V. Conclusion

This note has developed a sliding-mode controller which requires only output information for a class of uncertain linear systems. The controller comprises both linear and nonlinear components and is static in nature, i.e., no compensation/observation is included. The novelty of the approach is in the rationale and method used to synthesize the linear control component. The reachability condition is not required to be satisfied globally. Instead, sliding is only expected to take place within a subset of the state-space containing the origin referred to as the sliding patch. This region is shown to be rendered invariant by the control law. The linear static output feedback control component is synthesized using an LMI optimization. The resulting LMI formulation can be solved easily by standard commercially available software. The efficacy of the approach has been demonstrated on a numerical example taken from the sliding-modeliterature.

References

[1] S. K. Bag, S. K. Spurgeon, and C. Edwards, "Output feedback sliding-modedesign for linear uncertain systems," *Proc. Inst. Elect. Eng.*, pt. D, vol. 144, pp. 209–216, May 1997.

[2] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia, PA: SIAM, 1994.

[3] M. Chilali and P. Gahinet, "$\mathcal{H}_\infty$ design with pole placement constraints: An LMI approach," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 358–367, Mar. 1996.

[4] C. Edwards and S. K. Spurgeon, "sliding-mode stabilization of uncertain systems using only output information," *Int. J. Control*, vol. 62, no. 5, pp. 1129–1144, 1995.

[5] ——, "On the limitations of some variable structure output feedback controller designs," *Automatica*, vol. 36, pp. 743–748, 1999, to be published.

[6] R. El-Khazali and R. A. DeCarlo, "Variable structure output feedback control," in *Proc. American Control Conf.*, 1992, pp. 871–875.

[7] ——, "Output feedback variable structure control design using dynamic compensation for linear systems," in *Proc. American Control Conf.*, 1993, pp. 954–958.

[8] B. S. Heck and A. A. Ferri, "Application of output feedback to variable structure systems," *J. Guid. Control Dyn.*, vol. 12, pp. 932–935, 1989.

[9] B. S. Heck, S. V. Yallapragada, and M. K. H. Fan, "Numerical methods to design the reaching phase of output feedback variable structure control," *Automatica*, vol. 31, pp. 275–279, 1995.

[10] S. Hui and S. H. Żak, "Robust output feedback stabilization of uncertain dynamic systems with bounded controllers," *Int. J. Robust Nonlinear Control*, vol. 3, pp. 115–132, 1993.

[11] C. M. Kwan, "On variable structure output feedback controllers," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1691–1693, Nov. 1996.

[12] J. J. E. Slotine, J. K. Hedrick, and E. A. Misawa, "On sliding observers for nonlinear systems," *ASME: J. Dyn. Syst., Meas. Control*, vol. 109, pp. 245–252, Sept. 1987.

[13] S. K. Spurgeon and R. Davies, "A nonlinear control strategy for robust sliding-mode performance in the presence of unmatched uncertainty," *Int. J. Control*, vol. 57, no. 5, pp. 1107–1123, 1993.

[14] S. V. Yallapragada and B. S. Heck, "Reaching conditions in variable structure systems for output feedback control," in *Proc. American Control Conf.*, June 1990, pp. 32–35.

Local Stabilization of Discrete-Time Linear Systems with Saturating Controls: An LMI-Based Approach

J. M. Gomes da Silva, Jr. and S. Tarbouriech

***Abstract*—This note deals with the problem of local stabilization of linear discrete-time systems subject to control saturation. A linear matrix inequalitie-based framework is proposed in order to compute a saturating state feedback that stabilizes the system with respect to a given set of admissible initial states and, in addition, guarantees some dynamical performances when the system operates in the zone of linear behavior (i.e., when the controls are not saturated).**

***Index Terms*—Discrete-time systems, control saturation, linear matrix inequalities, local stability.**

I. Introduction

In the last years, the problem of the stabilization of linear systems subject to control saturation have been received the attention of many authors (see, for example, [1] and [2]). The interest in this problem is mainly motivated by the fact that the negligence of the control bounds can be source of limit cycles, parasitic equilibrium points, and even of the instability of the closed-loop system. The works found in the literature can be classified in three contexts of stability, namely the global, the semiglobal, and the local stability.

It is well known that the global stability can be achieved only when the open-loop system is not strictly unstable, i.e., in the discrete-time case, it has its poles inside or on the unite circle of the complex plane (see, for example, [3]–[5] and the references therein). However, the physical interest of the global stability is questionable since, in general, the system is restricted to operate in a limited zone of the state space. In this case, under the same hypothesis of open-loop stability, the semi-global approach seems to be more realistic. In particular, given any control bounds, it is possible to compute a linear state or output feedback (i.e. the saturation is avoided) guaranteeing the asymptotic stability of the closed-loop system with respect to (w.r.t) any bounded set of admissible initial conditions (see, for example, [6] and [7]). In our point of view, the main drawbacks of these two approaches are, first the open-loop stability requirements, secondly that, in general, the computed stabilizing control law does not provide significant improvement of the time-domain performance of the closed-loop system. Hence, when the open-loop system is unstable and/or some performance requirements should be satisfied, only the local stabilization is possible.

This note focuses on the local stabilization problem. Given a set of admissible initial conditions $\mathcal{X}_0$ to be stabilized, our objective is to compute a saturating state feedback control law that guarantees both the asymptotic convergence to the origin of all trajectories emanating from $\mathcal{X}_0$ and a certain degree of time-domain performance for the closed-loop system in a neighborhood of the origin. In this aim, we use a local representation of the saturated system deduced from the difference inclusions theory. This representation consists in a polytopic model valid in a certain polyhedral set in the state space. Based on this model, some conditions expressed as linear matrix inequalities (LMIs) and bilinear matrix inequalities (BMIs) are stated for determining a state feedback gain to satisfy both stabilization and performance

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requirements. Since the numerical solution of BMIs is a difficult task to accomplish, an LMI-framework, based on some relaxation schemes combined with an optimization problem, is proposed to handle the problem. Finally, the application of the proposed results in the semi-global stabilization context and in the design of piecewise control laws are presented.

