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Minimal Length Scale Models for Compact Stars

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Nature uses only the longest threads to weave her patterns, so each small piece of her fabric reveals the organization of the entire tapestry.

Richard Feynman

Abstract

We investigate the equation of state (EoS) for compact stars under the influence of a minimal length scale, utilizing two distinct approaches: the effective Generalized Uncertainty Principle (GUP) and noncommutative geometry via coherent states coordinates. Our application of the effective Kempf GUP formalism to the MIT Bag Model yields a modified EoS that aligns with the conventional theory as the GUP parameter approaches zero, revealing a maximal baryon density alongside a slight increase in the mass-radius relation of compact objects, indicating enhanced stability against gravitational collapse. These findings extend existing research on GUP-deformed Fermi gases. We then briefly outline the path towards an even more generalized GUP framework capable of integrating a variety of particles and interactions. Conversely, when applying the coherent states approach to the MIT Bag Model, we observe an overall qualitatively consistent behavior with the GUP model, though without quantitatively significantly impacting the EoS or mass-radius relations. Ultimately, our descriptions introduce minimal length scales and, consequently, the effects of quantum gravity in compact stars in a mathematically simple manner, suggesting their potential for extension to more complex systems.

Resumo

Investigamos a equação de estado (EoS) para estrelas compactas sob a influência de uma escala de comprimento mínima, utilizando duas abordagens distintas: o Princípio da Incerteza Generalizado (GUP) efetivo e a geometria não comutativa por meio de coordenadas de estados coerentes. Nossa aplicação do formalismo GUP efetivo de Kempf ao MIT Bag Model resulta em uma EoS modificada que se alinha com a teoria convencional à medida que o parâmetro GUP se aproxima de zero, revelando uma densidade bariônica máxima juntamente com um leve aumento na relação massa-raio de objetos compactos, indicando maior estabilidade ao colapso gravitacional. Esses achados estendem pesquisas existentes sobre gases de Fermi deformados pelo GUP. Em seguida, esboçamos brevemente o caminho para um modelo de GUP ainda mais generalizado, capaz de integrar uma variedade de partículas e interações. Por outro lado, ao aplicar a abordagem de estados coerentes ao MIT Bag Model, observamos um comportamento geral qualitativamente consistente com o modelo GUP, embora sem impactar significativamente de forma quantitativa a EoS ou as relações massa-raio. Em última análise, nossas descrições introduzem escalas de comprimento mínimas e, consequentemente, os efeitos da gravidade quântica em estrelas compactas de maneira matematicamente simples, sugerindo seu potencial para extensão a sistemas mais complexos.

Contents

Notation and Conventions

Throughout the entirety of the present work, we will utilize (unless explicitly stated otherwise) the subsequent notation and convention choices.

We use the Planck system of units $G = c = \hbar = k_B = 1$ for the sake of brevity and convenience.

The metric signature is $s = -2$, namely

$$\eta_{\mu\nu} = (+, -, -, -) \ .$$

The Einstein sum convention is used - repeated upper and lower Lorentz indices are summed over:

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu \ .$$

The notation $\mathbf{A}$ denotes a spatial vector of components with upper indices

$$\mathbf{A} = (A^1, A^2, A^3) \ .$$

Partial derivatives are denoted as $\frac{\partial}{\partial x^\mu} = \partial_\mu$, as such, we have

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \mathbf{x}} = \nabla \ .$$

A general four-dimensional Fourier transform is given by

$$f(x) = \int \frac{d^4 p}{(2\pi)^2} \, e^{-ipx} \tilde{f}(p) \ ,$$
$$\tilde{f}(p) = \int \frac{d^4 p}{(2\pi)^2} \, e^{ipx} f(x) \ .$$

Our choice of metric signature implies the three-dimensional Fourier transform

$$f(\mathbf{x}) = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \, e^{-i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}) \ ,$$
$$\tilde{f}(\mathbf{p}) = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \, e^{i\mathbf{p}\cdot\mathbf{x}} f(\mathbf{x}) \ .$$

For arbitrary n, the Dirac delta satisfies

$$\int d^n x \, e^{ipx} = (2\pi)^n \delta^n(p) \ .$$

1. Introduction

The concept of an indivisible and fundamental physical quantity has permeated human thought across millennia. Democritus, in the 5th century B.C., speculated on the existence of minimal entities comprising all matter, which he termed ἀτόμους (atomos), meaning "uncuttable" or "indivisible." This notion of indivisibility laid the philosophical cornerstone for our ongoing quest to understand the underlying structure of nature.

The ancient consideration of the atom as the smallest unit of matter finds its current natural and ultimate extension in the proposition that the fabric of spacetime itself might be composed of indivisible quanta. The emergence of Quantum Field Theory (QFT) in the 1930s further advanced this idea as a minimal length could serve as a solution to the infinities that plagued theoretical calculations.

Regularization techniques, such as cutoffs, were employed not merely as mathematical conveniences but were pondered as possibly reflecting a deeper physical reality. Yet, the idea of an intrinsic scale clashed with the principle of Lorentz invariance, introducing frame-dependent complications. Among the pioneers, Heisenberg contemplated a fundamentally discrete spacetime, proposing it as a natural source for a UV cutoff, a perspective he elaborated in his correspondence with notable contemporaries like Bohr and Pauli [1].

However, it was Matvei Bronstein, a visionary Russian physicist, who first grasped the full implications of a minimal length scale, extending beyond mere regularization methods in electrodynamics. Bronstein, now acknowledged for his foresight in recognizing the challenges of quantizing gravity, discerned the unique nature of gravitational interactions as early as 1936 [1,2]. He identified that, unlike other forces, gravity inherently resists the concentration of energy within a confined space due to the risk of forming a black hole. This insight pointed to an intrinsic limitation in the accuracy with which the gravitational field can be measured, foreshadowing the modern idea that spacetime may not be infinitely divisible.

Tragically, Bronstein's profound contributions went largely unnoticed during his lifetime, and his promising journey was cut short, as he was arrested in 1938 and ultimately executed in a Leningrad prison at the young age of 31 [1].

As Quantum Electrodynamics (QED) evolved into a more refined theory, once troubling infinities found their resolution, and the predictions of QED stood up to increasingly precise experiments. This progress allowed physicists to probe matter at progressively smaller scales without encountering a fundamental limit to resolution. Consequently, the early divergences that hinted at a discrete spacetime became less concerning, and such models lost favor, particularly due to their already mentioned conflict with the principle of Lorentz invariance.

It was Hartland Snyder, in 1947, who offered a mathematical framework that resonated with Heisenberg's earlier conjectures [3]. Distancing himself from the conventional use of a simple momentum space cutoff, which he deemed unfitting, Snyder pioneered a novel approach. By altering the canonical commutation relations between position and momentum operators, he elegantly devised a Lorentz-covariant noncommutative spacetime. This modification implicitly led to an increased Heisenberg uncertainty, subtly introducing the concept of a minimal possible resolution for observing structures - a notion that Snyder himself did not discuss in his seminal paper [1].

The critical influence of gravity in probing the minuscule scales of physics was finally brought to light by C. Alden Mead in 1964 [4, 5]. Through a series of insightful thought experiments, he elucidated how gravity amplifies Heisenberg's measurement uncertainty, inherently limiting our ability to measure distances with precision finer than the Planck length. Mead recognized that since gravity exerts a universal coupling, its effect, albeit often small, is an unavoidable factor in all experimental measurements [1].

In the 1970s, Hawking's groundbreaking work on black hole thermodynamics [6] unveiled what came to be known as the "trans-Planckian problem." The theoretically infinite blue shift of photons near a black hole's horizon posed the challenge of accounting for modes with energies surpassing the Planck scale when calculating a black hole's emission rate. In the wake of Hawking's discovery, significant contributions were made that deepened our grasp of black hole mechanics and the Planck scale - which inspired much of what followed in the study of minimal length scales. An excellent review on the concepts and main implementations of minimal length scales (up to 2013) may be found in [1] and a comprehensive historical coverage in [7]. An interesting study on the metaphysics and conceptual foundations of noncommutative geometries is made in [8].

Presently, reconciling the quantum field theories that constitute the Standard Model with the principles of General Relativity stands as one of the most profound challenges in theoretical physics. The obstacle is not the inability to quantize gravity itself, but rather that the traditional approach to doing so yields a perturbatively nonrenormalizable theory. At the heart of this issue lies the fact that Newton's constant is a dimensional quantity. This characteristic necessitates an unending sequence of counter-terms in the theory, which ultimately leads to the loss of predictive power.

It is indeed remarkable that the presence of a minimal length scale is an inherent feature of all attempts at describing fundamental theories of gravitation. For example, theories of quantum gravity such as String Theory and Quantum Loop Gravity all predict the existence of a fundamental unit of length [9].

While black holes, with their exceptional density and gravitational deformation, present an arena where the intertwining of Quantum Mechanics and General Relativity - and consequently the effects of quantum gravity - becomes most pronounced, it is compelling to consider whether compact stars, as the next densest objects known in the universe, might similarly reveal imprints of these fundamental interactions. This inquiry also provides a tangible context where the theoretical may meet the observable.

In what follows, we first present a brief introduction to the description of compact stars within classical four-dimensional spacetime in Chapter 2. The subsequent chapters delve into theoretical frameworks that introduce a minimal length scale to spacetime, first via the effective Kempf GUP formalism in Chapter 3, and then through the coherent states coordinate approach to noncommutative geometry in Chapter 4.

Our novel contributions are detailed in Chapters 5 and 6, where these quantum frameworks are applied to model descriptions of equations of state for compact stars in spaces endowed with a minimal length - possibly implementing the effects of quantum gravity. We study the thermodynamic consequences of such modifications and the implications they have in the mass-radius relations of stellar structure. Finally, our conclusions and final remarks are presented in Chapter 7.

2. Compact Stars in Conventional Spacetime

We shall commence by briefly introducing and deriving the laws and equations that govern the structure and existence of compact stars within the conventional framework of four-dimensional spacetime. This provides the necessary foundation and contrast for the later chapters, in which we will explore the implications of introducing a minimal length and non-commutative spacetime in our understanding of these compact objects.

2.1 The Tolman-Oppenheimer-Volkoff Equations

The Einstein Field Equations (EFE) constitute the core of General Relativity, dictating how matter and energy influence the curvature of spacetime. First derived in 1915 by Albert Einstein [10], they can be written as (assuming a vanishing cosmological constant, i.e., $\Lambda = 0$)

$$G_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{2.1}$$

where $\kappa = -8\pi$.

On the left-hand side of Eq. (2.1) we have the Einstein tensor $G_{\mu\nu}$, which describes the deformation of spacetime, given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \ , \tag{2.2}$$

with $g_{\mu\nu}$ as the metric tensor, and $R_{\mu\nu}$ and R being the Ricci tensor and Ricci scalar, respectively. The latter two terms provide measures of spacetime curvature and are derived from $g_{\mu\nu}$.

On the right-hand side of (2.1) we have the energy-momentum tensor $T_{\mu\nu}$, a symmetric second-rank tensor which describes how energy and matter are distributed within spacetime. A crucial feature of $T_{\mu\nu}$ is its conservation, mirroring the classical notion of energy and momentum conservation. This is expressed through its vanishing covariant divergence, that is,

$$\nabla^{\mu} T_{\mu\nu} = 0 \ . \tag{2.3}$$

This equation embodies the principle of local conservation of energy and momentum and ensures the internal consistency of the EFE.

Upon deriving the EFE, Einstein initially conjectured that, due to their non-linearity, they might have not had closed-form analytical solutions. This presumption, much to Einstein's surprise, was soon proven incorrect, when in 1916 the astrophysicist Karl Schwarzschild found the first exact solution to the EFE for the exterior field of a spherically symmetric, non-rotating mass, a solution that now bears his name [11]. This solution is of fundamental importance for its simplicity and symmetry. It stands as a cornerstone for studying compact objects in General Relativity.

The solution is derived in the vacuum context of the EFE, that is, the energy-momentum tensor effectively vanishes ($T_{\mu\nu} = 0$). Expressed in the spherical coordinates, the Schwarzschild metric takes the form

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{2.4}$$

where M is the mass of the spherically symmetric object, r is the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ denotes the solid angle.

While the Schwarzschild metric provides a profound understanding of the spacetime geometry outside of a spherically symmetric mass, it does not provide a description of the interior of such an object. To do this, one must abandon the assumption of a vacuum, thus introducing the concept of a matter-filled space where $T_{\mu\nu} \neq 0$. This necessitates a distinct solution to the Einstein field equations, which encompasses the physical properties of the matter distribution.

Frequently, as is the case for this work, the energy-momentum tensor is assumed to represent a perfect fluid. This is an idealized model of a fluid, which can be characterized entirely by its energy density ε and pressure p. The general form of the energy-momentum tensor for a perfect fluid is given by

$$T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu - p g_{\mu\nu} \ , \tag{2.5}$$

where u^μ is the four-velocity of the fluid satisfying the normalization condition $u_\mu u^\mu = 1$. In the rest frame of the fluid, where $u^\mu = (1, 0, 0, 0)$, the pressure becomes isotropic, and the energy-momentum tensor simplifies to the diagonal form

$$T_{\mu\nu} = \mathrm{diag}(\varepsilon, -p, -p, -p) \ . \tag{2.6}$$

The simplicity of the perfect fluid model makes it a powerful tool for a wide range of physical systems, particularly those related to stellar interiors.

Building upon the assumptions of the Schwarzschild metric and the perfect-fluid model of matter, a more comprehensive description of the interior of a spherically symmetric object was derived. In 1939, Richard C. Tolman, J. Robert Oppenheimer and George Volkoff, independently but nearly simultaneously, produced a set of differential equations that remain a pillar of modern astrophysics, known today as the Tolman-Oppenheimer-Volkoff (TOV) equations [12, 13].

The TOV equations, crucial in the study of stellar structure and stability, may be written as

$$\frac{dp(r)}{dr} = -\frac{[\varepsilon(r) + p(r)]\,[M(r) + 4\pi r^3 p(r)]}{r\,[r - 2M(r)]}\ , \tag{2.7}$$

$$M(r) = 4\pi \int_0^r \varepsilon(r')r'^2 dr'\ . \tag{2.8}$$

The first of these describes hydrostatic equilibrium, ensuring that the inward gravitational pull is precisely countered by the outward pressure gradient. The second equation, in integral form, represents the gravitational mass $M(r)$ enclosed within a sphere of radius r. It accounts for the cumulative effect of the local energy density $\varepsilon(r')$ at all radii r' from the center up to r.

However, to fully solve these equations, one requires a link between the energy density ε and pressure $p(r)$. This relation is provided by the Equation of State (EoS), which typically takes the form $\varepsilon(r) = \varepsilon\,(p(r))$. The EoS is indispensable as it encapsulates the thermodynamic properties of the stellar matter, tying together the microscopic physics of individual particles with the macroscopic properties of the star. The choice of EoS often depends on the specific astrophysical scenario under consideration, and each different EoS may lead to remarkably distinct stellar structures and behaviors.

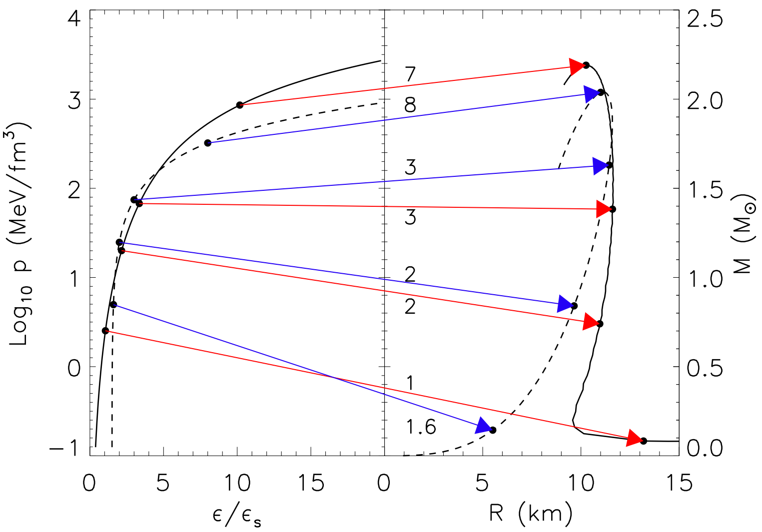

Figure 2.1: Two distinct EoS, through the TOV equations, lead to two distinct mass-radius relations. The arrows connect specific central energy density and pressure values with their corresponding $(M,\,R)$ points. Figure taken from [14].

2.2 The Equation of State

In the preceding section, we briefly introduced the TOV equations, which provide a basis for understanding the macroscopic behavior and properties of compact stars. We now shift our focus and explore the underlying microscopic physics described by the equation of state.

2.2.1 Conserved Quantities of the Lagrangian Formalism

Matter and energy interact with spacetime, bending it while being influenced by its curvature. Consequently, any theory of matter subjected to intense gravitational forces should be examined through the lens of curved spacetime.

The equivalence principle ensures that a local Lorentz frame can be constructed at any point in a star, serving as a reference frame. Interestingly, even at the highest densities of a neutron star on the brink of a black hole collapse, the relative metric variation over the average distance between baryons remains incredibly small. This negligible shift allows us to make a robust approximation where we treat the local Lorentz frame within a compact star as effectively infinite in relation to particle spacings. We may then treat matter at a specific density as if it were of infinite extent, disregarding the energy associated with boundaries compared to the bulk energy [15]. As such, we are allowed to use standard Quantum Field Theory (QFT) to microscopically describe matter within a compact star.

In QFT, as in classical field theory, the dynamics of a system with infinite degrees of freedom, such as a field ϕ, can be described using the Lagrangian density formalism. This formalism employs a functional (the Lagrangian density) $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ that describes the dynamics of the field and its derivatives. The behavior of the field ϕ can then be understood by considering the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \ . \tag{2.9}$$

From Noether's theorem, we know that each symmetry of the Lagrangian corresponds to a conserved current. For a global phase shift symmetry $\phi \to \phi + \delta\phi$, the Noether current (or current density) associated with the system's Lagrangian density is given by

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \ , \tag{2.10}$$

which, naturally, satisfies the continuity equation

$$\partial_\mu J^\mu = 0 \ . \tag{2.11}$$

From this, we can identify a conserved charge

$$N = \int_V d\mathbf{x} \, J^0 \ , \tag{2.12}$$

which may represent the total particle number N within the volume V.

