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# JULIA SETS ARE UNIFORMLY PERFECT

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ABSTRACT. We prove that Julia sets are uniformly perfect in the sense of Pommerenke (Arch. Math. **32** (1979), 192–199). This implies that their linear density of logarithmic capacity is strictly positive, thus implying that Julia sets are regular in the sense of Dirichlet. Using this we obtain a formula for the entropy of invariant harmonic measures on Julia sets. As a corollary we give a very short proof of Lopes converse to Brolin's theorem.

Let $\overline{\mathbf{C}}$ be the Riemann sphere. As usual we say that a set $A \subset \overline{\mathbf{C}}$ is an *annulus* if there exists, for some $0 < r < 1$, a conformal representation of $\{z \in \mathbf{C} | r < |z| < 1\}$ onto $A$. The number $\log(1/r)$ is called the *modulus* of $A$.

Given a set $K \subset \overline{\mathbf{C}}$, we say that an annulus $A$ divides $K$ if $K \cap A = \varnothing$ and $K$ intersects both connected components of the complement $A^c$ of $A$.

In [6] Pommerenke introduced the following definition: a set $K \subset \overline{\mathbf{C}}$ is said to be *uniformly perfect* if it contains more than one point and there exists $m > 0$ such that every annulus that divides $K$ has modulus $\leq m$. In particular, connected sets satisfy this definition since no annulus can divide them thus making the condition vacuous.

A uniformly perfect set $K$ is always *regular* (in the sense of Dirichlet), i.e., for every continuous function $\varphi: K \to \mathbf{R}$ there exists a continuous function $\varphi^*: \overline{\mathbf{C}} \to \mathbf{R}$ such that $\varphi^*/K = \varphi$ and $\varphi$ is harmonic in $K^c$. The function $\varphi^*$ (that is unique) will be called the *harmonic extension* of $\varphi$. In fact Pommerenke proved in [6] a much stronger property, namely, that denoting $d(\cdot, \cdot)$ the spherical metric on $\overline{\mathbf{C}}$ and denoting $\gamma(S)$ the logarithmic capacity of a compact set $S$, a compact set $K \subset \overline{\mathbf{C}}$ is uniformly perfect if and only if there exists $\delta > 0$ such that for all $a \in K$ and $r > 0$

$$\gamma(\{z \in K | d(z, a) \leq r\}) \geq \delta r.$$

It is well known that this property implies the regularity of $K$ (see, for instance, [8]).

Our first objective is to prove the following result that answers positively a question posed by Pommerenke in [7]. Afterwards we shall apply it to give a formula for the entropy of harmonic measures on Julia sets through which we shall recover, with much shorter proofs, some already known results relating

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harmonic measures with the maximizing (i.e., the entropy maximizing) measure of a rational map.