1) Notations: For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0, \forall i = 1, \ldots, n$. The elements of a matrix $A \in \Re^{m*n}$ are denoted by $A_{(i,l)}$, $i = 1, \ldots, m$, $l = 1, \ldots, n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A^T denotes the transpose of A. $\mathrm{diag}(x)$ denotes a diagonal matrix obtained from vector x. I_m denotes the m-order identity matrix and $1_m \triangleq [1 \ldots 1]^T \in \Re^m$. $\partial\mathcal{S}$ denotes the boundary of the set $\mathcal{S}$. **Co** $\{\cdot\}$ denotes a convex hull.

II. Problem Statement

Consider a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \Re^n$ and $u(k) \in \Re^m$ are respectively the state vector and the control vector. Matrices A and B are real constant matrices of appropriate dimensions. For system (1), we suppose that the following assumptions hold.

A1) The control vector is subject to amplitude constraints which define the polyhedral compact region $\Omega \subset \Re^m$

$$\Omega \triangleq \{u \in \Re^m; -\rho \preceq u \preceq \rho\}, \qquad \rho \succ 0. \tag{2}$$

A2) The pair (A, B) is controllable.

A3) The region of admissible initial states, denoted by $\mathcal{X}_0$, is known.

Consider the saturating feedback control law

$$u(k) = \mathrm{sat}(\mathrm{Fx(k)}) \tag{3}$$

where each component is defined, $\forall\, i = 1, \ldots, m$, as follows:

$$\mathrm{sat}(\mathrm{F_{(i)}x(k)}) \triangleq \begin{cases} -\rho_{(i)}, & \text{if } F_{(i)}x(k) < -\rho_{(i)} \\ F_{(i)}x(k), & \text{if } -\rho_{(i)} \leq F_{(i)}x(k) \leq \rho_{(i)} \\ \rho_{(i)}, & \text{if } F_{(i)}x(k) > \rho_{(i)}. \end{cases} \tag{4}$$

By applying this control law to system (1) the closed-loop system becomes *nonlinear*

$$x(k+1) = Ax(k) + Bsat(Fx(k)). \tag{5}$$

It is worth noticing that inside the domain $S(F, \rho)$ defined as

$$S(F, \rho) \triangleq \{x \in \Re^n; -\rho \preceq Fx \preceq \rho\} \tag{6}$$

the control inputs do not saturate and therefore, the evolution of the closed-loop system is described by the following *linear* model:

$$x(k+1) = (A + BF)x(k). \tag{7}$$

Outside $S(F, \rho)$, the control inputs saturate and the stability of the system must be analyzed by considering (5).

Under the above assumptions, the problem addressed in this note is the following:

Problem II.1: Compute a matrix F such that

1. All the trajectories of system (5) emanating from $\mathcal{X}_0$ converge asymptotically to the origin.
2. A certain degree of performance is guaranteed when the system operates inside the region of linear behavior $S(F, \rho)$.

III. Saturated System Model

In order to state the main results of the note, we define an appropriate representation for the saturated system. The basic idea is to represent the saturated system by a polytopic model. This kind of representation was first introduced in [8] and has been applied in the specific case of system (5) in [5], [9], and [10].

Note that each component of the control law defined by (4) can be also written as

$$u(k)_{(i)} = \mathrm{sat}(\mathrm{F_{(i)}x(k)}) = \alpha(\mathrm{x(k)})_{(i)}\mathrm{F_{(i)}x(k)} \tag{8}$$

where

$$\alpha(x(k))_{(i)} \triangleq \begin{cases} \dfrac{-\rho_{(i)}}{F_{(i)}x(k)}, & \text{if } F_{(i)}x(k) < -\rho_{(i)} \\ 1, & \text{if } -\rho_{(i)} \leq F_{(i)}x(k) \leq \rho_{(i)} \\ \dfrac{\rho_{(i)}}{F_{(i)}x(k)}, & \text{if } F_{(i)}x(k) > \rho_{(i)} \end{cases} \tag{9}$$

with $0 < \alpha(x(k))_{(i)} \leq 1$, $i = 1, \ldots, m$.

The coefficient $\alpha(x(k))_{(i)}$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, smaller is $\alpha(x(k))_{(i)}$, farther is the state vector from the region of linearity (6). Notice that $\alpha(x(k))_{(i)}$ is a function of $x(k)$. For a sake of simplicity, in the sequel we denote $\alpha(x(k))_{(i)}$ as $\alpha(k)_{(i)}$.

Define from the vector $\alpha(k) \in \Re^m$ a diagonal matrix $D(\alpha(k)) \triangleq \mathrm{diag}(\alpha(k))$. System (5) can be rewritten as

$$x(k+1) = (A + BD(\alpha(k))F)x(k) = \mathcal{A}_k x(k) \tag{10}$$

where at each instant k the matrix $\mathcal{A}_k$ is a function of $\alpha(k)$ and in consequence depends on $x(k)$.

Let now $0 < \underline{\alpha}_{(i)} \leq 1$ be a lower bound to $\alpha(k)_{(i)}$ and define the vector $\underline{\alpha} \triangleq [\underline{\alpha}_{(1)} \ldots \underline{\alpha}_{(m)}]^T$. The vector $\underline{\alpha}$ is associated to the following region in the state space:

$$S(F, \rho^\alpha) \triangleq \{x \in \Re^n; -\rho^\alpha \preceq Fx \preceq \rho^\alpha\} \tag{11}$$

where $\rho^\alpha_{(i)} \triangleq (\rho_{(i)}/\underline{\alpha}_{(i)})$. In fact, for all $x(k) \in S(F, \rho^\alpha)$, it follows that $1 \geq \alpha_{(i)} \geq \underline{\alpha}_{(i)}$.

Consider now all the possible m-order vectors such that the ith entry takes the value 1 or $\underline{\alpha}_{(i)}$. Hence, there exists a total of 2^m different vectors. By denoting each one of these vectors by γ_j, $j = 1, \ldots, 2^m$, define the following matrices:

$$\begin{aligned} D_j(\underline{\alpha}) &= D(\gamma_j) = \mathrm{diag}(\gamma_j) \\ A_j &= A + BD_j(\underline{\alpha})F. \end{aligned} \tag{12}$$

From the definition of matrices A_j, it follows that $\forall x(k) \in S(F, \rho^\alpha)$, $\mathcal{A}_k \in \mathbf{Co}\{A_j; j = 1, \ldots, 2^m\}$. Hence, system (5) can be locally represented by the polytopic model

$$x(k+1) = \sum_{j=1}^{2^m} \lambda_{j,k} A_j x(k) \tag{13}$$

with $\sum_{j=1}^{2^m} \lambda_{j,k} = 1$, $\lambda_{j,k} \geq 0$. In other words, at each instant k matrix $\mathcal{A}_k$ can be obtained as a linear convex combination of matrices A_j. It should be pointed out that model (13) represents the saturated system only in $S(F, \rho^\alpha)$. Actually, if $x(k) \in S(F, \rho^\alpha)$, the polytopic model (13) can be used to determine the state of the saturated system at the instant $(k+1)$.

