(q^{-1}) = q\gamma_C(q^{-1})A(q^{-1}). \end{aligned} \quad (42)$$

At a value of  $\theta$  at which the polynomials  $A$  and  $C$  are coprime, it follows from the theory of Diophantine equations (see e.g. [18]) that  $\alpha_A = \mathbf{0}$  and  $\alpha_C = \mathbf{0}$ , because  $\deg(q\gamma_A(q^{-1})) < \deg(A(q^{-1}))$  and  $\deg(q\gamma_C(q^{-1})) < \deg(C(q^{-1}))$ . Consider now a  $\theta$  at which there are common factors between  $A$  and  $C$  and let  $U(q^{-1}) \triangleq u_0 + u_1 q^{-1} + \dots + u_{n_u} q^{-n_u}$  denote the Greatest Common Divisor (GCD) of  $A$  and  $C$ . Then  $A = A_1 U$  and  $C = C_1 U$  for some coprime polynomials  $A_1$  and  $C_1$ . Then (42) is equivalent with  $q\gamma_A(q^{-1})C_1(q^{-1}) = q\gamma_C(q^{-1})A_1(q^{-1})$  where  $\deg(q\gamma_A) = n_a - 1$  and  $\deg(q\gamma_C) = n_c - 1$ . The set of all solutions of this equation is described by

$$q\gamma_A = \alpha_A^T \mathcal{B}_{0, n_a - 1} = A_1 T, \quad q\gamma_C = \alpha_C^T \mathcal{B}_{0, n_c - 1} = C_1 T \quad (43)$$

where  $T(q^{-1}) \triangleq t_0 + t_1 q^{-1} + \dots + t_{n_u - 1} q^{-n_u - 1}$  is a polynomial of degree  $n_u - 1$  with arbitrary coefficients. The left-kernel of  $\nabla_{\theta} H(q, \theta)$  is thus defined by those vectors  $\alpha^T = (\alpha_A^T | \alpha_B^T | \alpha_C^T)$  such that  $\alpha_A$  and  $\alpha_C$  are solution of (43), while  $\alpha_B$  is arbitrary. As stated earlier, we consider values of  $\theta$  at which  $\Gamma(\theta) > 0$ . At these values of  $\theta$ ,  $\alpha^T \nabla_{\theta} G(q, \theta) \neq 0$  for all vectors  $\alpha$  defined above and, by Theorem 4.2,  $I(\theta) > 0$  if and only if  $u(t)$  is such that  $\bar{E}[\alpha^T \nabla_{\theta} G(q, \theta) u(t)]^2 \neq 0$  for all such  $\alpha$ . For such  $\alpha$ , we have

$$\begin{aligned} \alpha^T \nabla_{\theta} G(q, \theta) u(t) &= \frac{1}{A^2} [-\alpha_A^T \mathcal{B}_{1, n_a} B + \alpha_B^T \mathcal{B}_{1, n_b} A] u(t) \\ &= \frac{1}{A^2} [-q^{-1} A_1 T B + \alpha_B^T \mathcal{B}_{1, n_b} A_1 U] u(t) \\ &= \frac{1}{AU} [-q^{-1} T B + \alpha_B^T \mathcal{B}_{1, n_b} U] u(t) \end{aligned}$$

where the coefficients of the polynomial  $T$ , of degree  $n_u - 1$ , as well as the coefficients of  $\alpha_B$  are completely free. Therefore



$\bar{E}[\alpha^T \nabla_\theta G(q, \theta) u(t)]^2 \neq 0$  if and only if the following pseudoregressor has full rank:

$$\psi(t) = \frac{1}{AU} \begin{pmatrix} -Bq^{-1} \\ \vdots \\ -Bq^{-n_u} \\ Uq^{-1} \\ \vdots \\ Uq^{-n_b} \end{pmatrix} u(t) = \frac{1}{AU} R \mathcal{B}_{1, n_b + n_u} u(t) \quad (44)$$