**Theorem.** *The Julia set $J(f)$ of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$ is uniformly perfect.*

*Proof*. Suppose, by contradiction, that there is a sequence of annuli $A_n$, $n = 1, 2, \dots$, dividing $K$ such that $\lim_{n\to+\infty} \operatorname{mod}(A_n) = \infty$. This property implies that for each $n$, a connected component $K_n$ of $A_n^c$ can be chosen so that $\lim_{n\to+\infty} \operatorname{diam} K_n = 0$. Denote by $K_n'$ the other connected component of $A_n^c$. Then $\inf_{n>0} \operatorname{diam} K_n' > 0$ because otherwise we could take a subsequence $\{K_{n_j}'\}$; with $\lim_{j\to+\infty} \operatorname{diam} K_{n_j}' = 0$ and points $p_j' \in K_{n_j}' \cap J(f)$, $p_j \in K_{n_j} \cap J(f)$ converging to points $p'$ and $p$ in $J(f)$, and then, from $\lim_{j\to+\infty} \operatorname{diam} K_{n_j}' = \lim_{j\to+\infty} \operatorname{diam} K_{n_j} = 0$ and $J(f) \subset K_{n_j} \cup K_{n_j}'$ for all $j$, it follows that $J(f) = \{p\} \cup \{p'\}$, which is impossible. Denote $D = \{z \in \mathbf{C} \mid |z| < 1\}$ and let $\varphi_n: D \to A_n \cup K_n$ be a conformal representation with $\varphi_n(0) \in K_n$. Then $\operatorname{mod}(D - \varphi_n^{-1}(K_n)) = \operatorname{mod}(A_n)$. Hence $\lim_{n\to+\infty} \operatorname{diam} \varphi_n^{-1}(K_n) = 0$ because $\lim_{n\to+\infty} \operatorname{mod}(A_n) = \infty$. Take $1 > r_n > \rho_n > 2\operatorname{diam}(K_n)$ satisfying $\lim_{n\to+\infty} r_n = 0$, $\lim_{n\to+\infty} \rho_n / r_n = 0$. Set $D_n' = \{z \mid |z| < \rho_n\}$. The family of functions $\varphi_n: D \to \overline{\mathbf{C}}$ is normal because $\inf_n \operatorname{diam} \varphi_n(D)^c = \inf_n \operatorname{diam} K_n' > 0$. Hence $\lim_{n\to+\infty} \operatorname{diam} \varphi_n(D_n') = 0$. But $\varphi_n(D_n')$ is an open set containing points of $J(f)$. Therefore the classical theory of Julia sets implies that there exist integers $t_n > 0$ such that $f^{t_n}(\varphi_n(D_n')) \supset J(f)$. Take $0 < c < \operatorname{diam} J(f)$, and let $m_n$ be the minimum positive integer such that $\operatorname{diam} f^{m_n}(\varphi_n(D_n')) \geq c$. Since $\lim_{n\to+\infty} \operatorname{diam} \varphi_n(D_n') = 0$, it follows that $\lim_{n\to+\infty} m_n = +\infty$. Moreover, since $\operatorname{diam} f^{m_n - 1}(\varphi_n(D_n')) < c$, it follows that $\operatorname{diam} f^{m_n}(\varphi_n(D_n')) < Lc$, where $L$ is the Lipschitz constant of $f$. Let $S$ be a set of four different points in $J(f)$. Take $c$ so small that every set of diameter $\leq Lc$ cannot contain two of them. Then $f^{m_n}(\varphi_n(D_n'))$ does not cover three points of $S$. Define $\psi_n: D \to \overline{\mathbf{C}}$ by $\psi_n(z) = f^{m_n} \varphi_n(r_n z)$. Let us show that the family $\{\psi_n\}$ is normal. It suffices to show that for all $n$, $\psi_n(D)$ does not cover three points (that may depend on $n$) of $S$. If $\rho_n / 2r_n < |z| < 1$ then $\rho_n/2 < |r_n z| < r_n$ and, since $\operatorname{diam} \varphi_n^{-1}(K_n) \leq \rho_n/2$, it follows that $r_n z \notin \varphi_n^{-1}(K_n)$ and $\varphi_n(r_n z) \notin K_n$. Then $\varphi_n(r_n z) \notin J(f)$ because $K_n = J(f) \cap \varphi_n(D)$. Hence $\psi_n(z) = f^{m_n} \varphi_n(r_n z) \notin f^{m_n}(J(f)) = J(f)$. Therefore $\psi_n(z) \notin S$ when $\rho_n/2r_n < |z| < 1$. On the other hand, if $|z| \leq \rho_n / 2r_n$, it follows that $|r_n z| \leq \rho_n/2 \leq \rho_n$ and then $\psi_n(z) = f^{m_n} \varphi_n(r_n z) \in f^{m_n} \varphi_n(D_n')$ that does not cover three points of $S$. This proves the normality of the family $\{\psi_n\}$. Then, given $\varepsilon > 0$, there exists a neighborhood $V$ of 0 such that $\operatorname{diam} \psi_n(V) \leq \varepsilon$ for all $n$. But for $n$ sufficiently large, $V \supset \{z \mid |z| < \rho_n / r_n\}$.

$$\begin{aligned} \varepsilon \geq \operatorname{diam} \psi_n(V) &\geq \operatorname{diam} \psi_n(\{z \mid |z| < \rho_n/r_n\}) \\ &= \operatorname{diam} f^{m_n} \varphi_n(D_n') \geq c. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this is a contradiction that proves the theorem.