IV. Local Asymptotic Stabilization

Problem II.1 can be interpreted as a problem of *Local Asymptotic Stabilization*. In fact, to solve it we should calculate a state feedback that guarantees the local stability of system (5) in a region that contains the set $\mathcal{X}_0$. Furthermore, when this system operates in the region of linearity, i.e., the closed-loop system is described by (7), a certain degree of time-domain performance should be guaranteed. This kind of specification can, in general, be achieved by placing the poles of $(A+BF)$ in a suitable region of the unit disk of the complex-plane [11]. In this way, if we are able to compute a matrix F such that a set $\mathcal{S}$ containing $\mathcal{X}_0$ is contractive w.r.t the saturated system (5) and the poles of $(A+BF)$ are located in a suitable region $\mathcal{D}$ of the unit disk then *Problem II.1* is solved.

A. Main Result

Consider the following data:

- a vector ρ of control bounds;
- a set of initial conditions $\mathcal{X}_0$ defined as an union of ellipsoidal sets and a polyhedral set described by its vertices

$$\mathcal{X}_0 = \mathcal{Z} \cup \left(\bigcup_{s=1}^{n_e} \mathcal{E}_s(P_s, 1) \right) \tag{14}$$

  with $\mathcal{E}_s(P_s,1) \triangleq \{x \in \Re^n; x^T P_s x \le 1\}$, $P_s = P_s^T > 0$, $\forall s = 1, \ldots, n_e$ and $\mathcal{Z} \triangleq \mathbf{Co}\{v_1, \ldots, v_{n_v}\}$, $v_i \in \Re^n$, $\forall i = 1, \ldots, n_v$;
- a region $\mathcal{D}$, contained in the unit disk of the complex plane, defined as [11]

$$\mathcal{D} \triangleq \{z \in \mathcal{C}; (H + zQ + \bar{z}Q^T) < 0\} \tag{15}$$

  where H and Q are $l * l$ symmetric real matrices and z is a complex number with its conjugate $\bar{z}$. We assume that if the poles of $(A+BF)$ are located in the region $\mathcal{D}$ the time-domain requirements in the zone of linear behavior of the system (5) are satisfied.

Proposition IV.1: If there exist matrices $W = W^T > 0$, $W \in \Re^{n*n}$, $Y \in \Re^{m*n}$ and a vector $\underline{\alpha} \in \Re^m$, satisfying the following matrix inequalities:

$$\begin{aligned}
&\text{(i)} \quad H_{(i,j)}W + Q_{(i,j)}(AW+BY) + Q_{(i,j)}(AW+BY)^T < 0 \\
&\qquad 1 \le i, l \le q \\
&\text{(ii)} \quad \begin{bmatrix} W & WA^T + Y^T D_j(\underline{\alpha}) B^T \\ AW + BD_j(\underline{\alpha})Y & W \end{bmatrix} > 0 \\
&\qquad \forall j = 1, \ldots, 2^m \\
&\text{(iii)} \quad \begin{bmatrix} W & \underline{\alpha}_{(i)} Y^T I_{m(i)}^T \\ \underline{\alpha}_{(i)} I_{m(i)} Y & \rho_{(i)}^2 \end{bmatrix} \ge 0 \quad \forall i = 1, \ldots, m \\
&\text{(iv)} \quad \begin{bmatrix} 1 & v_r^T \\ v_r & W \end{bmatrix} \ge 0 \quad \forall r = 1, \ldots, n_v \\
&\text{(v)} \quad \begin{bmatrix} P_s & I_n \\ I_n & W \end{bmatrix} \ge 0 \quad \forall s = 1, \ldots, n_e \\
&\text{(vi)} \quad 0 < \underline{\alpha}_{(i)} \le 1, \qquad i = 1, \ldots, m
\end{aligned} \tag{16}$$

then $F = YW^{-1}$ solves *Problem II.1* and the ellipsoid $\mathcal{E}(W^{-1}, 1) \triangleq \{x \in \Re^n; x^T W^{-1} x \le 1\}$ is a domain of asymptotic stability for system (5).

Proof: If there exist matrices $W = W^T > 0$, Y and a vector $\underline{\alpha}$ satisfying the matrix inequalities (i)–(vi) it follows that:

1. LMI (i) guarantees that all the eigenvalues of $(A+BF)$ are contained in region $\mathcal{D}$ [11];
2. from inequality (ii), one obtains

$$\sum_{j=1}^{2^m} \lambda_{j,k} \begin{bmatrix} W & WA^T + Y^T D_j(\underline{\alpha}) B^T \\ AW + BD_j(\underline{\alpha})Y & W \end{bmatrix} > 0 \tag{17}$$

   with $\sum_{j=1}^{2^m} \lambda_{j,k} = 1$, $\lambda_{j,k} \ge 0$;
3. inequality (iii) ensures that ellipsoid $\mathcal{E}(W^{-1},1)$ is contained in the region $S(F, \rho^\alpha)$ with $F = YW^{-1}$ [9];
4. LMIs (iv) and (v) guarantee that $\mathcal{X}_0$ defined by (14) is contained in the ellipsoid $\mathcal{E}(W^{-1},1)$ [12], [13].