where  $R \in \mathcal{R}^{(n_b + n_u) \times (n_b + n_u)}$  is a Sylvester matrix. Since  $A$ ,  $B$ ,  $C$  are coprime at all  $\theta$ , and  $U$  is the common factor of  $A$  and  $C$  at the considered  $\theta$ , it follows that  $B$  and  $U$  are coprime, and hence  $R$  in (44) is nonsingular. Therefore, by Theorem 5.1,  $\psi(t)$  in (44) is PE (and hence  $I(\theta) > 0$ ) if and only if  $u(t)$  is sufficiently rich of degree  $n_b + n_u(\theta)$ , where  $n_u(\theta)$  represents the number of common roots between  $A$  and  $C$  at the considered  $\theta$ . Since the maximum number of such common roots is  $\min\{n_a, n_c\}$ ,  $I(\theta)$  is positive definite at all identifiable  $\theta$  if and only if  $u(t)$  is SRk with  $k = n_b + \min\{n_a, n_c\}$ . ■

The corresponding result for an ARX model structure follows immediately as a corollary. We remind the reader that an ARX structure is globally identifiable at all  $\theta$  (see Definition 3.1) [15].

*Corollary 6.1:* For the ARX model structure  $A(q^{-1})y(t) = B(q^{-1})u(t) + e(t)$  with  $A(q^{-1})$  and  $B(q^{-1})$  as in (38) and (39), the information matrix  $I(\theta)$  is positive definite at all  $\theta$  if and only if  $u(t) \in \mathcal{U}_{n_b}$ . ■

This same condition - SR of order  $n_b$  - is known to be necessary and sufficient for informativity of data for ARX structures in open-loop identification [7].

*Box-Jenkins Model Structure:* Consider now the BJ model structure

$$y(t) = \frac{B(q^{-1})}{F(q^{-1})} u(t) + \frac{C(q^{-1})}{D(q^{-1})} e(t) \quad (45)$$

where  $B$  and  $C$  are as above, with  $F(q^{-1}) = 1 + f_1 q^{-1} + \dots + f_{n_f} q^{-n_f}$  and  $D(q^{-1}) = 1 + d_1 q^{-1} + \dots + d_{n_d} q^{-n_d}$ .

*Theorem 6.2:* For the BJ model structure (45), the information matrix  $I(\theta)$  is positive definite at all  $\theta$  at which the model structure is identifiable if and only if  $u(t)$  is SRk, where  $k = n_b + n_f$ .

*Proof:* The gradient vectors  $V_{uol}(q, \theta)$  and  $V_{eol}(q, \theta)$  defined in (20) are now partitioned into 4 blocks corresponding, successively, to the parameters of the polynomials  $B$ ,  $F$ ,  $C$ , and  $D$ . It is easy to see that the left-kernel of  $V_{eol}(q, \theta)$  (i.e. of  $\nabla_\theta H(q, \theta)$ ) is spanned by the set of vectors  $\alpha^T = (\alpha_B^T | \alpha_F^T | 0 \dots 0 | 0 \dots 0)$ . Therefore, by Theorem 4.2,  $I(\theta) > 0$  if and only if the following pseudoregressor is PE:

$$\psi_{B,F}(t) \triangleq \frac{1}{F^2} \begin{pmatrix} Fq^{-1} \\ \vdots \\ Fq^{-n_b} \\ -Bq^{-1} \\ \vdots \\ -Bq^{-n_f} \end{pmatrix} u(t) = \frac{1}{F^2} R \mathcal{B}_{1, n_b + n_f} u(t) \quad (46)$$

where  $R$  is a Sylvester matrix which is nonsingular at all values  $\theta$  at which  $B$  and  $F$  are coprime, i.e. at all  $\theta$  at which  $\Gamma(\theta) > 0$ . The result then follows from Theorem 5.1. ■

*Corollary 6.2:* For the OE model structure  $y(t) = (B(q^{-1})/F(q^{-1}))u(t) + e(t)$ , the richness condition on  $u(t)$  is identical to that for the BJ model structure. ■

## B. Closed-Loop Identification

We now apply the results of Theorem 4.3 to closed-loop identification of ARMAX and BJ model structures. We will show that the information matrix  $I(\theta)$  will be positive definite if the controller complexity achieves a prescribed degree; and when that is not the case, the degree of richness required of the reference excitation must precisely compensate for the deficit in controller complexity. This precisely quantifiable trade-off between controller complexity and degree of richness of the reference signal is another main contribution of this paper. For the controller  $K(q)$  of (2) we consider a coprime factorization  $K(q) = X(q^{-1})/Y(q^{-1})$ , with  $X(q^{-1}) = x_0 + x_1 q^{-1} + \dots + x_{n_x} q^{-n_x}$  and  $Y(q^{-1}) = 1 + y_1 q^{-1} + \dots + y_{n_y} q^{-n_y}$ .