Let us show some applications of the regularity of the Julia set. Recall that, given a regular compact set $K \subset \overline{\mathbf{C}}$ and a point $p \in \overline{\mathbf{C}}$, the *harmonic measure* $\mu_p$ is defined as the probability on the Borel $\sigma$-algebra of $K$ such that the integral with respect to $\mu_p$ of a continuous function $\varphi: K \to \mathbf{R}$ is given by

$$\int \varphi \, d\mu_p = \varphi^*(p)$$

where $\varphi^*:\overline{\mathbf{C}} \to \mathbf{R}$ is the harmonic extension of $\varphi$. If $p \notin K$ the support of $\mu_p$ is obviously the boundary of the connected component of $K^c$ that contains $p$, and it is well known that if $p$ and $q$ are in the same connected component of $K^c$ then $\mu_p$ and $\mu_q$ are equivalent and the Radon-Nykodim derivative $d\mu_p/d\mu_q$ is bounded and has a strictly positive infimum.

Observe that if $f:\overline{\mathbf{C}} \hookleftarrow$ is a rational map and $\varphi: J(f) \to \mathbf{R}$ is continuous, then $(\varphi \circ f)^* = \varphi^* \circ f$ because $\varphi^* \circ f$ is harmonic on the complement of $J(f)$ and $(\varphi^* \circ f)|J(f) = (\varphi \circ f)|J(f)$. Then

$$\int (\varphi \circ f)\, d\mu_p = \int \varphi\, d\mu_{f(p)}$$

because

$$\int (\varphi \circ f)\, d\mu_p = (\varphi \circ f)^*(p) = (\varphi^* \circ f)(p) = \varphi^*(f(p)) = \int \varphi\, d\mu_{f(p)}.$$

Hence, $\mu_p$ is $f$-invariant if and only if $f(p) = p$.

Given an attracting fixed point $p$ of $f$, define its basin $W^s(p)$ as the set of points $z$ such that $\lim_{n\to+\infty} f^n(z) = p$ and its immediate basin $B^s(p)$ as the connected component of $W^s(p)$ that contains $p$. Then $\mu_p$ is an invariant probability of $f$ whose support is the boundary $\partial B^s(p)$ of $B^s(p)$.

**Corollary 1.** *If $f:\overline{\mathbf{C}} \hookleftarrow$ is a rational map and $p$ is an attracting fixed point of $f$, the entropy $h_{\mu_p}(f)$ of $f$ with respect to $\mu_p$ is given by*

$$h_{\mu_p}(f) = \int \log \left( \sum_{x \in f^{-1}(p)\cap B^s(p)} \frac{d\mu_x}{d\mu_p} \right) d\mu_p$$

*where the points $x \in f^{-1}(p)$ are repeated according to its multiplicity.*

*Proof.* Define $J: \partial B_s(p) \to \mathbf{R}$ by

$$J = \sum_{x \in f^{-1}(p)\cap B^s(p)} \frac{d\mu_x}{d\mu_p}.$$

Let us prove that $J$ is the Jacobian of $\mu_p$, i.e., that

$$\mu_p(f(A)) = \int_A J\, d\mu_p \tag{1}$$

for every Borel set $A \subset \partial B^s(p)$ such that $f/A$ is injective. Once this is proved, the corollary follows from the formula