Suppose now that $x(k) \in \mathcal{E}(W^{-1},1)$. Since $\mathcal{E}(W^{-1},1) \subset S(F, \rho^\alpha)$, the state of the saturated system (5) at instant $(k+1)$ can be computed by using the polytopic model (13) with appropriate $\lambda_{j,k}$, $j = 1, \ldots, 2^m$ and matrices A_j defined from the coefficients of saturation $\underline{\alpha}_{(i)}$ and $F = YW^{-1}$. From (17) it follows that:

$$x(k)^T \left(\sum_{j=1}^{2^m} \lambda_{j,k} A_j \right)^T W^{-1} \left(\sum_{j=1}^{2^m} \lambda_{j,k} A_j \right) x(k) - x(k)^T W^{-1} x(k) < 0$$

that is,

$$x(k+1)^T W^{-1} x(k+1) - x(k)^T W^{-1} x(k) < 0.$$

Since this reasoning is valid $\forall x(k) \in \mathcal{E}(W^{-1},1)$, $x(k) \neq 0$, we can conclude that $\mathcal{V}(x(k)) \triangleq x(k)^T W^{-1} x(k)$ is a local strictly decreasing Lyapunov function for the saturated system (5) in $\mathcal{E}(W^{-1},1)$ and thus the ellipsoid $\mathcal{E}(W^{-1},1)$ is a contractive domain w.r.t system (5). Since $\mathcal{X}_0 \subset \mathcal{E}(W^{-1},1)$, the asymptotic convergence to the origin of all trajectories of system (5) emanating from $\mathcal{X}_0$ is guaranteed. The LMI (i) guarantees the performance in the region of linearity $S(F,\rho)$. □

B. LMI Framework

The variables to be found by applying Proposition (1) are W, Y, and $\underline{\alpha}$. However, inequalities (ii) and (iii) of (16) are bilinear (BMI) in the decision variables Y and $\underline{\alpha}$, whereas relations (i), (iv)–(vi) of (16) are linear (LMI) in W, Y and $\underline{\alpha}$.

An easy and straightforward way to overcome this problem is to fix, *a priori*, the value of the components of $\underline{\alpha}$. In this case, inequalities (ii) and (iii) become LMIs and, given $(\rho, \mathcal{X}_0, \mathcal{D})$, it is possible to solve constraints (i)–(vi) of *Proposition IV.1*, as a feasibility problem, with efficient numerical algorithms [12]. Of course, considering a fixed vector $\underline{\alpha}$ and the given data, it may actually be impossible to find a feasible solution. In fact, considering a scaling factor β, $\beta > 0$, the maximum homothetic set to $\mathcal{X}_0$, $\beta\mathcal{X}_0$, that can be stabilized by considering the fixed $\underline{\alpha}$, can be obtained by solving the following convex optimization problem with LMI constraints:

$$\begin{gathered} \max \beta \\ \text{subject to} \\ \beta > 0 \end{gathered}$$

$$\begin{cases} \begin{bmatrix} 1 & \beta v_i^T \\ \beta v_i & W \end{bmatrix} > 0 \quad \forall i = 1, \ldots, n_v \\ \begin{bmatrix} P_s & \beta I_n \\ \beta I_n & W \end{bmatrix} \ge 0 \quad \forall s = 1, \ldots, n_e \\ \text{relations (i), (ii), (iii), and (vi) of Proposition IV.1} \end{cases} \tag{18}$$

Henceforth, if the optimal value of β, $\beta^\star$, is greater than or equal to one, it means that it is possible to find a solution considering the fixed $\underline{\alpha}$ and the given data $(\rho, \mathcal{X}_0, \mathcal{D})$. We conjecture that smaller are the components of vector $\underline{\alpha}$, greater is the optimal value of the scalar β, that is, it is possible to stabilize larger domains of admissible initial states (see the numerical example in Section V-C). Note that the idea is to render the problem less conservative by allowing more control saturation. Hence, for a given region $\mathcal{D}$ and a region $\mathcal{X}_0$, we can consider an iterative scheme where we decrease the components of $\underline{\alpha}$ in each iteration until finding an optimal solution $(W^\star, Y^\star, \beta^\star)$ for (18) with $\beta^\star \geq 1$. In this case, two issues arise: how to choose the initial vector $\underline{\alpha}$ and how exactly to decrease the components of $\underline{\alpha}$ (if $\beta^\star < 1$ with the considered $\underline{\alpha}$). These issues can be considered as open problems and one simple way of handling them is to apply trial and error procedures.

Another solution consists in solving (18) by considering directly the problem with BMI constraints. However, as pointed in [14], the methods proposed in the literature for solving BMIs present exponential worst case complexities and therefore the required computational effort may be unreasonably large. Moreover, BMI-based problems are not convex, and thus, we cannot guarantee that the obtained solution is a global optimum. In order to overcome this computational difficulty in solving BMIs, we can approximate the solution of BMI optimization problems via polynomial-time algorithms, by using, for example, some relaxation schemes based on LMI relations (LMIR) (see for instance [14] and references therein). With this aim, we propose the following two-step iterative algorithm:

- **Step 1.** Given $\underline{\alpha}$, solve (18) for W, Y, and β (LMIR 1).
- **Step 2.** Given Y, solve (18) for W, $\underline{\alpha}$, and β (LMIR 2).

The iteration between these two steps stops when a desired precision for β is achieved. If $\beta^\star \geq 1$, it means that it is possible to stabilize system (5) for all initial conditions in $\mathcal{X}_0$ by considering the pole placement of $(A + BF)$ inside $\mathcal{D}$. In particular, all intermediate solutions with $\beta \geq 1$ are solutions to *Problem 1*. Hence, this kind of approach solves, in part, the problem of the choice of vector $\underline{\alpha}$ by using robust and available packages to solve LMIs [15].

Remark IV.1: It is worth noticing that if we start the algorithm with $\underline{\alpha} = 1_m$, the convergence to a solution $(\beta^\star, W^\star, Y^\star, \underline{\alpha}^\star)$, is ensured provided that the pair (A, B) is controllable. This follows from the fact that an optimal solution for LMIR 1 is also a feasible solution for LMIR 2 and vice versa. Of course, taking different initial vectors $\underline{\alpha}$ the proposed algorithm can converge to different values of $(\beta^\star, W^\star, Y^\star, \underline{\alpha}^\star)$.

Remark IV.2: The result of *Proposition IV.1* can be applied to stable or unstable open-loop systems. However, we should take into account that *Proposition IV.1* furnishes only a sufficient condition to solve *Problem II.1* by considering the data $(\rho, \mathcal{X}_0, \mathcal{D})$. Moreover, when the open-loop system is unstable the set $\mathcal{X}_0$ may be not contained in the controllable region of the system (1) with constrained controls. In this case there is effectively no solution to *Problem II.1*.