*ARMAX Model Structure:* Consider the ARMAX model structure (37). For simplicity, we shall consider only parameter values  $\theta$  at which the following assumption holds.

*Assumption 1:* The polynomials  $A(q^{-1})Y(q^{-1}) + B(q^{-1})X(q^{-1})$  and  $C(q^{-1})$  are coprime.

Notice that the subset of  $\theta$  values at which these polynomials have a common root has measure zero in the parameter space. They correspond to parameter values that cause a pole-zero cancellation between the closed-loop poles of the model and the zeros of the noise model. We then have the following result.

*Theorem 6.3:* Consider the ARMAX model structure (37) under feedback control with the stabilizing controller  $K(q) = X(q^{-1})/Y(q^{-1})$ , with Assumption 1 holding.

i) Let  $r(t) \equiv 0$ . Then the information matrix  $I(\theta)$  is positive definite at all  $\theta$  at which the model structure is identifiable if and only if

$$\max(n_x - n_a, n_y - n_b) \geq 0. \quad (47)$$

ii) Let  $\max(n_x - n_a, n_y - n_b) < 0$ . Then the information matrix  $I(\theta)$  is positive definite at all  $\theta$  at which the model structure is identifiable for almost all  $r(t) \in \mathcal{U}_k$  if and only if

$$k \geq \min(n_a - n_x, n_b - n_y). \quad (48)$$

*Proof:* For an ARMAX model structure, the filter  $V_{ecl}$  expressed in (22) becomes

$$V_{ecl} = \frac{H_0 S}{C^2 Y} \{-CY \nabla_\theta A - CX \nabla_\theta B + (AY + BX) \nabla_\theta C\}. \quad (49)$$

Let  $\alpha^T = (\alpha_A^T | \alpha_B^T | \alpha_C^T)$  denote any vector in the left-kernel of  $V_{ecl}$ . Using the same notations as in the proof of Theorem 6.1, it then follows that:

$$(q\gamma_A Y + q\gamma_B X)C = q\gamma_C (AY + BX). \quad (50)$$

By Assumption 1, the polynomials  $C$  and  $AY + BX$  are coprime. Since  $\deg(q\gamma_A Y + q\gamma_B X) < \deg(AY + BX)$  and

$\deg(q\gamma_C) < \deg(C)$ , it then follows from the theory of Diophantine equations [18] that the only solution of (50) is given by

$$\gamma_C = 0, \quad \text{and} \quad q\gamma_A Y + q\gamma_B X = 0. \quad (51)$$

It follows from the first part that any vector  $\alpha$  in the left kernel of  $V_{ecl}$  must have  $\alpha_C = \mathbf{0}$ . Consider now the right hand part of (51). Since  $X$  and  $Y$  are coprime, it follows again from the theory of Diophantine equations that if either  $\deg(Y) > \deg(q\gamma_B)$  or  $\deg(X) > \deg(q\gamma_A)$ , then the only solution of (51) is  $\gamma_B = \gamma_A = 0$ . Equivalently,  $V_{ecl}$  has full rank, and hence no external excitation is required, if and only if (47) holds.

Consider now the situation where  $\max(n_x - n_a, n_y - n_b) < 0$ . The general solution of  $q\gamma_A Y + q\gamma_B X = 0$  can then be written as

$$\gamma_A = XT, \quad \gamma_B = -YT \quad (52)$$

where  $T$  is an arbitrary polynomial of the form

$$T(q^{-1}) \triangleq t_1 q^{-1} + \dots + t_{n_t} q^{-n_t}, \quad \text{with} \quad (53)$$