$$h_{\mu_p}(f) = \int \log J\, d\mu_p$$

proved in [5]. To prove (1) denote $C^0(\partial B^s(p))$ and $C^0(\overline{B^s(p)})$ the spaces of continuous functions of $\partial B^s(p)$ and $\overline{B^s(p)}$ on $\mathbf{R}$ endowed with the norm of the supremum. Given a function $\varphi: \partial B^s(p) \to \mathbf{R}$ (resp. $\varphi: B^s(p) \to \mathbf{R}$), define $\mathscr{L}(\varphi): \partial B^s(p) \to \mathbf{R}$ (resp. $\mathscr{L}(\varphi): B^s(p) \to \mathbf{R}$) by $\mathscr{L}(\varphi)(x) = \sum_y \varphi(y)$ where the sum is taken over all the $y$'s in $f^{-1}(x) \cap \partial B^s(p)$ (resp. $y \in f^{-1}(x) \cap \overline{B^s(p)}$) repeated according to its multiplicity. Observe that $\mathscr{L}$ maps continuous functions in continuous functions. Then if $\varphi \in C^0(\partial B^s(p))$ then the harmonic extension $(\mathscr{L}(\varphi))^* \in C^0(\overline{B^s(p)})$ of $\mathscr{L}(\varphi)$ satisfies $(\mathscr{L}(\varphi))^* = \mathscr{L}(\varphi^*)$. To see this observe that $\mathscr{L}(\varphi^*)$ is harmonic in the complement of the critical values of $f$ because there it is locally the sum of the harmonic function $\varphi^*$ composed with the holomorphic branches of $(f|B^s(p))^{-1}$. Moreover $\mathscr{L}(\varphi^*)$ is obviously continuous; hence $\mathscr{L}(\varphi^*)$ is harmonic. Clearly we have $\mathscr{L}(\varphi^*)/\partial B^s(p) = \mathscr{L}\varphi$. Hence $\mathscr{L}(\varphi^*) = (\mathscr{L}(\varphi))^*$. Then, if $\varphi \in C^0(\partial B^s(p))$,

$$\begin{aligned}\int \mathscr{L}(\varphi)\, d\mu_p &= (\mathscr{L}(\varphi))^*(p) = \mathscr{L}(\varphi^*)(p)\\ &= \sum_{x\in f^{-1}(p)\cap B^s(p)} \varphi^*(x) = \sum_{x\in f^{-1}(p)\cap B^s(p)} \int \varphi\, d\mu_x\\ &= \int \varphi \left(\sum_{x\in f^{-1}(p)\cap B^s(p)} \frac{d\mu_x}{d\mu_p}\right) d\mu_p = \int \varphi J\, d\mu_p.\end{aligned}$$

From this equality it follows, by standard methods, that

$$\int \mathscr{L}(\varphi)\, d\mu_p = \int \varphi J\, d\mu_p$$

for every bounded measurable $\varphi : \partial B^s(p) \to \mathbf{R}$. Apply it to the case when $\varphi$ is the characteristic function of a Borel set $A \subset \partial B^s(p)$ such that $f/A$ is injective. Then $\mathscr{L}(\varphi)$ is the characteristic function of $f(A)$. Hence

$$\mu_p(f(A)) = \int \mathscr{L}(\varphi)\, d\mu_p = \int \varphi J\, d\mu_p = \int_A J\, d\mu_p,$$

completing the proof of the corollary.

For the next corollary recall that [4, 2, 5] if $\deg(f/B^s(p))$ denotes the degree of $f/B^s(p)$ (i.e., the number of preimages in $B^s(p)$ of any $x \in B^s(p)$ counted with multiplicity) then the topological entropy of $f|\partial B^s(p)$ is $\log\deg(f|B^s(p))$ and there exists a unique probability $\mu$ on $\partial B^s(p)$, invariant under $f|\partial B^s(p)$, such that $h_\mu(f) = \log\deg(f|B^s(p))$.