V. Applications

A. Semi-Global Stabilization

Consider the following assumption.

A4) All the eigenvalues of A are located inside or on the unit disk of the complex plane.

Under this assumption, from the result presented in [7] it follows that given any bounded set of initial conditions $\mathcal{X}_0$ and for any control bounds given by a vector ρ, it is possible to determine a control law $u(k) = Fx(k)$ such that for all initial conditions belonging to $\mathcal{X}_0$ ($x(0) \in \mathcal{X}_0$) the corresponding trajectories converge asymptotically to the origin *without control saturation*. In other words, system (1) is said to be semi-globally stabilizable. More specifically, it is proven in [7] that under assumptions *A1), A2), A3)*, and *A4)* there always exists $\epsilon > 0$ such that the parameter-dependent Riccati equation

$$P(\epsilon) = A^T P(\epsilon) A + \epsilon I_n - A^T P(\epsilon) B (B^T P(\epsilon) B + I_m)^{-1} \cdot B^T P(\epsilon) A \tag{19}$$

has a solution $P(\epsilon)$ and the control law $u(k) = Fx(k) = -(B^T P(\epsilon) B + I_m)^{-1} B^T P(\epsilon) A x(k)$ is such that

1) the eigenvalues of $(A + BF)$ are inside the unit disk;
2) the inclusion relation

$$\mathcal{X}_0 \subseteq \mathcal{E}(P(\epsilon), c) \subset S(F, \rho) \tag{20}$$

holds for some $c > 0$, with $\mathcal{E}(P(\epsilon), c) \triangleq \{x \in \Re^n; x^T P(\epsilon) x \leq c\}$.

A consequence of this result is the following.

Proposition V.1: Consider system (1) under Assumptions *A1), A2), A3)*, and *A4)*. Let $\mathcal{X}_0$ be defined as in (14), then

(a) There always exists a matrix $W = W^T > 0, W \in \Re^{n*n}$ and a matrix $Y \in \Re^{m*n}$ that verify the following set of LMIs.

$$\text{(i)} \quad \begin{bmatrix} W & WA^T + Y^T B^T \\ AW + BY & W \end{bmatrix} > 0.$$

$$\text{(ii)} \quad \begin{bmatrix} W & Y^T I_{m(i)}^T \\ I_{m(i)} Y & \rho_{(i)}^2 \end{bmatrix} \geq 0 \quad \forall i = 1, \ldots, m.$$

$$\text{(iii)} \quad \begin{bmatrix} 1 & v_r^T \\ v_r & W \end{bmatrix} \geq 0 \quad \forall r = 1, \ldots, n_v$$

$$\text{(iv)} \quad \begin{bmatrix} P_s & I_n \\ I_n & W \end{bmatrix} \geq 0 \quad \forall s = 1, \ldots, n_e. \tag{21}$$

(b) If (W, Y) is an admissible solution for the set of LMIs (21), then the control law $u(k) = Fx(k) = YW^{-1}x(k)$ guarantees that all the trajectories of system (1) emanating from $\mathcal{X}_0$ converge asymptotically to the origin without control saturation.

Proof: **(a).** Since there always exist a positive scalar ϵ and a matrix $P(\epsilon)$ solutions to (19) such that the inclusion relation (20) holds for some $c > 0$ with $F = -(B^T P(\epsilon) B + I_m)^{-1} B^T P(\epsilon) A$, it is easy to verify that $W = (P(\epsilon)/c)^{-1}$ and $Y = -c(B^T P(\epsilon) B + I_m)^{-1} B^T P(\epsilon) A P(\epsilon)^{-1}$ satisfy the set of LMIs (21). **(b).** The proof mimics the one of *Proposition IV.1*. □

It is worth noticing that, in our case, all the solutions to the semi-global stabilization problem obtained with the Riccati approach are contained in the set of solutions to the LMIs (21) but the converse does not hold. Furthermore, the LMI formulation allows to incorporate to the problem other control specifications. Convex optimization problems, with the LMIs (21) as constraints, can be formulated in order to find solutions to the semi-global problem that satisfy performance requirements.

Moreover, the solutions considered by the Riccati approach and by *Proposition V.1* suppose that the control does not saturate. This fact can lead to slow closed-loop dynamics. In this case, it can be useful to allow the control saturation in order to improve the speed of convergence of the trajectories to the origin. For example, we can use the result of *Proposition IV.1* and consider the following optimization problem:

$$\min \delta$$
$$\text{subject to}$$
$$\begin{cases} \text{(i)} \quad \begin{bmatrix} \delta W & WA^T + Y^T B^T \\ AW + BY & \delta W \end{bmatrix} > 0 \\ \text{relations (ii), (iii), (iv), (v), and (vi) of Proposition IV.1.} \end{cases} \tag{22}$$

Constraint (i) means that the considered region $\mathcal{D}$ is a disk of ray δ. Hence, minimization of δ implies the minimization of the spectral ray of $(A+BF)$. Furthermore, LMIR 1 and 2 described in Section IV-B can be applied considering the optimization problem (22) instead of (18). In this case the optimization problem (22) is a generalized eigenvalue problem (GEVP) [12].

B. Piecewise Control

The idea of the piecewise control is to apply higher feedback gains to the system as the state approaches the origin. This is an interesting way to deal with the problem of control saturation and, at the same time, to improve the rate of convergence of the closed-loop trajectories to the origin. The main problem of this kind of control law is to determine appropriate switching sets and the associated gains in order to avoid limit cycles or unstable behavior. In [13], for example, an interesting method is proposed to compute piecewise *linear* control laws for continuous-time linear systems. This approach is based on the solution to Riccati equations. We show now how to compute a piecewise *saturating* control law based on the condition given in *Proposition IV.1*.

Let N be the number of desired switching sets. The piecewise *saturating* control law can be computed as follows.

Step 1) Define N homothetical sets to $\mathcal{X}_0$ as follows:

$$\mathcal{X}_q = \beta_q \mathcal{X}_0, \quad 0 < \beta_q < 1; \quad q = 1, \ldots, N; \quad \beta_0 = 1$$
$$\mathcal{X}_N \subset \mathcal{X}_{N-1} \subset \ldots \subset \mathcal{X}_1 \subset \mathcal{X}_0.$$

Associate a vector of coefficients of saturation $\underline{\alpha}_q$ to each region $\mathcal{X}_q$.