$$n_t = \min\{n_a - n_x, n_b - n_y\}. \quad (54)$$

Thus, when  $\max(n_x - n_a, n_y - n_b) < 0$ , the left kernel of  $V_{ecl}$  is defined by vectors  $\alpha^T = (\alpha_A^T | \alpha_B^T | \mathbf{0}^T)$  such that  $\gamma_A = \alpha_A^T \mathcal{B}_{1,n_a}$  and  $\gamma_B = \alpha_B^T \mathcal{B}_{1,n_b}$  are constrained by (52) with  $T$  defined by (53), (54). By Theorem 4.3,  $I(\theta)$  is positive definite if and only if (29) holds for such  $\alpha$ . Using (22) and (41) it is easy to compute that, for an ARMAX model structure

$$\alpha^T \nabla_\theta W_y(q, \theta) = \alpha^T K(q) \nabla_\theta W_u(q, \theta) = -\frac{1}{C} \gamma_A. \quad (55)$$

Since  $\gamma_A = XT$  with  $T$  defined by (53), (54), we conclude that  $\bar{E}[\alpha^T \nabla_\theta W_y(q, \theta) r(t)]^2 \neq 0$  (or, equivalently,  $\bar{E}[\alpha^T K(q) \nabla_\theta W_u(q, \theta) r(t)]^2 \neq 0$ ) if and only if  $\bar{E}[T(q^{-1}) r(t)]^2 \neq 0$  for all  $T$  given by (53), (54), provided the points of support of the spectrum of  $r(t)$  do not coincide with possible zeroes of  $X$  on the unit circle. This holds for almost all  $r(t) \in \mathcal{U}_k$  if and only if  $k \geq \min\{n_a - n_x, n_b - n_y\}$ : see Theorem 5.1. ■

That an ARMAX model identified in closed loop is identifiable from noise information alone if the controller is sufficiently complex with respect to the model structure, as specified by condition (47), was already known: see Appendix C10.1 of [7]. What is novel and, we believe, remarkable in Theorem 6.3 is that, when that complexity condition is not satisfied by the controller, then the degree of richness required of the reference signal is precisely determined by how much that condition is violated. In other words, the degree of richness required of  $r(t)$  is precisely equal to the difference between the complexity required by expression (47) and the actual complexity of the controller.

*Corollary 6.3:* For the ARX model structure  $A(q^{-1})y(t) = B(q^{-1})u(t) + e(t)$  under feedback control with the stabilizing controller  $K(q) = X(q^{-1})/Y(q^{-1})$ , the richness conditions are identical to those given in Theorem 6.3 for the ARMAX model structure.

*Proof:* The proof follows immediately by setting  $C(q^{-1}) = 1$  everywhere in the proof of Theorem 6.3. ■

*BJ Model Structure:* Consider the BJ model structure (45). For simplicity, we shall again exclude parameter values  $\theta$  that cause a pole-zero cancellation between the closed-loop poles of the model and the zeros of the noise model. This corresponds to the following assumption.

*Assumption 2:* The polynomials  $F(z^{-1})Y(z^{-1}) + B(z^{-1})X(z^{-1})$  and  $C(z^{-1})$  are coprime.

*Theorem 6.4:* Consider the BJ model structure (45) under feedback control with the stabilizing controller  $K(q) = X(q^{-1})/Y(q^{-1})$ , with Assumption 2 holding.

i) Let  $r(t) \equiv 0$ . Then the information matrix  $I(\theta)$  is positive definite at all  $\theta$  at which the model structure is identifiable if and only if

$$\max(n_x - n_f, n_y - n_b) \geq n_d. \quad (56)$$

ii) Let  $\max(n_x - n_f, n_y - n_b) < n_d$ . Then the information matrix  $I(\theta)$  is positive definite at all  $\theta$  at which the model structure is identifiable for almost all  $r(t) \in \mathcal{U}_k$  if and only if

$$k \geq n_d + \min(n_f - n_x, n_b - n_y). \quad (57)$$

*Proof:* For the BJ model structure, the filter  $V_{ecl}$  of (22) becomes

$$V_{ecl} = \frac{SH_0}{C} \left\{ -\frac{KD}{F} \nabla_\theta B + \frac{KDB}{F^2} \nabla_\theta F + \frac{D}{C} \left( 1 + K \frac{B}{F} \right) \nabla_\theta C - \left( 1 + K \frac{B}{F} \right) \nabla_\theta D \right\}.$$