The invariance of the Lagrangian under spacetime translations gives rise to another conserved quantity - the canonical energy-momentum tensor of the theory. Formally expressed as

$$\mathcal{T}^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \ , \tag{2.13}$$

which, in accordance with (2.3), obeys

$$\partial_\mu\mathcal{T}^{\mu\nu} = 0 \ . \tag{2.14}$$

We recall from classical mechanics the definition of the canonical momentum π corresponding to a generalized coordinate (ϕ in this case)

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \ . \tag{2.15}$$

We may then obtain the Hamiltonian density from a Legendre transformation as

$$\mathcal{H} = \pi\,\partial^0\phi - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - \mathcal{L} \ . \tag{2.16}$$

This is precisely the (00) component of $\mathcal{T}^{\mu\nu}$. Integrating over three-space, we obtain the system's Hamiltonian

$$H = \int_V d\mathbf{x}\,\mathcal{H} = \int_V d\mathbf{x}\,\mathcal{T}^{00} \ . \tag{2.17}$$

Its value is the energy and it is a conserved quantity. The corresponding spacelike components ($i = 1, 2, 3$)

$$K^i = \int_V d\mathbf{x}\,\mathcal{K}^i = \int_V d\mathbf{x}\,\mathcal{T}^{0i} \tag{2.18}$$

are the momenta, with $\mathbf{K} = (K^1, K^2, K^3)$.

Together, (2.17) and (2.18) form a conserved four-vector, namely the energy-momentum vector

$$K_\mu K^\mu = E^2 - \mathbf{K}^2 = M^2 = \text{invariant} \ . \tag{2.19}$$

For the case of a perfect fluid, however, we recall that the off-diagonal components of the energy-momentum tensor are null. As such, other than the Hamiltonian density (2.16), the only other nonzero components of the canonical energy-momentum tensor (2.13) are

$$\mathcal{T}^{ii} = \frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\partial^i\phi + \mathcal{L} \ . \tag{2.20}$$

As evidenced by (2.6), these spatial (again, $i = 1, 2, 3$) diagonal components correspond to the pressure terms, thus we have

$$P = \int_V d\mathbf{x}\,\mathcal{P} = \int_V d\mathbf{x}\,\mathcal{T}^{ii} \ . \tag{2.21}$$

Having obtained these conserved quantities, we are now in a position to extract from them the thermodynamic quantities that govern the system's behavior by taking their expectation values. The specific meaning of these expectation values, however, might be model-dependent. As such, they will be discussed in detail in the following subsections, where we focus on the specifics of the MIT Bag Model.

2.2.2 The MIT Bag Model

Throughout this work, we focus specifically on quark stars, a particular class of compact stars that are composed entirely of quark matter. The conceptual foundations of such stars were established after recognizing that quarks, the elementary fermions that constitute the nucleons, exhibit asymptotic freedom. That is, under conditions of extreme densities or temperatures, quarks effectively become free of interaction, leading to a phase of matter where nucleons lose their individuality and quarks move within a significantly larger region of space.

Although the possibility of deconfined quark matter existing at the core of other star types has been considered (referred to as hybrid stars), our study will remain confined to stars where quark matter is the dominant phase. It is noteworthy that the universe itself is believed to have passed through the quark matter phase in its early stages.

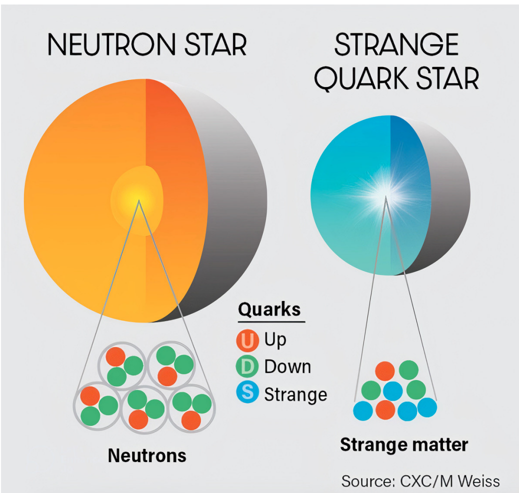

Figure 2.2: Matter in a neutron star (left) vs. a quark star (right).

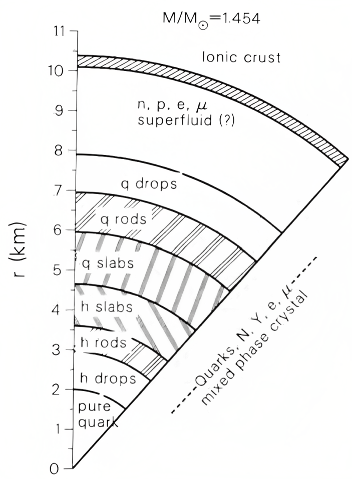

Figure 2.3: Hybrid star cross-section. From [15].

Despite the extensive advances in our understanding of the microscopic behavior of quarks, derived mainly from Quantum Chromodynamics (QCD), there are practical limitations when it comes to obtaining the equation of state of dense nuclear matter. This constraint leads us to the MIT Bag Model, a simpler approach based on the principle of asymptotic freedom, which treats quarks as free particles confined within a so-called bag.

The first description of the bag model appeared in 1967, in P.N. Bogoliubov's work - *Sur un modéle à quarks quasi-indépendants* [16]. In this preliminary description, Bogoliubov conceived a framework with three massless quarks constrained within a vacuum cavity. Despite being innovative, his model suffered from several shortcomings, the most significant of which was the violation of energy-momentum conservation.

The realization of the modern version of the bag model, however, occurred in 1974, at the hands of a group of scientists from the Massachusetts Institute of Technology [17]. Without initial knowledge of Bogoliubov's work, they inadvertently reinvented the bag model, incorporating important enhancements that resolved the issues of the original conception. This refined version, now widely known as the MIT Bag Model, introduced a phenomenological confining pressure, which not only rectified the energy-momentum conservation problem, but also provided a natural mechanism for confinement, thus giving the model a Lorentz-covariant form.

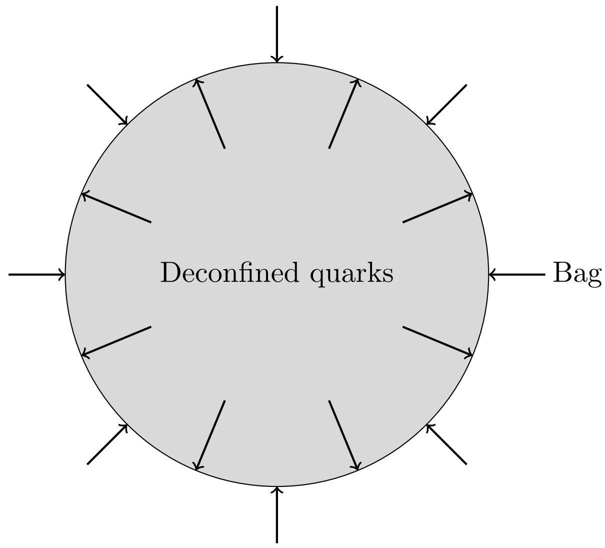

Figure 2.4: Representation of the MIT Bag Model.

In the MIT Bag Model, the bag pressure B serves to confine the free quarks within a particular volume. This pressure counteracts the outward pressure exerted by the quarks and ensures the overall stability of the system. As a result, the energy density ϵ and pressure $\mathscr{P}$ (we shall use this notation henceforth to avert ambiguity) of the quark matter are determined by this confining pressure as well as the kinetic energy of the quarks. In general the thermodynamic quantities of the model can be expressed through the expectation values of the normal ordered conserved quantities we introduced in the previous section:

$$\varepsilon = \frac{1}{(2\pi)^3} \langle :H: \rangle + B \ , \tag{2.22}$$

$$\mathscr{P} = \frac{1}{(2\pi)^3} \frac{\langle :P: \rangle}{3} - B \ , \tag{2.23}$$

$$\rho = \frac{1}{(2\pi)^3} \frac{\langle :N: \rangle}{3} \ , \tag{2.24}$$

where ρ is the baryon number density and the colons signify the normal ordering of the operators. In (2.23) the factor 3 divides $\langle{:}P{:}\rangle$ since the three identical pressure terms are taken into account, in (2.24) it divides $\langle{:}N{:}\rangle$ because there are three quarks in a Baryon. We derive these quantities explicitly in the next subsection. It should be noted that certain references might choose to implement B within the expectation values, so that it may not always appear explicitly as in equations (2.22) and (2.23).

2.2.3 EoS Thermodynamic Quantities

The Dirac Lagrangian Quantities

The MIT Bag Model considers a system of non-interacting (perfect fluid) fermions (quarks). Accordingly, these particles should be described by the free-field Dirac Lagrangian density, given by

$$\mathcal{L} = \sum_f \mathcal{L}_f \ , \tag{2.25}$$

where $\mathcal{L}_f$ is he free-field Dirac Lagrangian density for a quark of a given flavor f, namely

$$\mathcal{L}_f = \overline{\psi}_f \left(i\gamma^\mu \partial_\mu - m_f \right) \psi_f \ . \tag{2.26}$$

Here, m_f denotes the mass of the quark, ψ_f the Dirac field, $\overline{\psi}_f = \psi_f^\dagger \gamma^0$ the Dirac adjoint of the field. The Dirac matrices γ^μ dictate the behavior of spin-1/2 particles.

From (2.13) we see that the canonical energy-momentum tensor associated with this Lagrangian density must be

$$\mathcal{T}_f^{\mu\nu} = \frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \psi_f)} \partial^\nu \psi_f + \frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \overline{\psi}_f)} \partial^\nu \overline{\psi}_f - \eta^{\mu\nu} \mathcal{L}_f \ , \tag{2.27}$$

where

$$\frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \psi_f)} \partial^\nu \psi_f = i \overline{\psi}_f \gamma^\mu \partial^\nu \psi_f \tag{2.28}$$

and

$$\cancelto{0}{\frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \overline{\psi}_f)}} \partial^\nu \overline{\psi}_f = 0 \ . \tag{2.29}$$

We know from (2.6) that the energy-momentum tensor of a perfect fluid is diagonal. Therefore, we may explicitly derive the quantities H, P and N.

The $\mathcal{T}_f^{00}$ component leads to the Hamiltonian density H_f

$$\begin{aligned}
\mathcal{T}_f^{00} = \mathcal{H}_f &= i\overline{\psi}_f \gamma^0 \partial^0 \psi_f - \mathcal{L} \\
&= i\overline{\psi}_f \gamma^0 \partial^0 \psi_f - \overline{\psi}_f \left(i\gamma^\mu \partial_\mu - m_f \right) \psi_f \\
&= \cancel{i\overline{\psi}_f \gamma^0 \partial_0 \psi_f} - \cancel{i\overline{\psi}_f \gamma^0 \partial_0 \psi_f} - \overline{\psi}_f \left(i\gamma^i \partial_i - m_f \right) \psi_f \\
&= -\overline{\psi}_f \left(i\gamma^i \partial_i - m_f \right) \psi_f \ ,
\end{aligned} \tag{2.30}$$

where, considering all quark flavors, we have

$$\mathcal{H} = \sum_f \mathcal{H}_f \ . \tag{2.31}$$

We can then, as in (2.17), write the system's Hamiltonian as

$$H = -\sum_f \int_V d\mathbf{x}\, \overline{\psi}_f \left(i\gamma^i \partial_i - m_f \right) \psi_f = -\sum_f \int_V d\mathbf{x}\, \overline{\psi}_f \left(i\gamma \cdot \nabla - m_f \right) \psi_f \ . \tag{2.32}$$

Similarly, the diagonal spatial components $\mathcal{T}^{ii}$ lead to the $\mathcal{P}_f$ terms

$$\begin{aligned}
\mathcal{T}_f^{ii} = \mathcal{P}_f &= i\overline{\psi}_f \gamma^i \partial^i \psi_f + \mathcal{L} \\
&= -i\overline{\psi}_f \gamma^i \partial_i \psi_f + \overline{\psi}_f \left(i\gamma^\mu \partial_\mu - m_f \right) \psi_f \\
&= \overline{\psi}_f \left(i\gamma^0 \partial_0 - m_f \right) \psi_f
\end{aligned} \tag{2.33}$$

for all quark flavors, we have

$$\mathcal{P} = \sum_f \mathcal{P}_f \ . \tag{2.34}$$

We may therefore write the pressure terms as

$$P = \sum_f \int_V d\mathbf{x}\, \overline{\psi}_f \left(i\gamma^0 \partial_0 - m_f \right) \psi_f \ . \tag{2.35}$$

The third and last quantity considered, N, can be obtained from the Noether current (2.10) - in our case the Dirac current -

$$J_f^\mu = \frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \psi_f)} \delta\psi_f + \cancelto{0}{\frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \overline{\psi}_f)}} \delta\overline{\psi}_f = \frac{\partial \mathcal{L}_f}{\partial(\partial_\mu \psi_f)} \delta\psi_f \ . \tag{2.36}$$

The time component J^0 is then

$$J_f^0 = \frac{\partial \mathcal{L}_f}{\partial(\partial_0 \psi_f)} \delta\psi_f \ . \tag{2.37}$$

Here the derivative of the Lagrangian density is, as we saw already, π_f and the variation

$\delta\psi_f$ is - for a phase shift $\alpha = 1$ in the global symmetry transformation -

$$\delta\psi_f = i\alpha\psi_f = i\psi_f \ . \tag{2.38}$$

Therefore (2.35) gives

$$J_f^0 = i\psi_f^\dagger\psi_f \ . \tag{2.39}$$

The conserved particle number is thus

$$N = \sum_f \int_V d\mathbf{x}\, i\psi_f^\dagger\psi_f \ . \tag{2.40}$$

Second Quantization and Momentum Representation

The quantized Dirac fields may be expressed through Fourier transformations as the integrals in momentum space [18–20]:

$$\psi_f(x) = \frac{1}{(2\pi)^{3/2}} \int \sum_r \left(\frac{m_f}{E_\mathbf{p}}\right)^{(1/2)} \left[c_r(\mathbf{p})u_r(\mathbf{p})e^{-ip_\mu x^\mu} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ip_\mu x^\mu}\right] d\mathbf{p} \ , \tag{2.41}$$

$$\overline{\psi}_f(x) = \frac{1}{(2\pi)^{3/2}} \int \sum_r \left(\frac{m_f}{E_\mathbf{p}}\right)^{(1/2)} \left[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu}\right] d\mathbf{p} \ , \tag{2.42}$$

where the sum is taken over the two spin states and three color charges, and is concisely indexed by r. In the above, $E_\mathbf{p}$ represents the energy of the particle with momentum $\mathbf{p}$, that is,

$$E_\mathbf{p} = \sqrt{m_f^2 + \mathbf{p}^2} \ . \tag{2.43}$$

The operators $c_r(\mathbf{p})$ and $d_r(\mathbf{p})$ are the particle and antiparticle annihilation operators, respectively, while their Hermitian conjugates, $c_r^\dagger(\mathbf{p})$ and $d_r^\dagger(\mathbf{p})$, are their respective creation operators. They obey the following anticommutation relations [18–20]:

$$\left\{c_r(\mathbf{p}), c_{r'}^\dagger(\mathbf{p}')\right\} = \left\{d_r(\mathbf{p}), d_{r'}^\dagger(\mathbf{p}')\right\} = \delta(\mathbf{p}-\mathbf{p}')\delta_{rr'} \ , \tag{2.44}$$

$$\left\{c_r(\mathbf{p}), c_{r'}(\mathbf{p}')\right\} = \left\{c_r^\dagger(\mathbf{p}), c_{r'}^\dagger(\mathbf{p}')\right\} = 0 \ , \tag{2.45}$$

$$\left\{d_r(\mathbf{p}), d_{r'}(\mathbf{p}')\right\} = \left\{d_r^\dagger(\mathbf{p}), d_{r'}^\dagger(\mathbf{p}')\right\} = 0 \ , \tag{2.46}$$

$$\left\{c_r(\mathbf{p}), d_{r'}(\mathbf{p}')\right\} = \left\{c_r^\dagger(\mathbf{p}), d_{r'}^\dagger(\mathbf{p}')\right\} = 0 \ , \tag{2.47}$$

$$\left\{c_r(\mathbf{p}), d_{r'}^\dagger(\mathbf{p}')\right\} = \left\{c_r^\dagger(\mathbf{p}), d_{r'}(\mathbf{p}')\right\} = 0 \ . \tag{2.48}$$

The particle and antiparticle number operators are then respectively defined by

$$N_r(\mathbf{p}) = c_r^\dagger(\mathbf{p})c_r(\mathbf{p}) \ \text{ and } \ \overline{N}_r(\mathbf{p}) = d_r^\dagger(\mathbf{p})d_r(\mathbf{p}) \ . \tag{2.49}$$

The spinors $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$ are respectively the particle and antiparticle solutions of the Dirac equation, that is,

$$(\not{p}_\mu - m)u_r(\mathbf{p}) = (\not{p}_\mu + m)v_r(\mathbf{p}) = 0 \ , \tag{2.50}$$

characterizing spin-1/2 particles, such as quarks. Their conjugates, $\bar{u}_r(\mathbf{p}) = u_r^\dagger(\mathbf{p})\gamma^0$ and $\bar{v}_r(\mathbf{p}) = v_r^\dagger(\mathbf{p})\gamma^0$, are the associated adjoint spinors, in other words,

$$\bar{u}_r(\mathbf{p})(\not{p} - m) = \bar{v}_r(\mathbf{p})(\not{p} + m) = 0 \ . \tag{2.51}$$

We have used the Feynman notation $\not{p} = \gamma^\mu p_\mu$. The spinors satisfy the orthogonality relations [19, 20]:

$$u_r^\dagger(\mathbf{p})u_{r'}(\mathbf{p}) = v_r^\dagger(\mathbf{p})v_{r'}(\mathbf{p}) = \frac{E_\mathbf{p}}{m_f}\delta_{rr'} \ , \tag{2.52}$$

$$u_r^\dagger(\mathbf{p})v_{r'}(-\mathbf{p}) = v_r^\dagger(\mathbf{p})u_{r'}(-\mathbf{p}) = 0 \ . \tag{2.53}$$

The canonically quantized quantities H, P and N may then be obtained through the use of (2.41) and (2.42) in their definitions.