**Corollary 2.** $h_{\mu_p}(f) = \log\deg(f|B^s(p))$ *if and only if* $f^{-1}(p) \cap B^s(p) = p$.

*Proof.* If $f^{-1}(p) \cap B^s(p) = p$ then the formula of Corollary 1 immediately implies $h_{\mu_p}(f) = \log\deg B^s(p)$ because $d\mu_x|d\mu_p \equiv 1$ for all $x \in f^{-1}(p)\cap B^s(p)$. To prove the converse property observe that by Jensen inequality

$$\begin{aligned}h_{\mu_p}(f) &= \int \log\left(\sum_{x\in f^{-1}(p)\cap B^s(p)} \frac{d\mu_x}{d\mu_p}\right) d\mu_p\\ &\le \log \int \left(\sum_{x\in f^{-1}(p)\cap B^s(p)} \frac{d\mu_x}{d\mu_p}\right) d\mu_p = \log \sum_{x\in f^{-1}(p)\cap B^s(p)} \left(\int d\mu_x\right)\\ &= \log\deg(f|B^s(p))\end{aligned}$$

and the equality holds if and only if

$$\sum_{x\in f^{-1}(p)\cap B^s(p)} \frac{d\mu_x}{d\mu_p} = \deg(f|B^s(p))$$

$\mu_p$-almost everywhere. Denote $m = \deg(f|B^s(p))$ and let $S$ be the set of points in $f^{-1}(p) \cap B^s(p)$ repeated according to its multiplicity. Then

$$m\varphi^*(p) = \sum_{x\in S} \varphi^*(x) \tag{2}$$

for every $\varphi \in C^0(\partial B^s(p))$ because

$$\begin{aligned} m\varphi^*(p) &= \int m\varphi \, d\mu_p = \int \varphi \left(\sum_{x\in S} \frac{d\mu_x}{d\mu_p}\right) d\mu_p \\ &= \sum_{x\in S} \int \varphi \, d\mu_x = \sum_{x\in S} \varphi^*(x). \end{aligned}$$

Let us show that (2) implies that $x \in S$ implies $x = p$. Without loss of generality we shall assume that $\infty \in J(f)$. First we shall prove that (2) implies

$$p = \frac{1}{m} \sum_{x\in S} x. \tag{3}$$

Suppose that this is false. Then we can take a linear function $\psi: \mathbf{R}^2 = \mathbf{C} \to \mathbf{R}$ such that

$$\psi(p) \neq \frac{1}{m} \sum_{x\in S} \psi(x).$$

Given integers $n > 0$, $k > 0$, define $\varphi_{n,k} \in C^0(\partial B^s(p))$ by

$$\begin{aligned} \varphi_{n,k}(z) &= \psi(z) && \text{when } -k < \psi(z) < n, \\ \varphi_{n,k}(z) &= n && \text{when } \psi(z) \geq n, \\ \varphi_{n,k}(z) &= -k && \text{when } \psi(z) \leq -k. \end{aligned}$$

Then for each $k$, $\{\varphi_{n,k}\}_n$ is an increasing sequence and for each $n$, $\{\varphi_{n,k}\}_k$ is a decreasing sequence. Hence the same properties hold for $\{\varphi^*_{n,k}\}_n$, $\{\varphi^*_{n,k}\}_k$. Then, using that $\psi$ is harmonic, it is easy to see that

$$\psi = \lim_{k\to+\infty} \lim_{n\to+\infty} \varphi^*_{n,k}.$$

Then

$$\varphi^*_{n,k}(p) \neq \frac{1}{m} \sum_{x\in S} \varphi_{n,k}(x)$$

for $n, k$ sufficiently large, contradicting (2) and proving (3). But (3) implies