Step 2) For each $q = 0, \ldots, N$, solve an optimization problem of type (22) by considering $\mathcal{X}_q$ and $\underline{\alpha}_q$ as data and W_q, Y_q, δ as the associated optimal solution.

Step 3) For each $q = 0, \ldots, N$ define
- the feedback matrix: $F_q = Y_q W_q^{-1}$
- the switching set: $\mathcal{S}_q = \{x \in \Re^n; x^T W_q^{-1} x \leq 1\}$.

From *Proposition IV.1*, it follows that the application of the control law defined as

$$u(k) \triangleq \begin{cases} sat(F_0 x(k)), & \text{if } x(k) \in \mathcal{S}_0, x(k) \notin \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N\} \\ sat(F_1 x(k)), & \text{if } x(k) \in \mathcal{S}_1, x(k) \notin \{\mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_N\} \\ \vdots & \vdots \ \vdots \\ sat(F_N x(k)), & \text{if } x(k) \in \mathcal{S}_N \end{cases} \tag{23}$$

to the system (1) guarantees the asymptotic convergence to the origin of all the trajectories emanating from $\mathcal{X}_0$. Note that we consider a vector $\underline{\alpha}_q$ for each gain F_q. The idea is to accelerate even more the convergence by allowing the saturation at each time the trajectory enters a new region $\mathcal{S}_q$. Of course, if the effective saturation is not desired it suffices to consider $\underline{\alpha}_q = 1_m, \forall q = 0, \ldots, N$.

Remark V.1: Differently from [13], the ellipsoids do not need to be nested in our case. It should be noticed that, from the definition of the control law (23), if $x(k) \in \mathcal{S}_j$, it is not possible to switch to a previous saturating state feedback $sat(F_i x(k))$ with $i < j$ and, therefore, chattering cannot occur.

C. Numerical Examples

The numerical results presented in this section were obtained by using the MATLAB LMI Control Toolbox [15].

Example V.1: Consider the simplified model of the vertical dynamics of an helicopter borrowed from [16]. By considering a sampling period of $0.001s$ the matrices describing system (1) are the following:

$$A = \begin{bmatrix} 0.9964 & 0.0026 & -0.0004 & -0.0460 \\ 0.0045 & 0.9038 & -0.0188 & -0.3834 \\ 0.0097 & 0.0263 & 0.9379 & 0.1223 \\ 0.0005 & 0.0014 & 0.0968 & 1.0063 \end{bmatrix}$$
$$B = \begin{bmatrix} 0.0444 & 0.0167 \\ 0.2935 & -0.7252 \\ -0.5298 & 0.4726 \\ -0.0268 & 0.0241 \end{bmatrix}.$$

Notice that matrix A is unstable (the eigenvalues of A are: $1.0284 \pm 0.0098i, 0.9672, 0.8203$). The bounds on the control are given by $\rho = [3\ \ 2]^T$. The set of admissible initial conditions is an hypercube in $\Re^4$: $\mathcal{X}_0 = \{x \in \Re^4; -1 \leq x_{(i)} \leq 1, \forall i = 1, \ldots, 4\}$. The region $\mathcal{D}$ corresponds to a half disk centered at the origin with a ray $\delta < 1$

$$\mathcal{D} = \{z \in \mathcal{C}; \text{Re}\{z\} \geq 0 \text{ and } (\text{Re}\{z\})^2 + (\text{Im}\{z\})^2 \leq \delta^2, \delta < 1\}.$$

Notice that δ represents the spectral ray of $(A+BF)$. Smaller is δ, closer to the origin are the poles of $(A+BF)$ and greater tends to be the rate of the convergence of the trajectories to the origin.

Considering the data above, Table I shows the final values of $\underline{\alpha}$ and β obtained from the iterative algorithm proposed in Section IV-B from different initial vectors $\underline{\alpha}$ and scalars δ. The number of iterations needed in each case is denoted by $niter$. $\beta initial$ and $\beta final$ denote the optimal values of β obtained by the iterative algorithm from respectively $\underline{\alpha} initial$ and $\underline{\alpha} final$.

Regarding Table I, we can notice the following:

- Smaller are the components of $\underline{\alpha}$, greater is the β obtained from (18). This illustrates the fact that by allowing saturation we can stabilize the system for a larger set of initial conditions. Besides, more stringent is the performance requirement (smaller δ, in this case), smaller is the region of admissible initial states for which we can find a solution.
- In both cases $\delta = 0.9$ and $\delta = 0.8$, the better gain between $\beta initial$ and $\beta final$ is obtained for $\underline{\alpha} initial = 1_2$.
- For the case $\delta = 0.9$, the smaller is chosen $\underline{\alpha} initial$ (and therefore $\underline{\alpha} final$), the greater is $\beta initial$ (and therefore $\beta final$).
- For the case $\delta = 0.8$, it is not possible to find a solution $\beta \geq 1$.

These facts illustrate the tradeoff between the performance requirements (δ), the minimal size of the set of initial states (β) and the coefficient of saturation ($\underline{\alpha}$).

Example V.2: Consider the example treated in [6], for which system (1) is described by the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The bounds on the control are given by $\rho = 4$. The set of initial conditions is given by the hypercube $\mathcal{X}_0 \triangleq \{x \in \Re^4; |x_{(i)}| \leq 10, i = 1, 2, 3, 4\}$. Since the open-loop system is *not strictly unstable*, by applying the semi-global stabilization results developed in [6] the following gain is computed

$$F_1 = [0.0394 \quad 0.0840 \quad 0.0796 \quad 0.0283].$$

With this gain the convergence to the origin of all the trajectories emanating from $\mathcal{X}_0$ is ensured without control saturation.