The Hamiltonian (2.32) for a single quark flavor - H_f - (omitting integration limits for brevity) is thus written as

$$\begin{aligned}
H_f = &- \int d\mathbf{x} \left[\frac{1}{(2\pi)^{\frac{3}{2}}} \int \sum_{r=1}^{2} \left(\frac{m_f}{E_\mathbf{p}} \right)^{\frac{1}{2}} \left[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu} \right] d\mathbf{p} \right] \\
&\times (i\gamma \cdot \nabla - m_f) \left[\frac{1}{(2\pi)^{\frac{3}{2}}} \int \sum_{r'=1}^{2} \left(\frac{m_f}{E_{\mathbf{p}'}} \right)^{\frac{1}{2}} \left[c_{r'}(\mathbf{p}')u_{r'}(\mathbf{p}')e^{-ip'_\mu x^\mu} + d_{r'}^\dagger(\mathbf{p}')v_{r'}(\mathbf{p}')e^{ip'_\mu x^\mu} \right] d\mathbf{p}' \right] \\
&= \int d\mathbf{x} \, \frac{1}{(2\pi)^3} \int d\mathbf{p} \, d\mathbf{p}' \sum_{r,r'} \frac{m_f}{(E_\mathbf{p} E_{\mathbf{p}'})^{1/2}} \left[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu} \right] \\
&\times (-i\gamma \cdot \nabla + m_f) \left[c_{r'}(\mathbf{p}')u_{r'}(\mathbf{p}')e^{-ip'_\mu x^\mu} + d_{r'}^\dagger(\mathbf{p}')v_{r'}(\mathbf{p}')e^{ip'_\mu x^\mu} \right] \\
&= \int d\mathbf{x} \, \frac{1}{(2\pi)^3} \int d\mathbf{p} \, d\mathbf{p}' \sum_{r,r'} \frac{m_f}{(E_\mathbf{p} E_{\mathbf{p}'})^{1/2}} \left[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu} \right] \\
&\times \left\{ (\gamma \cdot \mathbf{p}' + m_f) \, c_{r'}(\mathbf{p}')u_{r'}(\mathbf{p}')e^{-ip'_\mu x^\mu} + (-\gamma \cdot \mathbf{p}' + m_f) \, d_{r'}^\dagger(\mathbf{p}')v_{r'}(\mathbf{p}')e^{ip'_\mu x^\mu} \right\} \ .
\end{aligned} \tag{2.54}$$

For the sake of convenience, we define

$$f(\mathbf{p}) \equiv \gamma \cdot \mathbf{p} + m_f \ , \tag{2.55}$$

so that (2.54) may be rewritten as

$$\begin{aligned}
H_f &= \int d\mathbf{x}\, \frac{1}{(2\pi)^3} \int d\mathbf{p}\, d\mathbf{p}' \sum_{r,r'} \frac{m_f}{(E_{\mathbf{p}} E_{\mathbf{p}'})^{1/2}} \left[c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{ip_\mu x^\mu} + d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip_\mu x^\mu} \right] \\
&\times \left\{ f(\mathbf{p}')\, c_{r'}(\mathbf{p}') u_{r'}(\mathbf{p}') e^{-ip'_\mu x^\mu} + f(-\mathbf{p}')\, d_{r'}^\dagger(\mathbf{p}') v_{r'}(\mathbf{p}') e^{ip'_\mu x^\mu} \right\} \\
&= \int d\mathbf{x}\, \frac{1}{(2\pi)^3} \int d\mathbf{p}\, d\mathbf{p}' \sum_{r,r'} \frac{m_f}{(E_{\mathbf{p}} E_{\mathbf{p}'})^{1/2}} \\
&\times \Bigg\{ \left[\bar{u}_r(\mathbf{p}) f(\mathbf{p}')\, u_{r'}(\mathbf{p}') \right] \left[c_r^\dagger(\mathbf{p}) c_{r'}(\mathbf{p}') \right] e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\, e^{i(p^0 - p'^0)x^0} \\
&+ \left[\bar{u}_r(\mathbf{p}) f(-\mathbf{p}')\, v_{r'}(\mathbf{p}') \right] \left[c_r^\dagger(\mathbf{p}) d_{r'}^\dagger(\mathbf{p}') \right] e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\, e^{i(p^0 + p'^0)x^0} \\
&+ \left[\bar{v}_r(\mathbf{p}) f(\mathbf{p}')\, u_{r'}(\mathbf{p}') \right] \left[d_r(\mathbf{p}) c_{r'}(\mathbf{p}') \right] e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\, e^{-i(p^0 + p'^0)x^0} \\
&+ \left[\bar{v}_r(\mathbf{p}) f(-\mathbf{p}')\, v_{r'}(\mathbf{p}') \right] \left[d_r(\mathbf{p}) d_{r'}^\dagger(\mathbf{p}') \right] e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\, e^{-i(p^0 - p'^0)x^0} \Bigg\} \, .
\end{aligned}$$

We now recall that

$$\int d\mathbf{x}\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}') \, , \tag{2.56}$$

and integrate over $\mathbf{x}$, obtaining

$$\begin{aligned}
H_f &= \int d\mathbf{p}\, d\mathbf{p}' \sum_{r,r'} \frac{m_f}{(E_{\mathbf{p}} E_{\mathbf{p}'})^{1/2}} \\
&\times \Bigg\{ \left[\bar{u}_r(\mathbf{p}) f(\mathbf{p}')\, u_{r'}(\mathbf{p}') \right] \left[c_r^\dagger(\mathbf{p}) c_{r'}(\mathbf{p}') \right] \delta(\mathbf{p}-\mathbf{p}') e^{i(p^0 - p'^0)x^0} \\
&+ \left[\bar{u}_r(\mathbf{p}) f(-\mathbf{p}')\, v_{r'}(\mathbf{p}') \right] \left[c_r^\dagger(\mathbf{p}) d_{r'}^\dagger(\mathbf{p}') \right] \delta(\mathbf{p}+\mathbf{p}') e^{i(p^0 + p'^0)x^0} \\
&+ \left[\bar{v}_r(\mathbf{p}) f(\mathbf{p}')\, u_{r'}(\mathbf{p}') \right] \left[d_r(\mathbf{p}) c_{r'}(\mathbf{p}') \right] \delta(\mathbf{p}+\mathbf{p}') e^{-i(p^0 + p'^0)x^0} \\
&+ \left[\bar{v}_r(\mathbf{p}) f(-\mathbf{p}')\, v_{r'}(\mathbf{p}') \right] \left[d_r(\mathbf{p}) d_{r'}^\dagger(\mathbf{p}') \right] \delta(\mathbf{p}-\mathbf{p}') e^{-i(p^0 - p'^0)x^0} \Bigg\} \, .
\end{aligned}$$

Now, integrating over $\mathbf{p}'$, we find (using $E_{\mathbf{p}} = E_{-\mathbf{p}}$)

$$\begin{aligned}
H_f &= \int d\mathbf{p} \sum_{r,r'} \frac{m_f}{E_{\mathbf{p}}} \\
&\times \Bigg\{ \underbrace{\left[\bar{u}_r(\mathbf{p}) f(\mathbf{p})\, u_{r'}(\mathbf{p}) \right]}_{\mathbf{I}} \left[c_r^\dagger(\mathbf{p}) c_{r'}(\mathbf{p}) \right] + \underbrace{\left[\bar{u}_r(\mathbf{p}) f(\mathbf{p})\, v_{r'}(-\mathbf{p}) \right]}_{\mathbf{II}} \left[c_r^\dagger(\mathbf{p}) d_{r'}^\dagger(-\mathbf{p}) \right] e^{2ip^0x^0} \\
&+ \underbrace{\left[\bar{v}_r(\mathbf{p}) f(-\mathbf{p})\, u_{r'}(-\mathbf{p}) \right]}_{\mathbf{III}} \left[d_r(\mathbf{p}) c_{r'}(-\mathbf{p}) \right] e^{-2ip^0x^0} + \underbrace{\left[\bar{v}_r(\mathbf{p}) f(-\mathbf{p})\, v_{r'}(\mathbf{p}) \right]}_{\mathbf{IV}} \left[d_r(\mathbf{p}) d_{r'}^\dagger(\mathbf{p}) \right] \Bigg\} \, .
\end{aligned}$$

We now analyze terms I-IV, we recall the spinor equations (2.50)-(2.51), the orthogonality relations (2.52)-(2.53) and definition (2.55) and write:

I-

$$\begin{aligned}\left[\bar{u}_r(\mathbf{p}) f\left(\mathbf{p}\right) u_{r'}(\mathbf{p})\right] &= \left[\bar{u}_r(\mathbf{p})\left(-\not{p}+\gamma^0 p_0 + m_f\right) u_{r'}(\mathbf{p})\right] \\ &= -\left[\underbrace{\bar{u}_r(\mathbf{p})\left(\not{p}-m_f\right)}_{=0} u_{r'}(\mathbf{p})\right] + p_0\left[u_r^\dagger(\mathbf{p}) u_{r'}(\mathbf{p})\right] \\ &= E_{\mathbf{p}}\left[\frac{E_{\mathbf{p}}}{m_f}\delta_{rr'}\right] \\ &= \frac{E_{\mathbf{p}}^2}{m_f}\delta_{rr'} \ ,\end{aligned} \tag{2.57}$$

II-

$$\begin{aligned}\left[\bar{u}_r(\mathbf{p}) f\left(\mathbf{p}\right) v_{r'}(-\mathbf{p})\right] &= \left[\bar{u}_r(\mathbf{p})\left(-\not{p}+\gamma^0 p_0 + m_f\right) v_{r'}(-\mathbf{p})\right] \\ &= -\left[\underbrace{\bar{u}_r(\mathbf{p})\left(\not{p}-m_f\right)}_{=0} v_{r'}(-\mathbf{p})\right] + p_0\underbrace{\left[u_r^\dagger(\mathbf{p}) v_{r'}(-\mathbf{p})\right]}_{=0} \\ &= 0 \ ,\end{aligned} \tag{2.58}$$

III-

$$\begin{aligned}\left[\bar{v}_r(\mathbf{p}) f\left(-\mathbf{p}\right) u_{r'}(-\mathbf{p})\right] &= \left[\bar{v}_r(\mathbf{p})\left(\not{p}-\gamma^0 p_0 + m_f\right) u_{r'}(-\mathbf{p})\right] \\ &= \left[\underbrace{\bar{v}_r(\mathbf{p})\left(\not{p}+m_f\right)}_{=0} u_{r'}(-\mathbf{p})\right] - p_0\underbrace{\left[v_r^\dagger(\mathbf{p}) u_{r'}(-\mathbf{p})\right]}_{=0} \\ &= 0 \ ,\end{aligned} \tag{2.59}$$

IV-

$$\begin{aligned}\left[\bar{v}_r(\mathbf{p}) f\left(-\mathbf{p}\right) v_{r'}(\mathbf{p})\right] &= \left[\bar{v}_r(\mathbf{p})\left(\not{p}-\gamma^0 p_0 + m_f\right) v_{r'}(\mathbf{p})\right] \\ &= \left[\underbrace{\bar{v}_r(\mathbf{p})\left(\not{p}+m_f\right)}_{=0} v_{r'}(\mathbf{p})\right] - p_0\left[v_r^\dagger(\mathbf{p}) v_{r'}(\mathbf{p})\right] \\ &= -E_{\mathbf{p}}\left[\frac{E_{\mathbf{p}}}{m_f}\delta_{rr'}\right] \\ &= -\frac{E_{\mathbf{p}}^2}{m_f}\delta_{rr'} \ .\end{aligned} \tag{2.60}$$

Taking (2.57)-(2.60) into account, we rewrite H_f as

$$
\begin{aligned}
H_f &= \int d\mathbf{p} \sum_{r,r'} \frac{m_f}{E_{\mathbf{p}}} \left\{ \frac{E_{\mathbf{p}}^2}{m_f} \delta_{rr'} \Big[c_r^\dagger(\mathbf{p}) c_{r'}(\mathbf{p}) \Big] - \frac{E_{\mathbf{p}}^2}{m_f} \delta_{rr'} \Big[d_r(\mathbf{p}) d_{r'}^\dagger(\mathbf{p}) \Big] \right\} \\
&= \int d\mathbf{p} \sum_r E_{\mathbf{p}} \Big[c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) - d_r(\mathbf{p}) d_r^\dagger(\mathbf{p}) \Big] \ . \qquad (2.61)
\end{aligned}
$$

We must now impose normal ordering to H_f, from definitions (2.49),

$$
{:}c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}){:} = c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) = N_r(\mathbf{p}) \ , \qquad (2.62)
$$

$$
{:}d_r(\mathbf{p}) d_r^\dagger(\mathbf{p}){:} = -d_r^\dagger(\mathbf{p}) d_r(\mathbf{p}) = -\overline{N}_r(\mathbf{p}) \ , \qquad (2.63)
$$

so that (2.61) becomes

$$
{:}H_f{:} = \int d\mathbf{p} \sum_r E_{\mathbf{p}} \Big[N_r(\mathbf{p}) + \overline{N}_r(\mathbf{p}) \Big] \ . \qquad (2.64)
$$

Now, for all quarks flavors, we write

$$
{:}H{:} = \sum_f \int d\mathbf{p} \sum_r E_{\mathbf{p}} \Big[N_r(\mathbf{p}) + \overline{N}_r(\mathbf{p}) \Big] \ . \qquad (2.65)
$$

Finally, we take the expectation value of (2.65), we first note (appendix A) that

$$
\sum_r \langle N_r(\mathbf{p}) \rangle = \gamma_f \, n(\mathbf{p}, \mu_f) \quad \text{and} \quad \sum_r \langle \overline{N}_r(\mathbf{p}) \rangle = \gamma_f \, n(\mathbf{p}, -\mu_f) \ , \qquad (2.66)
$$

where $\gamma_f = 2_{\text{spin}} \times 3_{\text{color}}$ is the flavor degeneracy, μ_f is the chemical potential and $n(\mathbf{p}, \pm\mu_f)$ are the Fermi-Dirac distributions for a given temperature T

$$
n(\mathbf{p}, \pm\mu_f) = \left[e^{\frac{E_{\mathbf{p}} \mp \mu_f}{T}} + 1 \right]^{-1} \ . \qquad (2.67)
$$

We obtain (now with explicit integration interval in spherical coordinates)

$$
\langle {:}H{:} \rangle = \sum_f \gamma_f \int_0^\infty 4\pi E_p \Big[n(p, \mu_f) + n(p, -\mu_f) \Big] p^2 dp \ . \qquad (2.68)
$$

An analogous and equally laborious procedure leads us from (2.35) and (2.40) respectively into the other two canonically quantized quantities [15]:

$$\langle :P: \rangle = \sum_f \gamma_f \int_0^\infty 4\pi E_p^{-1} \Big[n(p,\mu_f) + n(p,-\mu_f) \Big] p^4 dp \ , \tag{2.69}$$

$$\langle :N: \rangle = \sum_f \gamma_f \int_0^\infty 4\pi \Big[n(p,\mu_f) - n(p,-\mu_f) \Big] p^2 dp \ . \tag{2.70}$$

Having determined quantities (2.68)-(2.70), we can, at last, insert them in relations (2.22)-(2.24) respectively, whence we obtain the general MIT Bag Model EoS (in its thermodynamic quantities):

$$\varepsilon = \sum_f \frac{\gamma_f}{2\pi^2} \int_0^\infty E_p \Big[n(p,\mu_f) + n(p,-\mu_f) \Big] p^2 dp + B \ , \tag{2.71}$$

$$\mathscr{P} = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^\infty E_p^{-1} \Big[n(p,\mu_f) + n(p,-\mu_f) \Big] p^4 dp - B \ , \tag{2.72}$$

$$\rho = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^\infty \Big[n(p,\mu_f) - n(p,-\mu_f) \Big] p^2 dp \ . \tag{2.73}$$

These equations describe the thermodynamic properties of a static compact star - more specifically those comprised of non-interacting deconfined quark matter. When associated with the TOV equations, they provide a comprehensive theoretical framework for understanding the structure and dynamics of such stars. They are of fundamental relevance and will be used throughout the scope of this work.

2.2.4 The EoS in the Zero Temperature Limit

It is possible to derive an analytic solution for (2.71)-(2.73) in the particular case of zero temperature. This choice is physically justified if we are to model a compact start in which the nucleons are believed to be dissolved into quarks by the high pressure in the interior of the star: temperatures of neutron stars, shortly after birth, fall into the keV region - which is negligible on the nuclear scale [15]. For the aforementioned reasons, the analytic solutions we shall find in our present study will all be obtained within this assumption.