Let  $\alpha^T = (\alpha_B^T | \alpha_F^T | \alpha_C^T | \alpha_D^T)$  denote any vector in the left-kernel of  $V_{ecl}$ . Then, with the same notations as before, we find, after some calculations

$$\begin{aligned} \alpha^T V_{ecl} &= \frac{SH_0}{C} \left[ \left( \frac{B}{F} \gamma_F - \gamma_B \right) \frac{KD}{F} \right. \\ &\quad \left. + \left( \frac{D}{C} \gamma_C - \gamma_D \right) \left( 1 + K \frac{B}{F} \right) \right] \\ &= \frac{SH_0}{F^2 C^2 Y} [DCX(B\gamma_F - F\gamma_B) \\ &\quad + F(FY + BX)(D\gamma_C - C\gamma_D)] \\ &= 0. \end{aligned} \quad (58)$$

Assume that  $F$  and  $X$  have a common factor at the considered  $\theta$ ; let then  $M$ , of degree  $n_m$ , be the greatest common divisor of these two polynomials, i.e.  $F = \bar{F}M$  and  $X = \bar{X}M$ . Then

$$\alpha^T V_{ecl} = \frac{SH_0}{F\bar{F}C^2Y} [DC\bar{X}(B\gamma_F - F\gamma_B) + \bar{F}(FY + BX)(D\gamma_C - C\gamma_D)] = 0. \quad (59)$$

It follows from Assumption 1, the coprimeness of  $\bar{X}$  and  $Y$ , and the coprimeness of  $F$  and  $\bar{X}$  that the polynomials  $FY + BX$  and  $C\bar{X}$  are coprime. It then follows from the theory of Diophantine equations that the only solution of (58) is

$$Bq\gamma_F - Fq\gamma_B = 0, \quad \text{and} \quad Dq\gamma_C - Cq\gamma_D = 0 \quad (60)$$

if either  $n_c + n_x - n_m > n_f - n_m + n_d + n_c - 1$ , or  $\max\{n_f + n_y, n_b + n_x\} > n_d + n_f + n_b - 1$ . These conditions are equivalent to (56). Thus, suppose (56) holds and consider (60). At all values

of  $\theta$  at which  $\Gamma(\theta) > 0$ , the polynomials  $B$ ,  $F$  and  $C$ ,  $D$  are coprime. Since  $\deg(q\gamma_F) < \deg(F)$ , and  $\deg(q\gamma_C) < \deg(C)$ , it then follows from (60) that  $\gamma_B = \gamma_F = \gamma_C = \gamma_D = 0$ , and hence  $\alpha = \mathbf{0}$ , and the first part of the result follows.

Consider now the situation where  $\max\{n_x - n_f, n_y - n_b\} < n_d$ , and let  $\alpha$  be any vector in the kernel of  $V_{ecl}$ . The polynomials  $\gamma_B, \gamma_F, \gamma_C, \gamma_D$  corresponding to any such  $\alpha$  satisfy the equality (59), whose general solution is

$$D(B\gamma_F - F\gamma_B) = (FY + BX)T, \quad \bar{F}(D\gamma_C - C\gamma_D) = C\bar{X}T \quad (61)$$

where  $T$  is an arbitrary polynomial of the form (53) with

$$n_t = n_d + \min\{n_b - n_y, n_f - n_x\}. \quad (62)$$

By Theorem 4.3,  $I(\theta)$  is positive definite at each identifiable  $\theta$  if, for any such  $\alpha$ , we have  $\bar{E}[\alpha^T \nabla_\theta W_y(q, \theta) r(t)]^2 \neq 0$ . We have

$$\alpha^T \nabla_\theta W_y = \frac{1}{C^2} (D\gamma_C - C\gamma_D) = -\frac{\bar{X}T}{C\bar{F}} \quad (63)$$

where the last equality follows from (61). In this expression, the polynomial  $T$ , whose degree is given by (62), is free while all others are fixed. Therefore,  $\bar{E}[\alpha^T \nabla_\theta W_y(q, \theta) r(t)]^2 \neq 0$  if and only if  $\bar{E}[T r(t)]^2 \neq 0$  where  $T$  is given by (53) and (62), provided the points of support of the spectrum of  $r(t)$  do not coincide with possible zeroes of  $X$  on the unit circle. This holds for almost all  $r(t) \in \mathcal{U}_k$  if and only if  $k \geq n_d + \min\{n_b - n_y, n_f - n_x\}$ . ■

We observe that, just like in the case of an ARMAX model structure identified in closed loop, the degree of richness required of the external excitation signal  $r(t)$  is precisely equal to the difference between the complexity required by expression (56) and the actual complexity of the controller.