$$T(p) = \frac{1}{m} \sum_{x\in S} T(x) \tag{4}$$

for every Moebius map $T: \overline{\mathbf{C}} \hookleftarrow$ such that $T^{-1}(\infty) \in J(f)$ because $TfT^{-1}$ has $T(p)$ as an attracting fixed point whose preimages in this immediate basin are the points $\{T(x)|x \in S\}$ and $\infty \in J(TfT^{-1})$ (by the property $T^{-1}(\infty) \in J(f)$), and then, (3) implies (4). In particular (4) implies

$$\frac{1}{p - z_0} = \frac{1}{m} \sum_{x\in S} \frac{1}{x - z_0} \tag{5}$$

for all $z_0 \in J(f)$ (taking $T$ as $T(z) = (z - z_0)^{-1}$). The number of values of $z_0 \in \mathbf{C}$ for which this equality holds is either finite or holds for every $z_0 \in \mathbf{C}$. In the first case, since $J(f)$ is infinite, we can take $z_0 \in J(f)$ violating (5) and thus proving Corollary 2 by contradiction. In the second case, the left and right sides of (5) are identical as functions of $z_0 \in \mathbf{C}$. But the left side function has a unique pole at $z_0 = p$, and the right side has poles at all the $x$'s in $S$. Then, to be identical, we must have $p = x$ for all $x \in S$, proving the corollary.

Finally recall that a rational map $f: \overline{\mathbf{C}} \hookleftarrow$ of degree $d > 1$ has topological entropy $\log d$ and a unique invariant probability $\mu_{\max}$ (the maximizing measure) for which $h_{\mu_{\max}}(f) = \log d$ ([4, 2, 5]). When $f$ is a polynomial, $\infty$ is an attracting fixed point and $f^{-1}(\infty) = \{\infty\}$ (with multiplicity $d$). Hence, by Corollary 1, $h_{\mu_\infty}(f) = \log d$. We have thus proved the result of Brolin [1] stating that *for polynomials the maximizing measure is the harmonic measure with respect to* $\infty$. Using Corollaries 1 and 2 we can also prove the converse property (due to Lopes [3]): *if for a rational map* $f: \overline{\mathbf{C}} \hookleftarrow$ *its maximizing measure coincides with its harmonic measure with respect to* $\infty$ *then* $f$ *is a polynomial.* To prove this observe that $\mu_{\max} = \mu_\infty$ implies that $\mu_\infty$ is $f$-invariant. Hence $f(\infty) = \infty$. If $\infty \in J(f)$ then $\mu_\infty$ is the Dirac $\delta$ at $\infty$ and $h_{\mu_\infty}(f)$ would be 0. Hence $\infty \notin J(f)$. Being a fixed point, the property $\infty \notin J(f)$ implies that it is either an attracting fixed point, or the center of a Siegel disk. In the first case, from Corollary 1 and Jensen's inequality, it follows that

$$\log d = h_{\mu_\infty}(f) \leq \log \deg(f | B^s(\infty)) \leq \log d.$$

Hence, all the equalities hold, implying that the number of points in $f^{-1}(\infty) \cap B^s(\infty)$ counted with multiplicity is $d$ (because $\log \deg(f|B^s(\infty)) = \log d$) and, by Corollary 2, $f^{-1}(\infty) \cap B^s(\infty) = \{\infty\}$ (because $\log \deg(f|B^s(\infty)) = h_{\mu_\infty}(f)$). Since $f^{-1}(\infty)$ contains $d$ points (counted with multiplicity), it follows that $f^{-1}(\infty) = \{\infty\}$, which proves that $f$ is a polynomial. When $\infty$ is the center of a Siegel disk, $h_{\mu_\infty}(f) = 0$. This follows, for instance, from observing that the formula in Corollary 1 holds also (without changing the proof) replacing $B^s(p)$ by a Siegel disk with fixed point $p$. But since in this case $f^{-1}(p)$ intersected with the Siegel disk contains only $p$, it follows that $h_{\mu_p}(f) = 0$. Hence $h_{\mu_\infty}(f) = 0$, obviously contradicting $h_{\mu_\infty}(f) = \log d$.

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