In the limit $T \to 0$, the Fermi-Dirac distributions become step functions at energy

$$\sqrt{m_f^2 + p^2} = E_p = \mu_f = \sqrt{m_f^2 + p_f^2} \ , \tag{2.74}$$

where p_f is the Fermi momentum. In other words, we have

$$n(p, \pm\mu_f) \to \theta(p \mp p_f) \ . \tag{2.75}$$

This greatly simplifies our calculations and we may rewrite quantities (2.71)-(2.73) as

$$\varepsilon = \sum_f \frac{\gamma_f}{2\pi^2} \underbrace{\int_0^{p_f} \sqrt{m_f^2 + p^2}\, p^2 dp}_{I_1} + B \ , \tag{2.76}$$

$$\mathscr{P} = \sum_f \frac{\gamma_f}{6\pi^2} \underbrace{\int_0^{p_f} \frac{p^4}{\sqrt{m_f^2 + p^2}} dp}_{I_2} - B \ , \tag{2.77}$$

$$\rho = \sum_f \frac{\gamma_f}{6\pi^2} \underbrace{\int_0^{p_f} p^2 dp}_{I_3} \ . \tag{2.78}$$

The standard integrals I_1, I_2 and I_3 are given by [21]:

$$I_1 = \frac{1}{8}\Big[p_f E_{p_f}\left(2p_f^2 + m_f^2\right) - m_f^4 \omega_f\Big] \ , \tag{2.79}$$

$$I_2 = \frac{1}{8}\Big[p_f E_{p_f}\left(2p_f^2 - 3m_f^2\right) + 3m_f^4 \omega_f\Big] \ , \tag{2.80}$$

$$I_3 = \frac{1}{3} p_f^3 \ , \tag{2.81}$$

where we have used, for brevity, $\omega_f = \ln\left(\frac{p_f + E_{p_f}}{m_f}\right)$ and $E_{p_f} = \sqrt{m_f^2 + p_f^2}$. Expressions (2.76)-(2.78) therefore become

$$\varepsilon = \sum_f \frac{\gamma_f}{2\pi^2} \frac{1}{8}\Big[p_f E_{p_f}\left(2p_f^2 + m_f^2\right) - m_f^4 \omega_f\Big] + B \ , \tag{2.82}$$

$$\mathscr{P} = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{8}\Big[p_f E_{p_f}\left(2p_f^2 - 3m_f^2\right) + 3m_f^4 \omega_f\Big] - B \ , \tag{2.83}$$

$$\rho = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{3} p_f^3 \ . \tag{2.84}$$

These equations represent the thermodynamic quantities of the MIT Bag Model at zero temperature. They will serve as the baseline against which the novel analytic expressions derived in this study will be juxtaposed and analyzed.

Massless Non-interacting Quarks at Zero Temperature

Equations (2.82)-(2.84) may be further simplified if we additionally restrict ourselves to the zero mass limit approximation for quarks. This simplification implies

$$E_{p_f} = \mu_f = p_f \ . \tag{2.85}$$

Thus, the thermodynamic quantities are succinctly reduced to

$$\varepsilon = \sum_f \frac{\gamma_f}{2\pi^2} \frac{1}{4} \mu_f^4 + B \ , \tag{2.86}$$

$$\mathscr{P} = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{4} \mu_f^4 - B \ , \tag{2.87}$$

$$\rho = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{3} \mu_f^3 \ . \tag{2.88}$$

Equations (2.86) and (2.87) can be concisely written as

$$\mathscr{P} = \frac{1}{3} (\varepsilon - 4B) \ , \tag{2.89}$$

where one can observe that the external pressure exerted on the quark-filled bag vanishes for $\varepsilon = 4B$. The mass contained inside the bag of radius r is given by (2.8)

$$M(r) = \frac{4}{3} \pi \varepsilon r^3 \ , \tag{2.90}$$

which corresponds to the generic mass-radius relation of self-bound matter [22].

3. The Effective Kempf Formalism

In this chapter, we explore the concept of a Generalized Uncertainty Principle (GUP) as a mechanism for introducing minimal length into spacetime descriptions. We subsequently focus on the Kempf formalism, a specific instance of the GUP predicated on a deformed Heisenberg algebra. Finally, we study an effective model of the Kempf formalism for semiclassical limits and some of its existing applications to the fields of thermodynamics and astrophysics are exemplified. For the sake of clarity, we let $\hbar$ and G appear explicitly throughout this chapter.

3.1 The Generalized Uncertainty Principle

A Generalized Uncertainty Principle (GUP) is a proposed modification to the standard Heisenberg Uncertainty Principle

$$\Delta x \Delta p \geq \frac{\hbar}{2} \ . \tag{3.1}$$

While the Heisenberg Uncertainty Principle establishes a lower bound to the product of the uncertainties in position (Δx) and momentum (Δp) of a particle, the GUP introduces additional terms which become especially significant at scales close to the Planck length. Mathematically, the GUP may be generally expressed as

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \alpha(\Delta x)^2 + \beta(\Delta p)^2 + \gamma\right) \ , \tag{3.2}$$

where α, β and γ are positive parameters that induce deviations from conventional Quantum Mechanics.

3.1.1 Gravitational Motivations for a GUP

In field theories that do not involve gravitation, quantization is performed by imposing a quantum structure (e.g., non-trivial commutation relations) onto the classical theory. However, to quantize gravity, we expect that at distances of the order of the Planck length the concept of spacetime itself might need a radical revision. As such, one cannot exclude *a priori* that a quantum theory of gravity will require a modification of fundamental principles of Quantum Mechanics.

The Heisenberg Microscope with Classical Gravity

We first recall the Heisenberg microscope thought experiment, whence the Uncertainty Principle was first derived [23]. We consider a photon with λ wavelength moving in an x direction, the photon then scatters on a particle whose position on the x-axis we want to measure. The scattered photon that reaches the lens of the microscope must lie within an angle θ to produce an image so that we can measure the position of the particle.

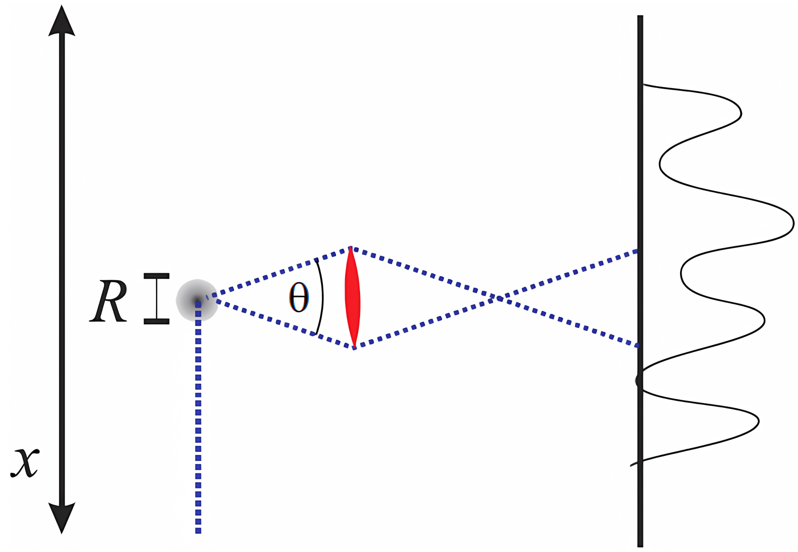

Figure 3.1: The Heisenberg microscope thought experiment. The photon moves along the x-axis and scatters off a probe within an interaction region of radius R, it is then detected by a microscope with opening angle θ. Figure taken from [1].

From classical optics, we know that the minimal resolution Δx is inversely proportional to the extension of the microscope's lens and proportional to the photon's wavelength, thus

$$\Delta x \approx \frac{\lambda}{\sin\theta}\ . \tag{3.3}$$

However, the measuring photon recoils when it scatters and transfers a momentum to the particle. As such, photons of higher energy (smaller wavelength) will interfere more in the particle's position. The the photon's linear momentum in the x direction is

$$\Delta p \approx \frac{h}{\lambda}\sin\theta\ . \tag{3.4}$$

Together, (3.3) and (3.4) result in the uncertainty

$$\Delta x \Delta p \approx h \gtrapprox \frac{\hbar}{2}. \tag{3.5}$$

We know today that Heisenberg's uncertainty is not just a peculiarity of a measurement method, but rather a fundamental property of the quantum nature of matter.

We may first estimate the effects of gravity in a very rough and heuristic way as in [4, 24]. We consider the Newtonian gravitational theory and assume that the photon behaves as a classical particle with an effective mass equal to its energy.

The time interval between the interaction of the photon with the particle and subsequent measurement is at least on the order of the time τ in which the photon travels the distance R, therefore $\tau \approx R$. The photon possesses an energy that exerts a gravitational pull on the particle whose position we wish to measure, so that the particle experiences a gravitational acceleration a of order

$$a \approx \frac{h}{\lambda} \frac{G}{R^2} \ . \tag{3.6}$$

Assuming that the particle is non-relativistic and much slower than the photon, the acceleration lasts about the duration the photon is in the region of strong interaction. From this we may write the particle's velocity as $v \approx aR$, or

$$v \approx \frac{h}{\lambda} \frac{G}{R} \ . \tag{3.7}$$

The particle then traverses a distance $L \approx v\tau$, or

$$L \approx \frac{h}{\lambda} G \ . \tag{3.8}$$

The direction of the photon is unknown within the angle θ, so that the direction of the acceleration and of the motion of the particle is also unknown. Projection onto the x-axis then yields

$$\Delta x \approx \frac{h}{\lambda} G \sin\theta \ . \tag{3.9}$$

We see, however, that we can substitute (3.4) in (3.9) and write

$$\Delta x \approx G \Delta p \ , \tag{3.10}$$

this implies the gravitational uncertainty

$$\Delta x \Delta p \approx G (\Delta p)^2 \ . \tag{3.11}$$

Assuming that the Heisenberg uncertainty (3.5) and the gravitational uncertainty (3.11) add linearly, we find

$$\Delta x \Delta p \gtrapprox \frac{\hbar}{2} \left(1 + \frac{2G}{\hbar} (\Delta p)^2 \right) \ . \tag{3.12}$$

Measurement of the Area of the Horizon of a Black Hole in Quantum Gravity

We now consider a thought experiment analogous to the Heisenberg microscope; however, this time we want to measure not the position of a particle, but the area of a non-rotating charged (i.e., Reissner-Nordström) black hole's horizon, as was done in [25].

For a Reissner-Nordström black hole of mass M and charge Q [26], the apparent horizon is, in Boyer–Lindquist coordinates, located at

$$R_h = GM \left[1 + \left(1 - \frac{Q^2}{GM^2} \right)^{\frac{1}{2}} \right] \ . \tag{3.13}$$

In conventional General Relativity an observer has no direct means to measure the the black hole's apparent horizon, as it emits no signal. From an operational point of view, in General Relativity, (3.13) must be treated rather as a definition of R_h than as an experimentally testable prediction [25]. In a quantum theory of gravitation, however, Hawking radiation permits an observer at infinity to receive a signal coming from the apparent horizon and thus perform, at least on a conceptual level, a direct measurement of its area.

The Heisenberg microscope-type experiment is the following: we consider an extremal ($Q^2 = GM^2$) Reissner-Nordström black hole, which has zero temperature. A photon of wavelength λ is sent from infinity and absorbed by the black hole, so that after absorption the black hole has mass $M + \Delta M$, with

$$\Delta M \approx \frac{h}{\lambda} \tag{3.14}$$

and is no longer extremal. It is expected that it will decay back to the extremal state. We consider the situation in which a single photon of wavelength λ is emitted back. The microscope then detects the emitted photon.

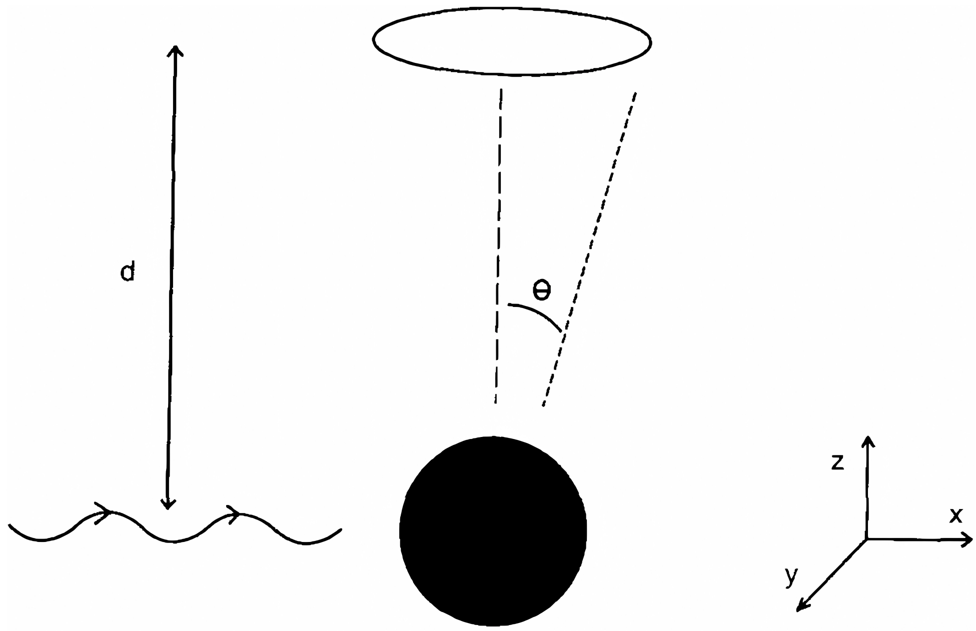

Figure 3.2: The described thought experiment. A photon moves along the x-axis and is absorbed by an extremal black hole; a microscope then detects the induced Hawking radiation, at a distance d along the z-axis; θ is the angular opening of the microscope. The projection of the black hole onto the (x, y)-plane is circle whose radius is measured in the experiment. Figure taken from [25].

For the same reasons as the usual Heisenberg microscope, uncertainties (3.3) and (3.4) are identically valid here, and so is the Heisenberg uncertainty (3.5).

Additionally, however, the mass of the black hole changes from $M + \Delta M$ to M during the emission process, and the radius of the horizon varies accordingly. At the moment of the measure, when the photon is emitted by the black hole, the measured quantity changes discontinuously. It makes no sense to ask whether the information carried by the detected photon refers to the black hole immediately before or after emission, or to any moment in between. The present uncertainty must be considered intrinsic to the measurement. This corresponds to a second - gravitational - source of error for R_h. For a Schwarzschild black hole, this uncertainty is

$$\Delta x \approx 2G\Delta M \ , \tag{3.15}$$

for a general Reissner-Nordström black hole it is

$$\Delta x \approx G\Delta M + \sqrt{(GM + G\Delta M) - GQ^2} - \sqrt{(GM)^2 - GQ^2} \gtrapprox 2G\Delta M \ . \tag{3.16}$$

From relations (3.14) and (3.4) we then obtain

$$\Delta x \approx G\frac{h}{\lambda} \gtrapprox \text{const.}\, G\Delta p \ , \tag{3.17}$$

which implies the uncertainty

$$\Delta x \Delta p \gtrapprox \text{const.}\, G\left(\Delta p\right)^2 \ , \tag{3.18}$$

where the relative constant cannot be determined in a model-independent manner [25].

We now add the gravitational uncertainty (3.18) and the Heisenberg uncertainty (3.5) linearly and find

$$\Delta x \Delta p \gtrapprox \frac{\hbar}{2}\left(1 + \text{const.}\, \frac{2G}{\hbar}(\Delta p)^2\right) \ . \tag{3.19}$$

We notice that, despite emerging from distinct theories, the GUPs obtained in (3.12) and (3.19) both lead to minimal length scales and have an almost identical form. Remarkably, this GUP form corresponds to the predicted results of String Theory [27, 28]. These results indicate that a minimum length (or maximal spacetime resolution) seemingly emerges naturally from any quantum theory of gravity; more examples of thought experiments can be found in [1, 24].

3.1.2 The Kempf Deformed Heisenberg Algebra

We follow an approach similar to [29], which we shall refer to as the Kempf formalism, where a special case of (3.2) - that resembles both (3.12) and (3.19) - is considered. Namely, we have $\alpha = \gamma = 0$ in one dimension, that is,

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta(\Delta p)^2\right) \ . \tag{3.20}$$

The curve of minimal uncertainty is then illustrated by Figure 3.3.

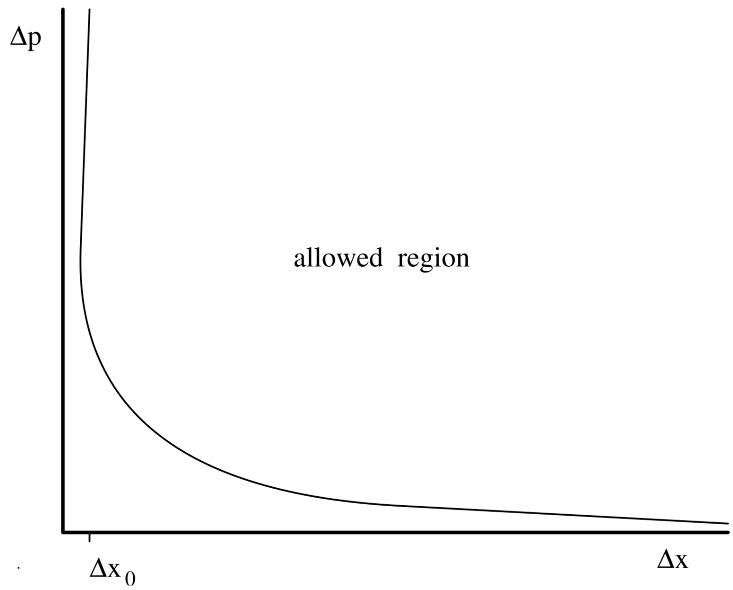

Figure 3.3: Modified uncertainty relation, implying a minimal length $\Delta x_0 > 0$. Figure taken from [29].

As expected, while in ordinary Quantum Mechanics Δx can be made arbitrarily small by letting Δp increase, this no longer holds if the GUP (3.20) is true, due to the $\beta(\Delta p)^2$ term. Thus, we have a maximal spacetime resolution.

In this one-dimensional case, relation (3.20) may be derived from the modified commutation relation:

$$[x, p] = i\hbar\left(1 + \beta p^2\right) \ , \tag{3.21}$$

which, in turn, establishes the minimum length scale

$$\Delta x \geq x_{min} = \hbar\sqrt{\beta} \ . \tag{3.22}$$

For the n-dimensional case, however, the deformed Heisenberg algebra is given by the commutation relations [29]:

$$[x_i, p_j] = i\hbar\,\delta_{ij}\left(1 + \beta\mathbf{p}^2\right) \ , \tag{3.23}$$

$$[p_i, p_j] = 0 \ , \tag{3.24}$$

$$[x_i, x_j] = 2i\hbar\beta\left(x_i p_j - x_j p_i\right) \ , \tag{3.25}$$

where (3.25) characterizes, of course, a noncommutative geometry.