*Corollary 6.4:* For the OE model structure  $y(t) = (B(q^{-1})/F(q^{-1}))u(t) + e(t)$ , under feedback control with the stabilizing controller  $K(q) = X(q^{-1})/Y(q^{-1})$ , the information matrix  $I(\theta)$  is positive definite at all  $\theta$  at which  $\Gamma(\theta) > 0$  if and only if  $K(q) \neq 0$ .

*Proof:* It is easy to see that for an OE model structure, the left kernel of  $V_{ecl}$  is defined by

$$\alpha^T V_{ecl} = \frac{SH_0X}{F^2Y} (B\gamma_F - F\gamma_B) = 0. \quad (64)$$

If  $SH_0X \neq 0$ , then the only solution to this Diophantine equation at values of  $\theta$  at which  $B$  and  $F$  are coprime is  $\gamma_B = 0$  and  $\gamma_F = 0$ . ■

This confirms a result obtained in [3] where it was shown that an OE model structure is identifiable in closed loop without external excitation as long as the controller is not identically zero.

## VII. NUMERICAL ILLUSTRATION

We illustrate our results with an ARMAX and a Box-Jenkins model structure, both in an open-loop and a closed-loop identification setup. We consider situations where the ‘true system’ is

in the model set; in such case, the positive definiteness of  $I(\theta_0)$  at the true value is a necessary condition for the existence of a unique global minimum of the identification criterion [15]. In all simulations presented in this section we have collected  $N = 10^5$  data samples. In order to validate our theoretical results, we replace the mathematical expectation  $\bar{E}[\cdot]$  in the expression (13) of  $I(\theta)$  by its approximation  $(1/N) \sum_{i=1}^N \psi(t, \theta) \psi^T(t, \theta)$ . The pseudoregressor  $\psi(t, \theta)$  is computed using (20) for the open-loop identification setup, and (22) for the closed-loop identification setup. The positive definiteness of  $I(\theta)$  is verified by using the MATLAB command `rank` with a tolerance of  $10^{-5}$ .

### A. ARMAX Model Structure

We consider the following ARMAX system as the true plant:

$$y(t) = \frac{b_{10}q^{-1} + b_{20}q^{-2}}{1 + a_{10}q^{-1}} u(t) + \frac{1 + c_{10}q^{-1}}{1 + a_{10}q^{-1}} e(t) \quad (65)$$

where  $a_{10} = -0.7$ ,  $b_{10} = 0.3$ ,  $b_{20} = 0.12$ ,  $c_{10} = -0.5$ , and  $e(t)$  denotes a realization of a zero-mean white noise signal with  $\sigma_e^2 = 0.09$ . We first perform an open-loop identification with an ARMAX model structure matching the true system, i.e. we take  $\theta = [a_1 \ b_1 \ b_2 \ c_1]^T$ . Considering that the polynomials  $A$  and  $C$  are coprime at  $\theta_0$ , we know by Theorem 6.1 that in order to obtain a positive definite information matrix, we need to excite the plant using an input signal  $u(t) \in \mathcal{U}_2$ , since  $n_b = 2$ . Indeed, for  $u(t) = 1$ , which is *SRE1*,  $I(\theta_0)$  is found to be singular, whereas for  $u(t) = \sin((\pi/6)t)$ , which is *SRE2*,  $I(\theta_0)$  is a full rank matrix.

Consider now that the plant (65) is operated in closed loop with the controller  $K = (1 - 0.175q^{-1})/(1 - q^{-1})$ , and that the same model structure is used. Since the condition (47) is satisfied, it follows from Theorem 6.3 that the true system can be identified without external excitation signal. Verification of the rank of the information matrix confirms this fact. By collecting enough data one can actually obtain a model of the plant with arbitrary accuracy, without any external excitation.