TABLE I
ALGORITHM PERFORMANCE

δ	$\underline{\alpha}_{initial}$	$\beta_{initial}$	$\underline{\alpha}_{final}$	β_{final}	n_{iter}
0.9	$[1\ \ 1]^T$	0.9339	$[0.7398\ \ 0.5387]^T$	1.4798	14
0.9	$[0.5\ \ 0.5]^T$	1.5580	$[0.0987\ \ 0.1250]^T$	1.9721	3121
0.9	$[0.3\ \ 0.3]^T$	1.8790	$[0.0972\ \ 0.1264]^T$	1.9718	1255
0.8	$[1\ \ 1]^T$	0.3283	$[0.7216\ \ 0.7125]^T$	0.4575	16
0.8	$[0.5\ \ 0.5]^T$	0.4988	$[0.2060\ \ 0.1491]^T$	0.5399	681
0.8	$[0.3\ \ 0.3]^T$	0.5185	$[0.1925\ \ 0.1635]^T$	0.5402	426

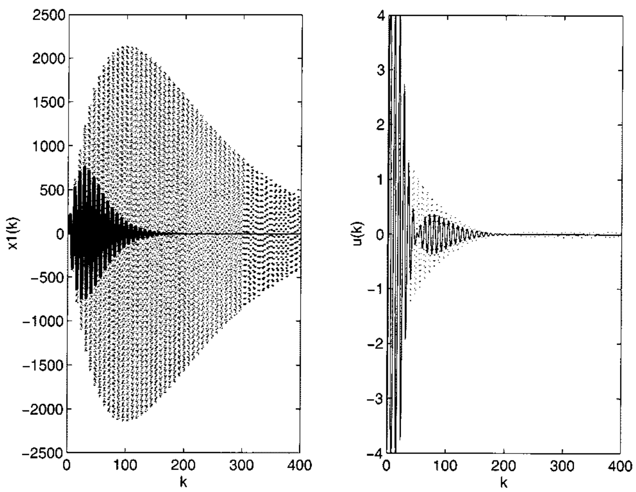

Fig. 1. Time-responses.

We apply now the proposed LMI-based approach to these data. By considering the optimization problem (22) with $\underline{\alpha} = 0.6$, we obtain the following feedback gain:

$$F_2 = [0.1534 \quad 0.3328 \quad 0.3207 \quad 0.1161].$$

From *Proposition IV.1*, the control law $u(k) = sat(F_2 x(k))$ stabilizes asymptotically the system for all initial conditions belonging to $\mathcal{X}_0$. Fig. 1 depicts the time-response of the first state variable and the control signal in both cases for the initial condition $x(0) = [-10\ \ 10\ \ 10\ \ 10]^T$. Note that with F_2 (solid line) we obtain a time-response less oscillatory and the convergence is faster than the one obtained with the application of F_1 (dotted line). Furthermore, the maximal amplitude peak of the state response is reduced with the application of the saturating control law (what is important if we have state amplitude constraints). Of course, with F_2 the control saturates. These facts illustrate that the time-response of the closed-loop system can be improved by allowing the saturation.

Example V.3: Consider the linearized model of an inverted pendulum studied in [13]. For a sampling period of $0.001s$ one gets the discretized model described by the following data:

$$A = \begin{bmatrix} 0.9995 & 0.0100 \\ -0.1000 & 0.9995 \end{bmatrix} \quad B = \begin{bmatrix} 0.0000 \\ 0.0100 \end{bmatrix}.$$

The bounds on the control are given by $\rho = 5$. The set of initial conditions is a disk centered at the origin with a ray equal to 1. With these data and considering $\underline{\alpha} = 1$ (i.e., saturation avoidance) the optimal solution of problem (22) gives

$$F_1 = [0.0022 \quad -1.5214].$$

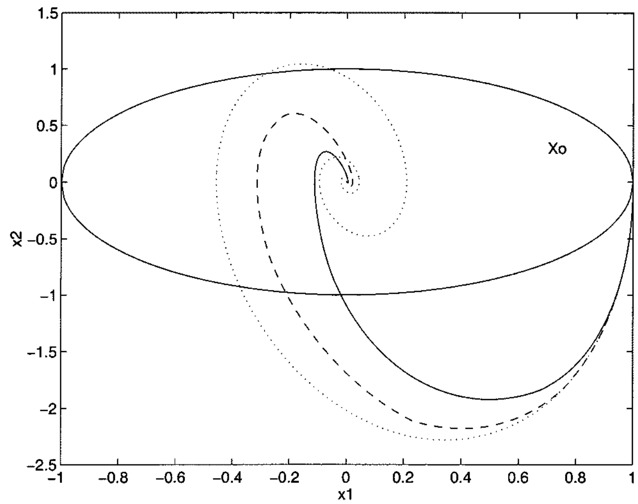

Fig. 2. State trajectories.

Consider now the application of the piecewise control law described in Section V-B with five switching sets, that is, $N = 4$. For this, we consider the following:

- $\beta_0 = 1, \beta_1 = 0.8, \beta_2 = 0.6, \beta_3 = 0.4, \beta_4 = 0.2$;
- $\underline{\alpha}_q = 0.6\ \forall q = 0, \ldots, 4$.

The obtained feedback gains F_{2q} and matrices P_{2q} defining the switching sets are the following:

$$\begin{aligned} F_{20} &= [-0.4062 \quad -2.4064] \\ F_{21} &= [-0.9649 \quad -2.8880] \\ F_{22} &= [-2.2507 \quad -3.6029] \\ F_{23} &= [-5.2998 \quad -4.7812] \\ F_{24} &= [-16.1831 \quad -7.2972] \end{aligned}$$

and

$$\begin{aligned} P_{20} &= \begin{bmatrix} 0.9852 & 0.1138 \\ 0.1138 & 0.0935 \end{bmatrix} \\ P_{21} &= \begin{bmatrix} 1.5290 & 0.2011 \\ 0.2011 & 0.1376 \end{bmatrix} \\ P_{22} &= \begin{bmatrix} 2.7083 & 0.3976 \\ 0.3976 & 0.2173 \end{bmatrix} \\ P_{23} &= \begin{bmatrix} 6.0954 & 0.9473 \\ 0.9473 & 0.3890 \end{bmatrix} \\ P_{24} &= \begin{bmatrix} 24.5113 & 3.3788 \\ 3.3788 & 0.9030 \end{bmatrix}. \end{aligned}$$

Fig. 2 depicts the state trajectories for an initial condition $x(0) = [1\ \ 0]^T$ by applying $u(k) = F_1 x(k)$ (dotted line), by applying the piecewise control law without saturation (dashed line) and by applying the piecewise control law resulting from matrices F_{2q} (solid line). Remark that the rate of convergence toward the origin is better with the saturating piecewise control law.