The relations (3.23)-(3.25) do not break rotational symmetry. Indeed, the generators of rotations may still be written in terms of position and momentum operators as

$$L_{ij} = \frac{1}{1 + \beta\mathbf{p}^2}(x_i p_j - x_j p_i) \ , \tag{3.26}$$

which in three dimensions corresponds to

$$L_k = \frac{1}{1+\beta \mathbf{p}^2} \epsilon_{ijk} x_i p_j \ , \tag{3.27}$$

generalizing the definition of angular momentum.

These algebra deformations imply profound modifications to the formalism of Quantum Mechanics. While a continuous momentum space is retained as seen from equation (3.24), the introduction of a minimal length as suggested by equation (3.25) necessitates the adoption of a quasi-position formalism. In this scenario, even elementary models such as the harmonic oscillator manifest considerable complexity and notable deviations when the energy scales approach or exceed $\sqrt{\beta}$ [29].

3.2 The Deformed Phase Space Statistical Method

It is expected that the divergences from conventional Quantum Mechanics due to the Kempf formalism will also manifest, even if in a subtler manner, in semiclassical systems. The nontrivial nature of these changes naturally raises the question of whether there is a simpler model to effectively incorporate this deformed Heisenberg algebra formalism, especially for systems where these modifications might have less drastic influences. In what follows, we adhere to the effective model derived within the framework of [30], in which such effects are implemented through a deformation of phase space volumes.

3.2.1 Deformation of the Poisson Brackets

From the Hamiltonian point of view of Classical Mechanics, the canonical equations of motion may be represented through Poisson brackets where the position coordinates x_i and the conjugate momenta p_j obey the Poisson algebra, namely

$$\{x_i, p_j\} = \delta_{ij} \ , \tag{3.28}$$

$$\{p_i, p_j\} = 0 \ , \tag{3.29}$$

$$\{x_i, x_j\} = 0 \ . \tag{3.30}$$

These relationships can be understood physically as enabling the simultaneous measurement of a particle's position and momentum. They imply the exact determination of coordinates in the corresponding phase space, without uncertainty. Transitioning to Quantum Mechanics from this perspective is direct; the classical dynamical variables are replaced with their corresponding Hermitian operators in Hilbert space and Poisson brackets are substituted with Dirac commutators. Consequently, the initial Poisson algebra transforms into the Heisenberg algebra

$$[x_i, p_j] = i\hbar\,\delta_{ij} \ , \tag{3.31}$$

$$[p_i, p_j] = 0 \ , \tag{3.32}$$

$$[x_i, x_j] = 0 \ , \tag{3.33}$$

which, of course, implies the Heisenberg Uncertainty Principle.

A GUP, which induces a deformation to the Heisenberg algebra, should therefore also induce deformation to its classical limit, the Poisson algebra. We consider general deformations to the commutation relations (3.31)-(3.33), such that

$$[x_i, p_j] = i\hbar\, f_{ij}(x,p) \ \longrightarrow\ \{x_i, p_j\} = f_{ij}(x,p) \ , \tag{3.34}$$

$$[p_i, p_j] = i\hbar\, h_{ij}(x,p) \ \longrightarrow\ \{p_i, p_j\} = h_{ij}(x,p) \ , \tag{3.35}$$

$$[x_i, x_j] = i\hbar\, g_{ij}(x,p) \ \longrightarrow\ \{x_i, x_j\} = g_{ij}(x,p) \ , \tag{3.36}$$

where the deformation functions f_{ij}, g_{ij} and h_{ij} are restricted according to the properties of commutators and brackets: bilinearity, the Leibniz rules and the Jacobi identity. Additionally, according to the Darboux theorem [31], it is always possible to choose auxiliary canonically conjugate variables $x_i(X,P)$ and $p_i(X,P)$ such that they satisfy relations (3.34)-(3.36).

In the particular case of the Kempf formalism. The commutation relations (3.23)-(3.25), induce the deformed Poisson brackets:

$$\{x_i, p_j\} = (1+\beta p^2)\delta_{ij} \ , \tag{3.37}$$

$$\{p_i, p_j\} = 0 \ , \tag{3.38}$$

$$\{x_i, x_j\} = 2\beta\,(p_i x_j - p_j x_i) \ . \tag{3.39}$$

3.2.2 Deformation of Differential Volumes

From the perspective of classical Statistical Mechanics, the coordinates (x_i, p_i) of a particle define a phase space. Since both position and momentum vary with time, the dynamical behavior of the system can be viewed as a continuous trajectory of the phase point in the phase space. In quantum Statistical Mechanics, however, the particle has no well-defined trajectory in the phase space. The Heisenberg Uncertainty Principle (3.1), and therefore the Heisenberg algebra, effectively implies a discretization of the phase space in minimal volumes. A modification of the Heisenberg Uncertainty Principle (i.e., a GUP), therefore, deforms such volumes.

In [30] in particular, the phase space deformation effects are analyzed for the case of a partition function of a quantum system that is then taken to the semiclassical limit. For a non-deformed Heisenberg algebra, we have the transition

$$Z = \sum_n e^{-E_n/T} \quad \longrightarrow \quad Z = \int e^{-H(X,P)/T}\, d^N X\, d^N P \ . \tag{3.40}$$

In the case of a deformed algebra, we know that the expression for the quantum partition function will remain unaltered, since a sum over deformed volumes will have the exact same form as the sum over non-deformed ones. However, this is not the case for a classical partition function. Now we must deal with an integration over phase space that deviates from the original continuous partition function in (3.40) by a Jacobian factor

$$J = \frac{\partial(x,p)}{\partial(X,P)} \tag{3.41}$$

that distorts the phase space, relating the canonical variables of the non-deformed algebra (which we called X and P) to the ones of the deformed algebra x and p. Namely, we have

$$Z = \sum_n e^{-E_n/T} \quad \longrightarrow \quad Z = \int e^{-H(x,p)/T}\, \frac{d^N x\, d^N p}{J} \ . \tag{3.42}$$

The Jacobian of the transformation between non-deformed and deformed algebras can be written purely as combinations of deformed Poisson brackets (3.34)-(3.36), and, in the particular case of the Kempf deformed algebra of a $2N$-dimensional phase space, we have [30, 32]

$$J = \prod_{i=1}^{N} \{x_i, p_i\} = \left(1+\beta p^2\right)^N \ . \tag{3.43}$$

This is an important result (which is demonstrated up to $N = 3$ in Appendix B) that gives us the possibility to calculate (for general deformations) the continuous function without introducing canonically conjugated auxiliary variables.

In other words, we may effectively incorporate the effects of a GUP into semiclassical $2N$-dimensional systems by applying the following transformation to their non-deformed counterparts:

$$d^N x\, d^N p \quad \longrightarrow \quad \frac{d^N x\, d^N p}{(1+\beta p^2)^N} \ , \tag{3.44}$$

which represents a distortion in the differential volumes of phase space and can be shown to be invariant under time evolution from the Liouville theorem [28, 32]. The effective formalism described here has been derived equivalently as a deformation of the Planck constant h in [33, 34].

The weight factor in (3.44) effectively cuts off integrations over p beyond $p = \frac{1}{\sqrt{\beta}}$. Indeed, this can be seen in the plot for the factor of the three-dimensional case, $(1+\beta p^2)^{-3}$, in Figure 3.4.

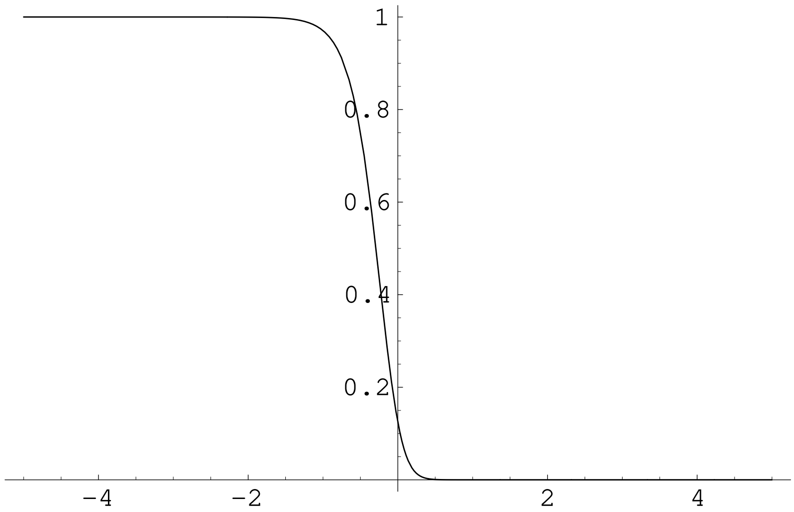

Figure 3.4: Plot of $[1+\beta p^2]^{-3}$ (y-axis) vs. $[\log_{10}\sqrt{\beta}p]$ (x-axis). Figure taken from [32].

Through deformations of the densities of state for ideal non-relativistic and ultra-relativistic gases, [35, 36] derive corrections for the usual thermodynamic relations. Additionally [35], specifically within the context of compact star configurations, applies such gas models to the Newtonian gravity-pressure balance equation, [37] does the same but applies it to the TOV equations instead, both suggest small corrections to the Chandrasekhar limit.

4. The Coherent States Coordinate Approach

In the present chapter, we introduce a noncommutative geometry model based on the coherent states formalism of Quantum Mechanics. While a modification of the Dirac commutation relations induces a minimal length, the Uncertainty Principle remains unchanged. We first introduce a $2+1$-dimensional version of the formalism consisting of a noncommutative plane and commuting time coordinate. Subsequently, we present a generalized version of this approach for arbitrary even-dimensional spacetimes, which preserves Lorentz invariance.

4.1 Prelude to the Coherent States Approach

For the sake of clarity, we provide a brief introduction to the concepts surrounding the idea of noncommutative geometries and then study the simpler formalism of a noncommutative plane. Since the time coordinate commutes with the other two noncommuting spatial coordinates, the model studied in this section is not Lorentz-invariant.

4.1.1 Motivations for a Consistent Formalism

Although the GUP framework presented in the previous chapter indeed led to a noncommutative geometry, this need not always be the case. That is, while both approaches may induce a minimal length scale, one can both have a GUP in a commutative geometry and a noncommutative geometry where the usual Uncertainty Principle holds. Although formally the Kempf formalism is an instance of noncommutative geometry, literature often reserves this term for theories such as the one presented in the following [1], where the only modified Dirac commutation relation is (3.33).

The fundamental idea of noncommutative geometries is that, upon quantization, spacetime coordinates turn into noncommuting Hermitian operators. The simplest general form the commutators of such operators assume is of the kind

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \ , \tag{4.1}$$

where $\theta^{\mu\nu}$ is a real-valued antisymmetric tensor of rank two with dimension of length squared. Similarly to the parameter β seen in the third chapter, the components of $\theta^{\mu\nu}$ determine the minimal length of the theory, imposing a maximal resolution on spacetime. Usual Quantum Mechanics is recovered if $\theta^{\mu\nu} \to 0$.

Attempts to formulate classical or quantum dynamics over a noncommutative spacetime are a non-trivial effort. The most common path to avoid such difficulties is to replace the noncommuting coordinate operators with standard coordinates and introduce a different multiplication rule between ordinary functions, commonly known as the Moyal product, or the $\star$-product. The application of the Moyal product to QFT is then straightforward: take a commutative QFT and simply replace the ordinary product with the Moyal product in the action. The quadratic terms are not modified since the Moyal product adds only surface contributions. Consequently, only the interaction terms are modified.

The simplicity introduced by the Moyal product formulation is, however, deceptive. In spite of the apparent simplicity provided, this formalism induces a series of complications. Specifically, the Moyal product implements the intrinsic nonlocality of the original noncommutative geometry into nomcommutative QFT through nonlocal interactions. Such a nonlocality then emulates the nonexistence of points of noncommutative geometry. However, in the particular case of the Moyal product, these vertices may present technical and conceptual issues.

From a technical perspective, the only way to perform calculations of measurable quantities is to expand the model in powers of theta. At any finite order in theta the model is virtually a conventional local field theory with additional vertices, thus losing the memory of its original nonlocality. The resulting Feynman amplitudes continue to have the same ultraviolet singularities as in the commutative case. Conceptually, the new vertices induce the said nonplanar graphs, leading to mixing between ultraviolet and infrared divergences (UV/IR mixing). Whether UV/IR mixing is a flaw of theory or a feature of nature is still unclear [38–40].

In other words, a perturbation treatment on the Moyal product QFT is not only unable to get rid of the UV singularities, as originally intended, but introduces a new kind of phenomenon which appears be in conflict with the expectations of renormalization group theory [38, 41]. Therefore, an approach that does not present these issues, or at least demonstrating that they are not intrinsic to noncommutative geometries, is desirable.

4.1.2 The Noncommutative Plane

We follow the ideas and procedures of [41,42], where the simplest instance of noncommutative geometry in what we call the coherent states formalism, namely, the noncommutative plane, was first investigated. For better intelligibility, we denote the noncommutative position operators by upper case letters and the commutative coordinates by lower case letters.

A noncommutative plane is described by space coordinates that satisfy the commutation relations

$$\left[X^i, p_j\right] = i\delta^i_j \ , \tag{4.2}$$

$$\left[p_i, p_j\right] = 0 \ , \tag{4.3}$$

$$\left[X^i, X^j\right] = i\theta\epsilon^{ij} \ , \tag{4.4}$$

where ϵ^{ij} is the two-dimensional Levi-Civita symbol. Therefore, the non-commutative plane is divided into minimal areas of size θ, one can no longer speak in terms of points.

From (4.4), we know that there are no common eigenstates for the noncommutative space coordinates and no coordinate representation can be defined. Consequently, wave functions or fields defined over points can also no longer be defined. Coherent states are the sharpest coordinate states that can be defined for noncommuting coordinates - they are minimal-uncertainty states and enable one to define mean values of coordinate operators.

To apply the coherent state approach, we construct an appropriate set of ladder operators built from the noncommutative plane coordinates. These operators should satisfy the usual commutation rules of the QM creation and annihilation operators. The mean values of any operator over coherent states are commutative quantities.

We thus introduce a set of ladder operators

$$A \equiv \frac{1}{\sqrt{2}}(X^1 + iX^2) \ , \tag{4.5}$$

$$A^\dagger \equiv \frac{1}{\sqrt{2}}(X^1 - iX^2) \ , \tag{4.6}$$

which satisfy the commutation relation

$$\left[A, A^\dagger\right] = \theta \ . \tag{4.7}$$

Here, A and $A^\dagger$ correspond to the coordinate annihilation and creation operators respectively. Coherent states are the eigenstates of such operators. The clear advantage of defining (4.5) and (4.6) is that, unlike for the noncommuting position coordinates, we have eigenstates such that

$$A\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle \ , \tag{4.8}$$

$$\left\langle\alpha\right| A^\dagger = \left\langle\alpha\right| \overline{\alpha} \ . \tag{4.9}$$

The normalized coherent states $|\alpha\rangle$ are explicitly given by

$$\left|\alpha\right\rangle \equiv \exp\left(\frac{\overline{\alpha}A - \alpha A^\dagger}{\theta}\right)\left|0\right\rangle = \exp\left(-\frac{\alpha\overline{\alpha}}{2\theta}\right)\exp\left(-\frac{\alpha}{\theta}A^\dagger\right)\left|0\right\rangle \ , \tag{4.10}$$

where the vacuum state is annihilated by A and we have used the Baker-Campbell-Hausdorff formula. Naturally, the states $|\alpha\rangle$ obey the completeness relation

$$\frac{1}{\pi\theta}\int d\alpha\, d\overline{\alpha}\ \left|\alpha\right\rangle\left\langle\alpha\right| = 1 \ . \tag{4.11}$$

The fundamental idea of this approach is to associate to the set of noncommuting coordinates - through their expected values over coherent states - a set of commuting ones, in the same manner that classical and quantum variables are related in Quantum Mechanics. These mean coordinates are

$$\begin{aligned} \langle\alpha| X^1 |\alpha\rangle &= \sqrt{2}\,\langle\alpha| \frac{A+A^\dagger}{2} |\alpha\rangle \\ &= \sqrt{2}\,\frac{\alpha+\overline{\alpha}}{2} \\ &= \sqrt{2}\,\mathrm{Re}(\alpha)\ , \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \langle\alpha| X^2 |\alpha\rangle &= \sqrt{2}\,\langle\alpha| \frac{A-A^\dagger}{2i} |\alpha\rangle \\ &= \sqrt{2}\,\frac{\alpha-\overline{\alpha}}{2i} \\ &= \sqrt{2}\,\mathrm{Im}(\alpha)\ . \end{aligned} \tag{4.13}$$

The two-vector $\mathbf{x} = (\mathrm{Re}(\alpha), \mathrm{Im}(\alpha))$ corresponds to the mean-position of the particle over the noncommutative plane.

We may then associate an ordinary function of mean-position coordinates $F(\alpha)$ or $F(\mathbf{x})$ to the operator-valued functions $F(X^1, X^2)$ through the expectation value

$$F(\alpha) \equiv \langle\alpha| F(X^1, X^2) |\alpha\rangle\ . \tag{4.14}$$

Now, we can provide a definition for the noncommutative Fourier transform, which is given by [41]

$$F(\alpha) = \frac{1}{2\pi} \int d^2p\, \tilde{f}(\mathbf{p})\, \langle\alpha| \exp\left(ip_j X^j\right) |\alpha\rangle\ , \tag{4.15}$$

where the mean-valued term corresponds to the noncommutative plane wave. The expectation value of the noncommutative plane wave function may in turn be rewritten as [41, 42]

$$\langle\alpha| \exp\left(ip_j X^j\right) |\alpha\rangle = \exp\left(-\frac{\theta}{4}\mathbf{p}^2 + i\,\mathbf{p}\cdot\mathbf{x}\right)\ , \tag{4.16}$$

where $\mathbf{p} = (p_1, p_2)$ is the canonical conjugate of $\mathbf{x}$ and can thus be though of as a mean linear momentum.