We now consider that the plant is controlled by  $K = 0.25/(1 - q^{-1})$ , and that the same model structure is used as before. In this case, the controller is not complex enough with respect to the chosen model structure and an additional excitation is necessary for identification. From Theorem 6.3, we know that  $I(\theta_0) > 0$  for  $r(t) \in \mathcal{U}_k$  with  $k = \min\{n_a - n_x, n_b - n_y\} = 1$ . The simulation with  $r(t) = 1$  confirms the theoretical result, i.e.  $\text{rank}(I(\theta_0)) = 4$ .

We now perform an identification experiment with the same experimental conditions ( $K = 0.25/(1 - q^{-1})$  and  $r(t) = 1$ ), but using an over-parametrized ARMAX model structure  $\theta = [a_1 \ a_2 \ b_1 \ b_2 \ b_3 \ c_1]^T$ . We observe that, with such combination of controller and reference excitation,  $I(\theta_0)$  is singular, with  $\text{rank}(I(\theta_0)) = 5$ . Theorem 6.1 tells us that, in such case, identifiability requires an excitation signal  $r(t)$  that is *SR2*. Indeed, for  $r(t) = \sin((\pi/6)t)$  our simulations show that  $I(\theta_0)$  is positive definite, i.e.  $\text{rank}(I(\theta_0)) = 6$ . This confirms that the required richness on the excitation signal depends on the model structure that is used, and not on the true plant.

## B. BJ Model Structure

We consider the following BJ system:

$$y(t) = \frac{b_{10}q^{-1}}{1 + f_{10}q^{-1}}u(t) + \frac{1 + c_{10}q^{-1}}{1 + d_{10}q^{-1} + d_{20}q^{-2}}e(t) \quad (66)$$

where  $f_{10} = -0.45$ ,  $b_{10} = 0.3$ ,  $c_{10} = -0.75$ ,  $d_{10} = -1.2$ ,  $d_{20} = 0.36$  and  $e(t)$  is defined as in Section VII-A. To perform the open-loop identification using a BJ model structure with  $\theta = [b_1 \ c_1 \ d_1 \ d_2 \ f_1]^T$  a signal  $u(t) \in \mathcal{U}_k$  with  $k = n_b + n_f = 2$  is needed. An inspection of the rank of the information matrix, for  $u(t) = \sin((\pi/6)t)$  confirms that  $I(\theta_0)$  is positive definite. To verify the necessity condition of Theorem 6.2,  $I(\theta_0)$  is computed for  $u(t) = 1$ . As anticipated, the information matrix is singular in this case.

Assume a situation where the plant is to be identified in closed loop with the controller  $K = 0.25/(1 - q^{-1})$ . For a model structure  $\theta = [b_1 \ c_1 \ d_1 \ d_2 \ f_1]^T$  and the given controller, we have  $n_d = 2$ ,  $n_b = 1$ ,  $n_f = 1$ ,  $n_x = 0$  and  $n_y = 1$ . According to Theorem 6.4, the degree of richness required of  $r(t)$  is  $k = n_d + \min(n_f - n_x, n_b - n_y) = 2$ . Indeed, for  $r(t) = \sin((\pi/6)t)$ ,  $I(\theta_0)$  is found to be positive definite, whereas we have found  $\text{rank}(I(\theta_0)) = 4$  for  $r(t) = 1$ .

## VIII. CONCLUSION

The information matrix plays a fundamental role in system identification, given that it combines information about the identifiability of the model structure and about the informativity of the data set. In addition, when the system is in the model set, the information matrix evaluated at the convergence value of the identification criterion is, up to a scalar factor, the inverse of the parameter covariance matrix. Our first main contribution has been to provide necessary and sufficient conditions for the transfer of excitation from a vector-filtered scalar input signal to the corresponding regression vector. With this new result, we have derived necessary and sufficient conditions on the richness of the excitation signal and on the controller complexity (in closed-loop identification) that make the information matrix full rank at all values of the parameter space where the model structure is identifiable. Our results apply to general model structures, in open-loop and in closed-loop identification. We have applied our results to all model structures that are commonly used in prediction error identification. In closed-loop identification, these results provide a remarkable and quantifiable tradeoff between controller complexity and degree of richness of the external excitation.

## ACKNOWLEDGMENT

The authors wish to thank R. Hildebrand and L. Haine for some useful hints for the proof of Theorem 5.1.

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