VI. CONCLUSION

The major contribution of this note resides in the use of a local polytopic representation of the saturation nonlinearity for studying the multiobjective problem of both local stabilization and performance requirements satisfaction with respect to a linear system with saturating controls. Thanks to this representation and the use of relaxation schemes, numerical efficient techniques based on an LMI-framework are proposed in order to compute a effectively saturating state feedback control law that solves the problem.

Since efficient algorithms and software to solve LMI-based problems are available, the proposed method represents an interesting and easy-implementable way to compute saturating control laws. Moreover, the proposed LMI framework allows to treat uncertain systems and to incorporate to the problem other control requirements and state amplitude constraints.

REFERENCES

[1] D. S. Bernstein and A. N. Michel, "Special issue: Saturating actuators," *Int. J. Robust Nonlinear Control*, vol. 5, no. 5, pp. 375–540, 1995.

[2] A. A. Stoorvogel and A. Saberi, "Special issue: Control problems with constraints," *Int. J. Robust Nonlinear Control*, vol. 9, no. 10, pp. 583–734, 1999.

[3] C. C. H. Ma, "Unstability of linear unstable system with inputs limits," *J. Dynam. Syst., Meas. Control*, no. 113, pp. 742–744, 1991.

[4] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2411–2425, Dec. 1994.

[5] C. Burgat and S. Tarbouriech, "Stability and control of saturated linear systems," in *Non-Linear Systems*, A. J. Fossard and D. Normand-Cyrot, Eds. London, U.K.: Chapman & Hall, 1996.

[6] Z. Lin and A. Saberi, "Semi-global exponential stabilization of linear discrete-time systems subject to input saturation via linear feedback," *Syst. Control Lett.*, vol. 24, pp. 125–132, 1995.

[7] Z. Lin, A. Saberi, and A. A. Stoorvogel, "Semiglobal stabilization of linear discrete-time systems subject to input saturation via linear feedback—An ARE-based approach," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1203–1207, Aug. 1996.

[8] A. P. Molchanov and E. S. Pyatnitskii, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory," *Syst. Control Lett.*, vol. 13, pp. 59–64, 1989.

[9] J. M. Gomes da Silva, Jr., A. Fischman, S. Tarbouriech, J. M. Dion, and L. Dugard, "Synthesis of state feedback for linear systems subject to control saturation by an LMI-based approach," in *Proc. 2nd IFAC Symp. Robust Control*, Budapest, Hungary, 1997, pp. 229–234.

[10] S. Tarbouriech and J. M. Gomes da Silva, Jr., "Admissible polyhedra for discrete-time linear systems with saturating controls," in *Proc. 1997 Amer. Control Conf.*, vol. 6, Albuquerque, USA, 1997, pp. 3915–3919.

[11] M. Chilali and P. Gahinet, "H_∞ design with pole placement constraints: An LMI approach," *IEEE Trans. Automat. Contr.*, vol. 41, Mar. 1996.

[12] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied Mathematics* Philadelphia, PA, 1994.

[13] G. F. Wredenhagen and P. R. Bélanger, "Piecewise-linear LQ control for systems with input constraints," *Automatica*, vol. 30, no. 3, pp. 403–416, 1994.

[14] K. C. Goh, M. G. Safonov, and J. H. Ly, "Robust synthesis via bilinear matrix inequalities," *Int. J. Robust Nonlinear Control*, vol. 6, no. 9/10, pp. 1079–1095, 1996.

[15] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*. Natic, MA: The Math Works Inc., 1995.

[16] W. E. Schmitendorf, "Designing stabilizing controllers for uncertain systems using the Riccati equation," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 376–378, Apr. 1988.

Simultaneous Stabilization of Two Discrete-Time Plants Using a 2-Periodic Controller

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***Abstract*—A generic method for designing a 2-periodic controller for the simultaneous placement of the closed-loop poles of two single-input–single-output discrete shift-invariant plants at the origin is presented. The method consists of first recasting the simultaneous pole-placement problem as one of solving a coupled pair of linear polynomial equations involving three unknown polynomials, and then obtaining the controller parameters in terms of the coefficients of these polynomials. The isolated cases for which such pole placement is not possible have been listed. Simulation results show that the performances of systems thus compensated are superior to their performances when compensated using the higher periodicity controllers suggested in literature.**

***Index Terms*—Periodic controller, simultaneous stabilization.**

I. INTRODUCTION

As is well known, the simultaneous stabilization of two SISO, linear, time-invariant (LTI) plants using an LTI controller is not possible, except when the poles and zeros of the plants satisfy a certain interlacing property [1], [2]. In this note, we examine if a discrete periodic controller can achieve the same for two discrete shift-invariant plants. Now, although there exists a considerable number of works in the literature that investigate different aspects of the capabilities of periodic controllers [3]–[10] and the references therein), such a problem appears to have been considered only in [4]. (It may be noted that the problem of simultaneous pole placement of continous-time plants has been considered in [3]. The controller used there is of generalized sampled data hold-function type.) In [4], a controller C for the simultaneous stabilization of M shift-invariant plants, G_i, $i = 1, 2, \ldots, M$, has been obtained by patching up the dead-beat controllers C_i corresponding to each G_i in the following fashion:

$$\begin{aligned} C(N) &= C_i, \quad \text{for} \quad N_{i-1} \le N < N_i \\ &= 0, \quad \text{for} \quad N = N_M \\ &= C(N + N_M) \end{aligned}$$

where $N_i = n_1 + n_2 + \cdots + n_i$, and n_i is the number of sampling periods required by C_i to bring the output of G_i to zero. Now, since n_i depends on the order of G_i, the total period N_M may become a large number, and the main shortcoming of this controller is that for $N_M - n_i$ instants of the total period, the output of G_i will deviate from zero (and may actually build up substantially) before being brought to zero during the n_i instants when C_i is in force. Besides, since each C_i may, by itself, be unstable, it is important to note that its internal states must be reset to zero every time it is taken off the active loop.

In this note, we aim to stabilize two plants simultaneously using only one 2-periodic controller, chosen, following [5], in the controller-canonical form. The corresponding closed-loop characteristic equations for the two plants, as obtained following [5] and [6], are then simplified to yield a simultaneous pair of linear polynomial equations involving three unknown polynomials. The necessary and sufficient conditions for the existence of these polynomials along with a method for

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