The Hausdorff decomposition used to obtain (4.16) yields an additional factor in the definition of the plane wave for the noncommutative plane. This is the crucial aspect of the coherent states model, i.e., the noncommutativity is introduced through the modified Fourier transform of ordinary functions, namely

$$F(\mathbf{x}) = \frac{1}{2\pi} \int d^2p\, \tilde{f}(\mathbf{p}) \exp\left(-\frac{\theta}{4}\mathbf{p}^2 + i\,\mathbf{p}\cdot\mathbf{x}\right)\ . \tag{4.17}$$

This modified Fourier transform shows that noncommutativity produces a Gaussian damping factor. In other words, the present approach induced noncommutativity to QFT by replacing ordinary plane waves with Gaussian wave packets.

More specifically, we may introduce fields in the noncommutative plane through the modified versions of their momentum expansions of second quantization. In particular, the fermionic fields are given by

$$\psi_f(\mathbf{x},t) = \frac{1}{2\pi}\int \sum_r \left(\frac{m_f}{E_{\mathbf{p}}}\right)^{\frac{1}{2}} \left[c_r(\mathbf{p})u_r(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}+iE_{\mathbf{p}}t} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-iE_{\mathbf{p}}t}\right] e^{-\frac{\theta}{4}\mathbf{p}^2}d\mathbf{p} \ , \tag{4.18}$$

$$\overline{\psi}_f(\mathbf{x},t) = \frac{1}{2\pi}\int \sum_r \left(\frac{m_f}{E_{\mathbf{p}}}\right)^{\frac{1}{2}} \left[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-iE_{\mathbf{p}}t} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}+iE_{\mathbf{p}}t}\right] e^{-\frac{\theta}{4}\mathbf{p}^2}d\mathbf{p} \ , \tag{4.19}$$

which correspond to the usual 2+1-dimensional momentum representation of conventional QFT modified by a new Gaussian term dependent on the minimal length of the theory. The bosonic fields, in turn, become

$$\phi(\mathbf{x},t) = \frac{1}{2\pi}\int \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}+iE_{\mathbf{p}}t} + b^\dagger(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-iE_{\mathbf{p}}t}\right] e^{-\frac{\theta}{4}\mathbf{p}^2}d\mathbf{p} \ , \tag{4.20}$$

$$\phi^\dagger(\mathbf{x},t) = \frac{1}{2\pi}\int \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a^\dagger(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-iE_{\mathbf{p}}t} + b(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}+iE_{\mathbf{p}}t}\right] e^{-\frac{\theta}{4}\mathbf{p}^2}d\mathbf{p} \ , \tag{4.21}$$

where the operators $a(\mathbf{p})$ and $b(\mathbf{p})$ are respectively the ordinary bosonic particle and antiparticle annihilation operators, $a^\dagger(\mathbf{p})$ and $b^\dagger(\mathbf{p})$ are their respective creation operators.

This formalism elegantly renders QFT UV-finite and cures the singular behavior of Feynman propagators, this can be explicitly seen, for example, from the momentum propagator for a boson of mass m found in [41, 42]:

$$G\left(\mathbf{p}^2\right) = \frac{1}{E_{\mathbf{p}}^2 - \mathbf{p}^2 - m^2}e^{-\frac{\theta}{2}\mathbf{p}^2} \ . \tag{4.22}$$

The propagator (4.22) displays the expected UV cutoff obtained introduced by the noncommutative coordinates. In other words, the noncommutativity of spatial coordinates leads to an exponential cutoff in the Green function at large momenta.

Interestingly, since noncommutativity is introduced only through the new definition of the Fourier transform subsequently used for second quantization, the Lagrangian and Hamiltonian densities of this noncommutative model maintain the same form as those of ordinary QFT. This is particularly useful if we wish to find values derived from the symmetries of these quantities.

4.2 The Even-Dimensional Lorentz-Invariant Approach

We now follow the procedure devised in [38], in which the formalism developed in the last section for the noncommutative plane is generalized for any even-dimensional spaces. This formalism now allows the consideration of, for example, noncommutative Minkowski spacetimes.

In order for a given parameter to be a proper fundamental length, it must be Lorentz-invariant, in other words, it should be invariant under Lorentz transformations in any dimension. This must be the case because, in the absence of Lorentz invariance, one could in principle define a boost that effectively deforms this minimal length to arbitrarily small scales, undermining its very definition. Lorentz invariance requires time to be included among the noncommuting coordinates.

We begin by defining a set with an even number D of Hermitian coordinate operators X^μ (with $\mu = 1, 2, \dots, D$)

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \ , \tag{4.23}$$

where $\theta^{\mu\nu}$ is defined to be Lorentz covariant antisymmetric tensor, with its components corresponding to maximal resolutions of spacetime. The other commutation relations are the same as the ones in usual QM.

We then apply the property that any antisymmetric matrix can be brought, through an appropriate rotation, to the block-diagonal form

$$\theta^{\mu\nu} = \operatorname{diag}\left(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_{D/2}\right) \ , \tag{4.24}$$

where

$$\hat{\theta}_j \equiv \theta_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{4.25}$$

In the case of odd-dimensional spacetimes the last element of (4.24) would be null; as such, there would always be one coordinate that commutes with all others in these spacetimes. Thus, the covariance of (4.24) allows us to represent noncommutativity of all spacetime coordinates as a set of noncommutative planes. That is, the noncommutative spacetime can always be foliated in a way that the noncommutativity is restricted to these individual planes.

Since we are considering even-dimensional spacetimes, we may chose the represent spacetime coordinates with $D/2$ two-vectors $\mathbf{X}_j$ as

$$\begin{aligned} X^\mu &= \left(X^1, X^2, \dots, X^{D-1}, X^D\right) \\ &= \left(\mathbf{X}_1, \dots, \mathbf{X}_{D/2}\right) \ , \end{aligned} \tag{4.26}$$

where $\mathbf{X}_j \equiv (\mathscr{X}_{1j}, \mathscr{X}_{2j})$ are the two-vectors with $(\mathscr{X}_{1j}, \mathscr{X}_{2j})$ coordinates that define the

j-th noncommutative plane satisfying

$$[\mathscr{X}_{1j}, \mathscr{X}_{2j}] = i\,\theta_j \ . \tag{4.27}$$

We now construct the appropriate set of ladder operators as in the previous plane formalism. In the block-diagonal basis, we define the creation and annihilation operators for the i

$$A_j \equiv \frac{1}{\sqrt{2}}\left(\mathscr{X}_{1j} + i\mathscr{X}_{2j}\right) \ , \tag{4.28}$$

$$A_j^\dagger \equiv \frac{1}{\sqrt{2}}\left(\mathscr{X}_{1j} - i\mathscr{X}_{2j}\right) \ , \tag{4.29}$$

which obey the commutation relation

$$\left[A_j, A_k^\dagger\right] = \delta_{jk}\,\theta_j \ . \tag{4.30}$$

The normalized coherent states for the ladder operators are explicitly given by

$$|\alpha\rangle \equiv \prod_j \exp\left(\frac{\overline{\alpha}_j A_j - \alpha_j A_j^\dagger}{\theta_j}\right)|0\rangle \ . \tag{4.31}$$

We then proceed as in the previous section by associating commutative coordinated to the noncommuting ones through their expectation values over coherent states. We have

$$\begin{aligned}\langle\alpha|\,\mathscr{X}_{1j}\,|\alpha\rangle &= \sqrt{2}\,\langle\alpha|\,\frac{A_j + A_j^\dagger}{2}\,|\alpha\rangle \\ &= \sqrt{2}\,\frac{\alpha_j + \overline{\alpha}_j}{2} \\ &= \sqrt{2}\,\mathrm{Re}(\alpha_j) \ ,\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\langle\alpha|\,\mathscr{X}_{2j}\,|\alpha\rangle &= \sqrt{2}\,\langle\alpha|\,\frac{A_j - A_j^\dagger}{2i}\,|\alpha\rangle \\ &= \sqrt{2}\,\frac{\alpha_j - \overline{\alpha}_j}{2i} \\ &= \sqrt{2}\,\mathrm{Im}(\alpha_j) \ .\end{aligned} \tag{4.33}$$

We write $\mathbf{x}_j = (\mathrm{Re}(\alpha_j), \mathrm{Im}(\alpha_j))$ for the mean-position two-vector in the j-th noncommutative plane.

Now, we may associate the ordinary function $F(\alpha_j)$ or $F(\mathbf{x}_j)$ to the operator-valued function $F(\mathbf{X}_j)$ through the expectation value

$$F(\alpha_j) \equiv \langle\alpha| \, F(\mathbf{X}_j) \, |\alpha\rangle \ . \tag{4.34}$$

The noncommutative Fourier transform is then defined by [38]

$$F\left(\alpha\right) = \int \prod_{j=1}^{D/2} \left(\frac{d\mathbf{p}_j}{2\pi}\right) \tilde{f}\left(\mathbf{p}_1 \ , \dots , \mathbf{p}_{D/2}\right) \langle\alpha| \exp\left[i \sum_{j=1}^{D/2} \mathbf{p}_j \cdot \mathbf{X}_j \right] |\alpha\rangle \ , \tag{4.35}$$

where the $\mathbf{p}_j$ are the momentum two-vector associated with the j-th noncommutative plane coordinates. Thus, the explicit noncommutative plane wave is found, once again through the Hausdorff formula, to be [38]

$$\langle\alpha| \exp\left[i \sum_{j=1}^{D/2} \mathbf{p}_j \cdot \mathbf{X}_j \right] |\alpha\rangle = \exp\left[\sum_{j=1}^{D/2} \left(-\frac{1}{4}\theta_j \mathbf{p}_j^2 + i\,\mathbf{p}_j \cdot \mathbf{x}_j \right) \right] \ . \tag{4.36}$$

We may the rewrite (4.35) as

$$F\left(\mathbf{x}\right) = \int \prod_{j=1}^{D/2} \left(\frac{d\mathbf{p}_j}{2\pi}\right) \tilde{f}\left(\mathbf{p}_1 \ , \dots , \mathbf{p}_{D/2}\right) \exp\left[\sum_{j=1}^{D/2} \left(-\frac{1}{4}\theta_j \mathbf{p}_j^2 + i\,\mathbf{p}_j \cdot \mathbf{x}_j \right) \right] \ , \tag{4.37}$$

where the additional Gaussian damping term smears the plane waves into packets.

This general method then renders QFT UV-finite and, more explicitly, the Feynman scalar propagator in momentum space is similarly found to be [38]:

$$G\left(\mathbf{p}_1, \dots , \mathbf{p}_{D/2}\right) = \frac{1}{(2\pi)^D} \frac{1}{\left[\sum_{j=1}^{D/2} \mathbf{p}_j^2\right] + m^2} \exp\left[-\frac{1}{2} \sum_{j=1}^{D/2} \theta_j \, \mathbf{p}_j^2 \right] \ . \tag{4.38}$$

While the propagator in (4.38) solves the issue of singularities, we see that the exponential term may not always be Lorentz-invariant, since the parameters θ_j are coupled to two-vectors. It can then be shown, as done explicitly in [38], that the requirement of Lorentz-invariance leads to the constraint

$$\theta_j = \theta \ , \tag{4.39}$$

which is intuitively in agreement with the notion that no particular spacetime dimensions should be privileged.

The bosonic propagator in momentum space then becomes the Lorentz-invariant [38]:

$$G\left(p^\mu p_\mu\right) = \frac{1}{(2\pi)^D} \frac{1}{p^\mu p_\mu + m^2} \, e^{-\frac{\theta}{2} p^\mu p_\mu} \ . \tag{4.40}$$

We further note that the formalism developed in this section, in spite of including time among the noncommuting coordinates, has not been found to present any violations of unitarity, as shown in [38] for one-loop calculations.

As in the previous section, the noncommutativity of spacetimes coordinates in this generalized model is introduced only through the new definition of the Fourier transform and the previous conclusions and utilities remain valid, only now in arbitrary even-dimensional systems and in Lorentz-invariant form.

We see, as a natural consequence of the use of mean values to define spacetime positions, that the effect of noncommutativity corresponds to the substitution of Dirac delta distributions by Gaussian distributions. This idea has also been conceptually applied as an effective model of noncommutativity to several gravitational scenarios commonly associated with the presence of singularities.

Specifically in the context of black holes, this effective model has been used, for example, in [43–49]. With significant physical consequences such as the existence of a minimal nonzero mass to which a black hole can shrink, a finite maximum temperature that a black hole can reach before cooling down to absolute zero, the absence of any curvature singularity, etc. A comprehensive review on noncommutative geometry models for black holes (at least up to 2009) can be found in [50].

5. Effective GUP Model EoS

We now apply the effective Kempf GUP formalism described in the third chapter to the EoS of compact stars. Specifically, we induce a deformation on the thermodynamic quantities of the MIT Bag Model that we propose serves as an effective semiclassical description of deconfined quark matter in a space with minimal length. We subsequently investigate the zero temperature limit, for which we find analytical solutions that we further apply to the TOV equations. Finally, we briefly propose and outline a path towards a generalized version of this model here devised which is not restricted to Kempf deformations and permits the inclusion an arbitrary variety of particle species and interaction terms.

5.1 The Kempf-Deformed MIT Bag Model

5.1.1 Deformed EoS Thermodynamic Quantities

First, we recall that the general form of the MIT Bag Model given by (2.22)-(2.23) is dependent on the mean-valued quantities $\langle{:}H{:}\rangle$, $\langle{:}P{:}\rangle$ and $\langle{:}N{:}\rangle$. In order to be explicitly calculated, as we saw in the second chapter of this work, all of these quantities must be obtained from their respective Lagrangian density symmetries in second quantization. This process involves integrating the expectation values of the QFT expressions over the space occupied by the involved particles (in this case noninteracting fermions) and all their possible momentum states (which is also an integral since we are dealing with the large number limit).

Next, we recall, from the third chapter, that the effective consequence of the deformation of a Heisenberg algebra due to a GUP is a deformation of the differential volumes involved in the integration of quantities in the semiclassical limit. In other words and more intuitively, a change to $\Delta x \Delta p$ has inherent implications for $dx\, dp$.

We notice that the thermodynamic quantities of the EoS are obtained from integrations that satisfy the requirements for the application of the effective GUP formalism. These integrations over position and momentum naturally introduce differential volumes that can be distorted through a Jacobian term to coherently implement the effects of the new underlying algebra. We may therefore develop an effective GUP model of the MIT Bag Model.

We explicitly write the expectation value of the normal-ordered free-field Dirac Hamiltonian in second quantization $\langle{:}H{:}\rangle$, similarly to what was done in (2.54):

$$\langle{:}H{:}\rangle = \frac{1}{(2\pi)^3} \iint \sum_{f,r,r'} \frac{\gamma_f\, m_f}{(E_{\mathbf{p}} E_{\mathbf{p}'})^{1/2}} \Big\langle : \Big[c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{ip_\mu x^\mu} + d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip_\mu x^\mu} \Big]$$
$$\times \left(-i\gamma \cdot \nabla + m_f\right) \Big[c_{r'}(\mathbf{p}') u_{r'}(\mathbf{p}') e^{-ip'_\mu x^\mu} + d_{r'}^\dagger(\mathbf{p}') v_{r'}(\mathbf{p}') e^{ip'_\mu x^\mu} \Big] : \Big\rangle \, d^3x\ d^3p\ d^3p'. \quad (5.1)$$

Now, the spatial differential element d^3x may be associated with either d^3p or d^3p' (we arbitrarily choose d^3p) so that we may induce the volume deformation through transformation (3.44) as

$$d^3x\ d^3p \ \longrightarrow\ \frac{d^3x\ d^3p}{(1+\beta p^2)^3}\ . \tag{5.2}$$

We note that there is no ambiguity in our choice here - either association of the spatial element with one of the two momentum ones will yield exactly the same result. Theory guarantees that this must be the case, since p and p' refer to ψ and $\overline{\psi}$, which represent the same particle species and therefore the same set of momentum states. In the end, these integration variables must run over the same momentum space and we must have coherent momentum labels (as is explicitly shown in the second chapter) so that the canonical anticommutation relations (2.44)-(2.48) are satisfied and we have a consistent theory. This is also in agreement with the foundational principle of the uncertainty relation, which stipulates that uncertainties are inherently paired with corresponding coordinate-momentum indices.

Therefore, we write the effective GUP version of (5.1), which we call $\langle{:}H{:}\rangle_\beta$, as

$$\begin{aligned}\langle{:}H{:}\rangle_\beta = \frac{1}{(2\pi)^3}\iint\sum_{f,r,r'}\frac{\gamma_f\, m_f}{(E_{\mathbf{p}}E_{\mathbf{p}'})^{1/2}}\Big\langle{:}\Big[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu}+d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu}\Big]\\ \times(-i\gamma\cdot\nabla+m_f)\Big[c_{r'}(\mathbf{p}')u_{r'}(\mathbf{p}')e^{-ip'_\mu x^\mu}+d_{r'}^\dagger(\mathbf{p}')v_{r'}(\mathbf{p}')e^{ip'_\mu x^\mu}\Big]{:}\Big\rangle\frac{d^3x\ d^3p\ d^3p'}{(1+\beta p^2)^3}\ .\end{aligned} \tag{5.3}$$

This new expression differs from the one found in the second chapter only by the deformation term. We may then perform the integrations over x and p' and simplifications up to (2.68) exactly as we have done in the second chapter, since they are not affected by the new term. We thus find, now in spherical coordinates and with explicit integration limits,

$$\langle{:}H{:}\rangle_\beta = \sum_f\gamma_f\int_0^\infty 4\pi E_p\Big[n(p,\mu_f)+n(p,-\mu_f)\Big]p^2\frac{dp}{(1+\beta p^2)^3}\ . \tag{5.4}$$

Naturally, this same procedure can be applied to $\langle{:}P{:}\rangle$ and $\langle{:}N{:}\rangle$, it is then straightforward that we analogously find:

$$\langle{:}P{:}\rangle_\beta = \sum_f\gamma_f\int_0^\infty 4\pi E_p^{-1}\Big[n(p,\mu_f)+n(p,-\mu_f)\Big]p^4\frac{dp}{(1+\beta p^2)^3}\ , \tag{5.5}$$

$$\langle{:}N{:}\rangle_\beta = \sum_f\gamma_f\int_0^\infty 4\pi\Big[n(p,\mu_f)-n(p,-\mu_f)\Big]p^2\frac{dp}{(1+\beta p^2)^3}\ . \tag{5.6}$$

Having now determined the new deformed quantities (5.4)-(5.6), we can find the modified MIT Bag Model thermodynamic quantities through the deformed version of relations

(2.22)-(2.24), namely

$$\varepsilon_\beta = \frac{1}{(2\pi)^3} \langle :H: \rangle_\beta + B \ , \tag{5.7}$$

$$\mathscr{P}_\beta = \frac{1}{(2\pi)^3} \frac{\langle :P: \rangle_\beta}{3} - B \ , \tag{5.8}$$

$$\rho_\beta = \frac{1}{(2\pi)^3} \frac{\langle :N: \rangle_\beta}{3} \ . \tag{5.9}$$

Substituting (5.4)-(5.6) into (5.7)-(5.9) respectively, we obtain the effective GUP model for the general MIT Bag Model EoS thermodynamic quantities:

$$\varepsilon_\beta = \sum_f \frac{\gamma_f}{2\pi^2} \int_0^\infty E_p \Big[n(p,\mu_f) + n(p,-\mu_f) \Big] p^2 \frac{dp}{\left(1+\beta p^2\right)^3} + B \ , \tag{5.10}$$

$$\mathscr{P}_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^\infty E_p^{-1} \Big[n(p,\mu_f) + n(p,-\mu_f) \Big] p^4 \frac{dp}{\left(1+\beta p^2\right)^3} - B \ , \tag{5.11}$$

$$\rho_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^\infty \Big[n(p,\mu_f) - n(p,-\mu_f) \Big] p^2 \frac{dp}{\left(1+\beta p^2\right)^3} \ . \tag{5.12}$$

These represent an effective description of a bag of deconfined noninteracting quark matter in a space with a minimal length regulated by the parameter β (more specifically, for our choice of units, $x_{min} = \sqrt{\beta}$) induced by a GUP. As expected, the limit $\beta \to 0$ returns us to the conventional quantities (2.71)-(2.73). There are no further specific restrictions imposed over these quantities other than the ones already incorporated by the usual MIT Bag Model.

We may therefore insert any physically reasonable set of parameters in (5.10)-(5.12) and numerically obtain the EoS quantities, and thus, in turn, apply such results to numerically solve the TOV equations. However, in order to obtain greater physical insight on the conceptual changes introduced by this model and to be able to later compare it with existing literature, we chose to turn our attention to analytical solutions. More specifically, we shall focus on the zero temperature limit.

5.1.2 Deformed EoS in the Zero Temperature Limit

Now, we investigate the deformed MIT Bag Model in the $T \to 0$ limit. As we have already seen, the Fermi-Dirac distributions become step functions at energy $E_p = \mu_f$. We have thus the effective GUP model equivalent of the quantities found in the second chapter, we now have:

$$\varepsilon_\beta = \sum_f \frac{\gamma_f}{2\pi^2} \int_0^{p_f} \sqrt{m_f^2 + p^2}\, p^2 \frac{dp}{(1+\beta p^2)^3} + B \;, \tag{5.13}$$

$$\mathscr{P}_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^{p_f} \frac{p^4}{\sqrt{m_f^2 + p^2}} \frac{dp}{(1+\beta p^2)^3} - B \;, \tag{5.14}$$

$$\rho_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^{p_f} p^2 \frac{dp}{(1+\beta p^2)^3} \;, \tag{5.15}$$

where setting $\beta = 0$ returns relations (2.76)-(2.78). All of these can be integrated to find analytical solutions.

In particular, for the sake of illustration, we chose to integrate the simplest expression, the baryon number density (5.15):

$$\rho_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{8\beta^{\frac{3}{2}}} \left[\frac{\beta^{\frac{1}{2}} p_f \left(\beta p_f^2 - 1\right)}{\left(\beta p_f^2 + 1\right)^2} + \tan^{-1}\left(\sqrt{\beta}\, p_f\right) \right] \;. \tag{5.16}$$

We promptly make two relevant observations. First, we see that if we take the limit

$$\lim_{\beta\to 0} \frac{1}{8\beta^{\frac{3}{2}}} \left[\frac{\beta^{\frac{1}{2}} p_f \left(\beta p_f^2 - 1\right)}{\left(\beta p_f^2 + 1\right)^2} + \tan^{-1}\left(\sqrt{\beta}\, p_f\right) \right] = \frac{p_f^3}{3} \;, \tag{5.17}$$

relation (5.16) reduces to

$$\rho_0 = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{3} p_f^3 \;, \tag{5.18}$$

which, as expected, is simply relation (2.84) - this is a requirement for the consistency of the formalism. Second, that if we instead consider the limit where the Fermi momentum p_f goes to infinity, we get

$$\lim_{p_f\to \infty} \frac{1}{8\beta^{\frac{3}{2}}} \left[\frac{\beta^{\frac{1}{2}} p_f \left(\beta p_f^2 - 1\right)}{\left(\beta p_f^2 + 1\right)^2} + \tan^{-1}\left(\sqrt{\beta}\, p_f\right) \right] = \frac{1}{8\beta^{\frac{3}{2}}} \frac{\pi}{2} \;, \tag{5.19}$$

in other words,

$$\rho_{\beta_{p_f\to\infty}} = \sum_f \frac{\gamma_f}{6\pi^2} \frac{\pi}{16} \beta^{-3/2} \;. \tag{5.20}$$

What (5.20) shows us is that the baryon number density now converges to a maximum limit (whereas it would diverge in the conventional case). Effectively, the GUP introduces an asymptotic cutoff to the possible momentum configurations. Such behavior is expected from this formalism, as we had inferred from Figure 3.

Now, knowing that the values of βp^2 are expected to be much smaller than 1, for the sake of simplicity in future applications, we can approximate relations (5.13)-(5.15) up to the first order in β. Namely, we have

$$\varepsilon_\beta = \sum_f \frac{\gamma_f}{2\pi^2} \int_0^{p_f} \sqrt{m_f^2 + p^2}\, p^2 \left(1 - 3\beta p^2\right) dp + B \ , \tag{5.21}$$

$$\mathscr{P}_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^{p_f} \frac{p^4}{\sqrt{m_f^2 + p^2}} \left(1 - 3\beta p^2\right) dp - B \ , \tag{5.22}$$

$$\rho_\beta = \sum_f \frac{\gamma_f}{6\pi^2} \int_0^{p_f} p^2 \left(1 - 3\beta p^2\right) dp \ . \tag{5.23}$$

Comparing (5.21)-(5.23) to (2.76)-(2.78), we may write

$$\varepsilon_\beta = \varepsilon_0 - 3\beta \sum_f \frac{\gamma_f}{2\pi^2} \int_0^{p_f} \sqrt{m_f^2 + p^2}\, p^4 dp \ , \tag{5.24}$$

$$\mathscr{P}_\beta = \mathscr{P}_0 - 3\beta \sum_f \frac{\gamma_f}{6\pi^2} \int_0^{p_f} \frac{p^6}{\sqrt{m_f^2 + p^2}} dp \ , \tag{5.25}$$

$$\rho_\beta = \rho_0 - 3\beta \sum_f \frac{\gamma_f}{6\pi^2} \int_0^{p_f} p^4 dp \ . \tag{5.26}$$

We then integrate the new expressions to find:

$$\varepsilon_\beta = \varepsilon_0 + \beta \sum_f \frac{\gamma_f}{2\pi^2} \frac{1}{16} \left[p_f E_{p_f} \left(-8p_f^4 - 2p_f^2 m_f^2 + 3m_f^4\right) - 3m_f^6 \omega_f \right] \ , \tag{5.27}$$

$$\mathscr{P}_\beta = \mathscr{P}_0 + \beta \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{16} \left[p_f E_{p_f} \left(-8p_f^4 + 10p_f^2 m_f^2 - 15m_f^4\right) + 15m_f^6 \omega_f \right] \ , \tag{5.28}$$

$$\rho_\beta = \rho_0 - \beta \sum_f \frac{\gamma_f}{6\pi^2} \frac{3}{5} p_f^5 \ , \tag{5.29}$$

where we have defined $\omega_f = \ln\left(\frac{p_f + E_{p_f}}{m_f}\right)$ for brevity. Again, making $\beta \to 0$ now returns us to the relations (2.82)-(2.84). We may then use the approximate analytical solutions (5.27)-(5.29) to (within reason, given the limitations of a first-order approximation) analyze the behavior of the thermodynamic quantities of the GUP-deformed MIT Bag Model. The results we obtained for this case are summarized in Figures 5.1 to 5.3, where we represent β directly through the associated squared minimal length scale.

From Figures 5.1 and 5.2 we notice that the introduction of the GUP has the effect of, compared to the conventional MIT Bag Model ($\beta = 0$), reducing the values of both the energy density and pressure for the same baryon number density.

We argue that the reduction in the energy density values can be physically justified by the fact that the GUP, as we have noted before, essentially restricts the available momentum states for quarks within the bag. This means that extremely high-momentum (short-wavelength) excitations, which are commonplace in the standard model, get suppressed in the presence of the GUP. This suppression leads to fewer particle excitations in the bag, thereby reducing the overall energy density.

On the other hand, fewer excitations also mean that the quarks exert less outward pressure on the bag boundaries, thus explaining the decreased pressure. The reduction in pressure, given the same baryon density, would initially suggest that introducing a GUP effectively softens the equation of state in relation to the baryon number density.

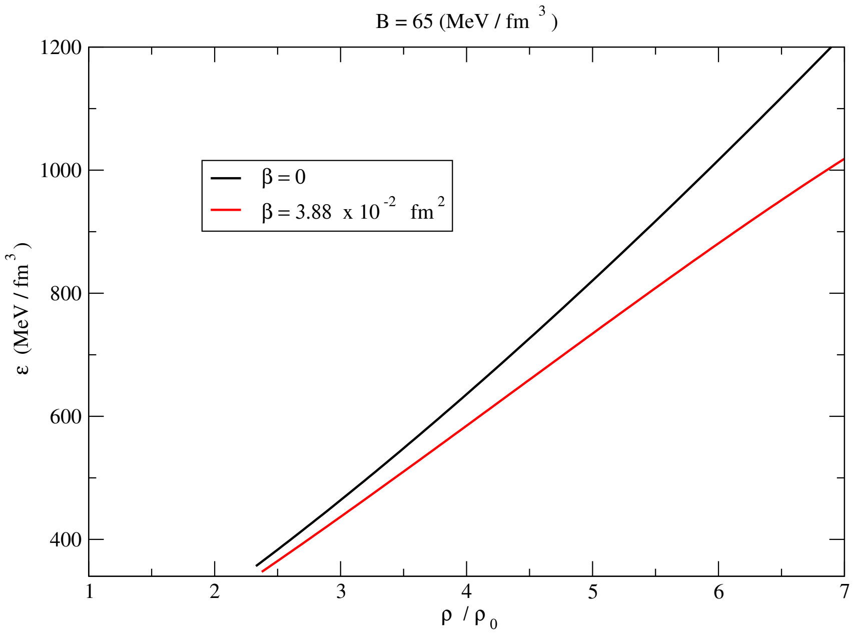

Figure 5.1: Energy density (we represent ε_β simply as ε) vs. ratio of the baryonic number density (we denote ρ_β by ρ) and the saturation density ($\rho_0 = 0.153\,\mathrm{fm}^{-3}$).

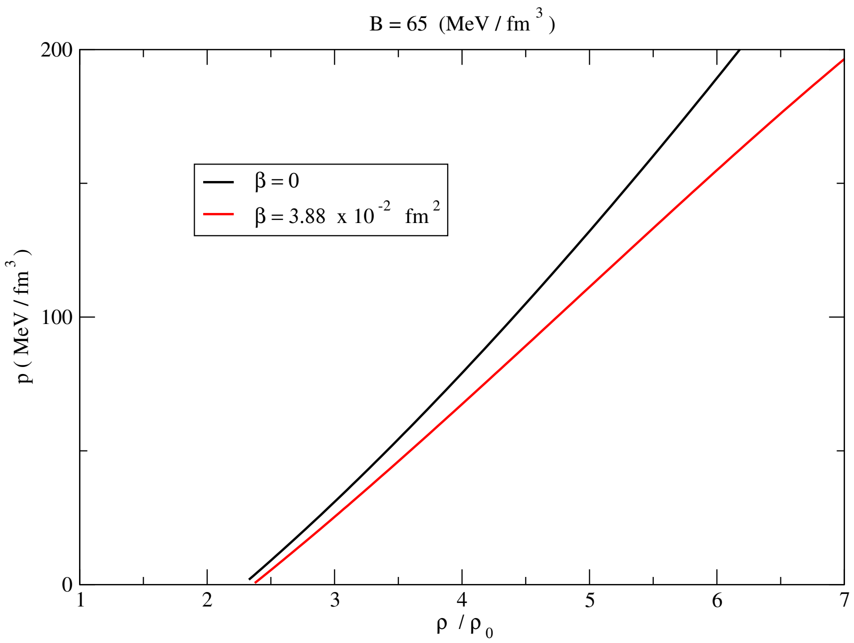

Figure 5.2: Pressure (we represent $\mathscr{P}_\beta$ simply as p) vs. ratio of the baryonic number density (ρ) and the saturation density ($\rho_0 = 0.153\,\mathrm{fm}^{-3}$).

However, the relative reductions in energy density occur at a higher rate than those in pressure. This results in the behavior observed in Figure 5.3, where we see that, for a given energy density, the pressure actually increases compared to the standard MIT Bag Model. This in turn suggests a stiffer EoS, which provides greater resistance to gravitational collapse, meaning we could theoretically find higher mass limits for such compact objects.

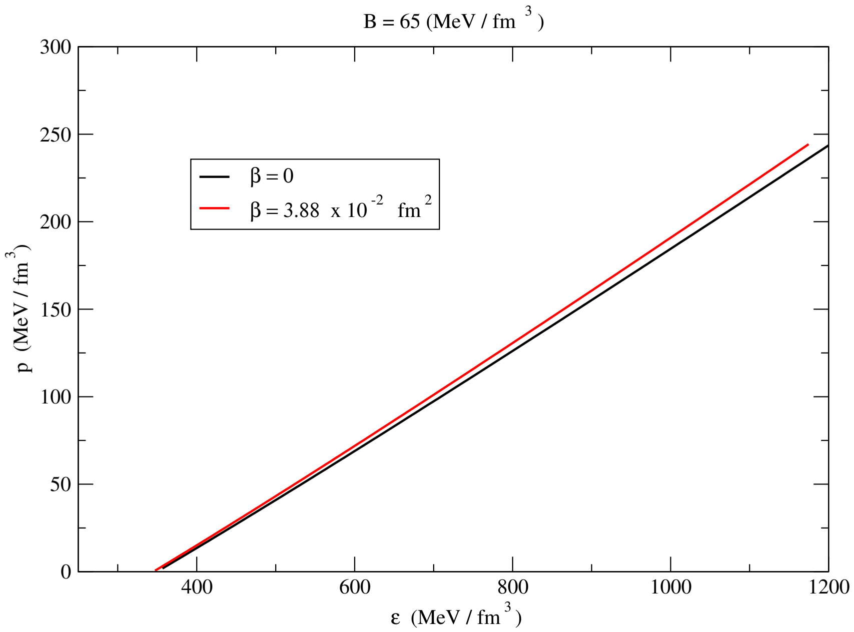

Figure 5.3: Pressure (p) vs. energy density (ε).

In all of the EoS plots, we notice that the effects of the GUP are progressively intensified in higher energy regimes. This is in good accordance with the foundational idea behind the GUP that quantum gravitational effects become more pronounced on higher energy scales. Mathematically, this is due to the momentum dependence of the deformation imposed on the Heisenberg algebra.

It is also worth noting that in both Figure 5.1 and Figure 5.2, in addition to the angular divergences between the old and new relations, there is also a slight shift in the lowest value for the baryon number density for each. This is expected from (5.29).

To better analyze the effects of the introduction of a GUP to compact objects, we must investigate the consequences of the application of the newly obtained EoS to the TOV equations. This process is straightforward from the thermodynamic values we previously obtained, our results are given by Figure 5.4 for diverse values of β within the order of usual observational bounds [51–53].

In Figure 5.4, we see that indeed the mass-radius relation is slightly increased when we introduce the GUP, the effects to both the maximal mass and radius of the star are exacerbated accordingly for greater values of β.

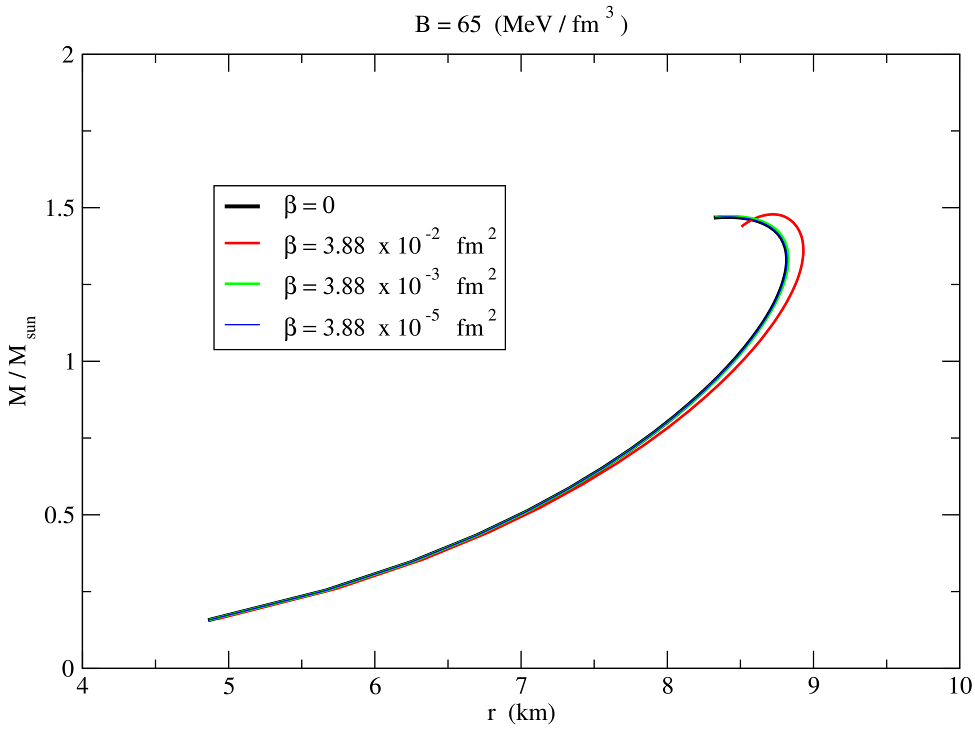

Figure 5.4: Mass in solar masses $\left(\frac{M}{M_{\text{sun}}}\right)$ vs. radius (r) of the compact star.

The new relations found in this subsection are also particularly interesting since the zero temperature limit of the MIT Bag Model is considerably similar to the degenerate Fermi gas model, which is frequently studied as a thermodynamic benchmark in literature. Indeed, if after taking the $T \to 0$ limit, one also ignores the bag constant in (2.76)-(2.78) and the sum over quark flavors, one should have (with proper mass terms) a model equivalent to that of the noninteracting degenerate Fermi gas model.

5.1.3 The Deformed Zero Temperature Massless Limit

For the sake of completeness and conceptual insight, we briefly investigate the effective GUP MIT Bag Model in the massless limit approximation. We may simply take the limit $m_f \to 0$ in expressions (5.27)-(5.29) and recall that this implies $E_{p_f} = \mu_f = p_f$, we find

$$\varepsilon_\beta = \sum_f \frac{\gamma_f}{2\pi^2}\frac{1}{4}\mu_f^4\left(1 - \frac{\beta}{2}\mu_f^2\right) + B\ , \tag{5.30}$$

$$\mathscr{P}_\beta = \sum_f \frac{\gamma_f}{6\pi^2}\frac{1}{4}\mu_f^4\left(1 - \frac{\beta}{2}\mu_f^2\right) - B\ , \tag{5.31}$$

$$\rho_\beta = \sum_f \frac{\gamma_f}{6\pi^2}\frac{1}{3}\mu_f^3\left(1 - \frac{9\beta}{5}\mu_f^2\right)\ , \tag{5.32}$$

where setting $\beta = 0$ gives us back relations (2.86)-(2.88). We notice, however, that (5.30) and (5.31) can be combined to write

$$\mathscr{P}_\beta = \frac{1}{3}\left(\varepsilon_\beta - 4B\right) \ . \tag{5.33}$$

This is simply (with the new thermodynamic definitions) equation (2.89)! The relationship between $\mathscr{P}_\beta$ and ε_β is exactly the same as that for $\mathscr{P}$ and ε. This implies that, had we plotted (5.33) in the same manner as we did for Figure 5.3, we would have perceived no angular difference. We may expect small vertical or horizontal shifts due to a modification of the allowed values of the quantities, but the plots would be tangentially parallel in every point.

We conclude this section by observing that, if we reduce our MIT Bag Model to the degenerate Fermi gas model (set $B = 0$ in (5.13)-(5.15) and considering a single particle flavor, for example), and apply the ultra-relativistic limit approximation, our expressions for this constrained version of the EoS have the same form as the ones found for the GUP-modeled degenerate ultra-relativistic Fermi gas in the literature [35–37]. This demonstrates the coherence of our model with previous studies and, more importantly, that we have generalized previous findings. Relations (5.13)-(5.15) alone are generalizations of the ones already existent in the literature, and, as such, so are the considerably less constrained relations (5.10)-(5.12).

5.2 Towards a General EoS Deformation Model

Although in the last section we have investigated the effects of the Kempf GUP in the MIT Bag Model EoS, the general structure of this deformed EoS approach is not restricted to this specific application. In principle, we can find GUP-deformed models for any EoS as long as we have proper Lagrangian densities with symmetries that lead to the thermodynamic quantities - which was the starting point of our work. We have such generality due to the fact that our reasoning was developed through the application of the second quantization formalism. Additionally, we may consider any GUP that satisfies the properties of those discussed in Appendix A.

To see how an arbitrary GUP affects a general EoS, we first recall that, while the deformation formalism of [30] was specifically used in the context of the Kempf GUP, the Jacobian term can be written as

$$J = \prod_{i=1}^{N} f_{ii}(x, p) \ , \tag{5.34}$$

as long as the new deformed canonical commutation relation $[x_i, p_j] = i\hbar\, f_{ij}(x, p)$ is diagonal (we note that dependence on x conversely implies a maximal momentum resolution).

Naturally, since we do not expect any specific dimensions of space to be particularly privileged in relation to the others, the Jacobian term further simplifies to

$$J = [f_{ii}(x,p)]^N \tag{5.35}$$

and transformation (3.44) may be written in the more general form

$$d^N x \, d^N p \;\longrightarrow\; \frac{d^N x \, d^N p}{[f_{ii}(x,p)]^N} \ . \tag{5.36}$$

In the second chapter, we have studied the symmetries of the fermionic free-field Lagrangian density and how they lead to the thermodynamic quantities (2.22)-(2.24). While these equations are specific to the MIT Bag Model, the overall dependency of these quantities on the mean-valued terms $\langle{:}H{:}\rangle$, $\langle{:}P{:}\rangle$ and $\langle{:}N{:}\rangle$ of their corresponding model is more general [15].

A more complex EoS, however, may involve not only additional particle species beyond fermions but also interaction terms between such particles. Consequently, a comprehensive deformation model would need to account for these intricacies. Here, to illustrate how we can implement a general deformation model to more sophisticated systems, we study the mean-valued quantity associated with the energy density, namely, the Hamiltonian.

In the case of a system of fermions, the Hamiltonian (2.32) is suitable for noninteracting particles. We may consider (2.32) to be a sum over any fermion species rather than over quark flavors and write

$$H_0 = \sum_i \int d^3x_i \, \overline{\psi}_i(x_i) \left(-i\gamma \cdot \nabla + m_i\right) \psi_i(x_i) \ . \tag{5.37}$$

This is often referred to as the free-field Hamiltonian or the kinetic Hamiltonian.

For systems where the fermions interact, we will then have the total Hamiltonian

$$H = H_0 + H_{\text{int}} \ , \tag{5.38}$$

where H_{int} corresponds to the interaction term. In the case of two-body interactions, H_{int} can be written in the general form [54, 55]

$$H_{\text{int}} = \frac{1}{2!} \sum_{i,j;i\neq j} \iint d^3x_i \, d^3x_j \, \overline{\psi}_i(x_i) \overline{\psi}_j(x_j) V(\mathbf{x}_i, \mathbf{x}_j) \psi_j(x_j) \psi_i(x_i) \ , \tag{5.39}$$

where $V(\mathbf{x}_i, \mathbf{x}_j)$ is the potential describing the interaction between the two different particles.

The thermodynamic quantity ε of the EoS will then depend on the normal-ordered expectation value

$$\langle :H: \rangle = \langle :H_0: \rangle + \langle :H_{\text{int}}: \rangle \ , \tag{5.40}$$

where for two-body interactions we have

$$\langle :H_{\text{int}}: \rangle = \frac{1}{2!} \sum_{i,j;i \neq j} \iint d^3x_i \, d^3x_j \ \langle :\overline{\psi}_i(x_i) \overline{\psi}_j(x_j) V(\mathbf{x}_i, \mathbf{x}_j) \psi_j(x_j) \psi_i(x_i): \rangle \ . \tag{5.41}$$

Now, to compute the deformed energy density of this interacting fermion system, we must first find the deformed quantity

$$\langle :H: \rangle_\beta = \langle :H_0: \rangle_\beta + \langle :H_{\text{int}}: \rangle_\beta \ . \tag{5.42}$$

After performing the momentum expansion for the wave functions, it is easy to see that the first term of the left-hand side of (5.42) is simply the more general version of (5.3) subject to the deformation

$$d^3x_i \, d^3p_i \ \longrightarrow \ \frac{d^3x_i \, d^3p_i}{\left[f_{ii}(x_i, p_i)\right]^3} \ . \tag{5.43}$$

For the interaction term, however, the integrations over the positions of the two particles will now define two distinct differential volumes. After performing the momentum expansions, we must then have, for the same reasons discussed for the term (5.2), a deformation of the kind

$$d^3x_i \, d^3p_i \ d^3x_j \, d^3p_j \ \longrightarrow \ \frac{d^3x_i \, d^3p_i \ d^3x_j \, d^3p_j}{\left[f_{ii}(x_i, p_i)\right]^3 \left[f_{jj}(x_j, p_j)\right]^3} \ . \tag{5.44}$$

This is dimensionally expected and establishes an asymptotic cutoff for the momenta of both particles involved.

The expression given in (5.42) can, however, be treated as general and does not need to correspond to systems of fermions, but in principle to any particles, with arbitrary n-body interactions. The term $\langle :H_0: \rangle_\beta$ should then represent the sum of all free-field Hamiltonians, and we expect the deformations to be of the same kind as (5.43). On the other hand, we expect an n-body interaction to suffer a deformation of the kind

$$d^3x_1 \, d^3p_1 \ldots d^3x_n \, d^3p_n \ \longrightarrow \ \frac{d^3x_1 \, d^3p_1 \ldots d^3x_n \, d^3p_n}{\left[f_{11}(x_1, p_1)\right]^3 \ldots \left[f_{nn}(x_n, p_n)\right]^3} \ . \tag{5.45}$$

Since relations (5.42), (5.44) and (5.45) do not *a priori* impose any restriction on the kinds of particles described, we may use them in arbitrary EoS with any proper GUP to construct the quantity ε_β. We estimate that similar procedures should be available for other thermodynamic quantities, which can possibly provide a clearer view of the effects of a GUP-induced minimal length in more complex models of compact objects.

6. Noncommutative Geometry Model EoS

Now, we apply the coherent states coordinate model of noncommutative geometry studied in the fourth chapter to the EoS of compact stars. More specifically, we use the new definition of the Fourier expansion of the wave function in the derivation of the MIT Bag Model EoS. We analyze the limits of zero temperature and of zero mass and briefly compare them to our previous findings.

6.1 The Noncommutative MIT Bag Model

In order to implement the coherent states coherent approach to the MIT Bag Model, we must first observe that the general $2n$-dimensional modified Fourier transform given in (4.37) reduces, for the case of fermions in four-dimensional Minkowski spacetime, to

$$\psi_f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \sum_r \left(\frac{m_f}{E_{\mathbf{p}}}\right)^{\frac{1}{2}} \left[c_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ip_\mu x^\mu} + d_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) e^{ip_\mu x^\mu}\right] e^{-\frac{\theta}{4} p^\mu p_\mu} d\mathbf{p} \ , \quad (6.1)$$

$$\overline{\psi}_f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \sum_r \left(\frac{m_f}{E_{\mathbf{p}}}\right)^{\frac{1}{2}} \left[c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{ip_\mu x^\mu} + d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip_\mu x^\mu}\right] e^{-\frac{\theta}{4} p^\mu p_\mu} d\mathbf{p} \ , \quad (6.2)$$

as stated in [56] and demonstrated for bosonic fields in Appendix C.

It is then straightforward to apply the momentum expansions (6.1) and (6.2) to the definitions of $\langle :H: \rangle$, $\langle :P: \rangle$ and $\langle :N: \rangle$, construct $\langle :H: \rangle_\theta$, $\langle :P: \rangle_\theta$ and $\langle :N: \rangle_\theta$, and subsequently obtain the noncommutative-modeled thermodynamic quantities ε_θ, $\mathscr{P}_\theta$ and ρ_θ. We may first note, however, that the new damping term $e^{-\frac{\theta}{4} p^\mu p_\mu}$, in our Mikowski spacetime, involves the Lorentz scalar invariant

$$p^\mu p_\mu = (p^0)^2 - \mathbf{p}^2 \ . \quad (6.3)$$

Since the coherent states coordinates approach does not alter the dispersion relation (2.43) (i.e. $(p^0)^2 = E_{\mathbf{p}}^2 = \mathbf{p}^2 + m_f^2$) [57, 58], we may simply write

$$p^\mu p_\mu = m_f^2 \quad (6.4)$$

and move the exponential term outside of the integrals of (6.1) and (6.2). We should note that this observation may not be as evident here as it is for conventional QFT, since it is a known consequence of many minimal length theories to introduce a modified dispersion relation [1]. We also observe that (2.43) is valid only for on-shell particles, and since the Fourier expansion is over real momentum states referring to a real particle, this procedure is guaranteed to work. In the case of, for example, a Feynman propagator, such as we have seen in the fourth chapter and is studied in [38, 41, 42, 56], virtual particles are involved and they need not satisfy (2.43), so integrations may not be so trivial.

Once again, for the sake of illustration, we explicitly substitute (6.1) and (6.2) in the definition of the expectation value of the normal-ordered free-field Dirac Hamiltonian $\langle{:}H{:}\rangle$ and write

$$\langle{:}H{:}\rangle_\theta = \frac{1}{(2\pi)^3}\iint \sum_{f,r,r'} \frac{\gamma_f\, m_f\, e^{-\frac{\theta}{4}p^\mu p_\mu} e^{-\frac{\theta}{4}p'^\mu p'_\mu}}{\left(E_{\mathbf{p}}E_{\mathbf{p}'}\right)^{1/2}} \Big\langle{:}\Big[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu}\Big]$$
$$\times\left(-i\gamma\cdot\nabla + m_f\right)\Big[c_{r'}(\mathbf{p}')u_{r'}(\mathbf{p}')e^{-ip'_\mu x^\mu} + d_{r'}^\dagger(\mathbf{p}')v_{r'}(\mathbf{p}')e^{ip'_\mu x^\mu}\Big]{:}\Big\rangle\, d^3x\ d^3p\ d^3p'\ , \tag{6.5}$$

where we then rewrite the Lorentz scalar invariant as (6.3) and get

$$\langle{:}H{:}\rangle_\theta = \frac{1}{(2\pi)^3}\iint \sum_{f,r,r'} \frac{\gamma_f\, m_f\, e^{-\frac{\theta}{2}m_f^2}}{\left(E_{\mathbf{p}}E_{\mathbf{p}'}\right)^{1/2}} \Big\langle{:}\Big[c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ip_\mu x^\mu} + d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip_\mu x^\mu}\Big]$$
$$\times\left(-i\gamma\cdot\nabla + m_f\right)\Big[c_{r'}(\mathbf{p}')u_{r'}(\mathbf{p}')e^{-ip'_\mu x^\mu} + d_{r'}^\dagger(\mathbf{p}')v_{r'}(\mathbf{p}')e^{ip'_\mu x^\mu}\Big]{:}\Big\rangle\, d^3x\ d^3p\ d^3p'\ . \tag{6.6}$$

Again, we may perform the integrations over x and p' and simplifications up to (2.68) exactly as we have done in the second chapter. We find, in spherical coordinates and with explicit integration limits,

$$\langle{:}H{:}\rangle_\theta = \sum_f \gamma_f\, e^{-\frac{\theta}{2}m_f^2}\int_0^\infty 4\pi E_p\Big[n(p,\mu_f) + n(p,-\mu_f)\Big]p^2dp\ . \tag{6.7}$$

Similarly, we apply same procedure to $\langle{:}P{:}\rangle$ and $\langle{:}N{:}\rangle$, and we easily find:

$$\langle{:}P{:}\rangle_\theta = \sum_f \gamma_f\, e^{-\frac{\theta}{2}m_f^2}\int_0^\infty 4\pi E_p^{-1}\Big[n(p,\mu_f) + n(p,-\mu_f)\Big]p^4dp\ , \tag{6.8}$$
$$\langle{:}N{:}\rangle_\theta = \sum_f \gamma_f\, e^{-\frac{\theta}{2}m_f^2}\int_0^\infty 4\pi \Big[n(p,\mu_f) - n(p,-\mu_f)\Big]p^2dp\ . \tag{6.9}$$

We may now write the noncommutative-modeled thermodynamic quantities of the modified MIT Bag Model as:

$$\varepsilon_\theta = \frac{1}{(2\pi)^3}\langle{:}H{:}\rangle_\theta + B\ , \tag{6.10}$$
$$\mathscr{P}_\theta = \frac{1}{(2\pi)^3}\frac{\langle{:}P{:}\rangle_\theta}{3} - B\ , \tag{6.11}$$
$$\rho_\theta = \frac{1}{(2\pi)^3}\frac{\langle{:}N{:}\rangle_\theta}{3}\ . \tag{6.12}$$

Substituting (6.7)-(6.9) into (6.10)-(6.12) respectively, we obtain the noncommutative-modeled MIT Bag Model EoS:

$$\varepsilon_\theta = \sum_f \frac{\gamma_f\, e^{-\frac{\theta}{2}m_f^2}}{2\pi^2} \int_0^\infty E_p \Big[n(p,\mu_f) + n(p,-\mu_f)\Big] p^2 dp + B \ , \tag{6.13}$$

$$\mathscr{P}_\theta = \sum_f \frac{\gamma_f\, e^{-\frac{\theta}{2}m_f^2}}{6\pi^2} \int_0^\infty E_p^{-1} \Big[n(p,\mu_f) + n(p,-\mu_f)\Big] p^4 dp - B \ , \tag{6.14}$$

$$\rho_\theta = \sum_f \frac{\gamma_f\, e^{-\frac{\theta}{2}m_f^2}}{6\pi^2} \int_0^\infty \Big[n(p,\mu_f) - n(p,-\mu_f)\Big] p^2 dp \ , \tag{6.15}$$

where taking the limit $\theta \to 0$ returns us to the usual commutative model.

We see that relations (6.13)-(6.15) are simply the quantities of the MIT Bag Model (2.71)-(2.73) modified by the constant parameter $e^{-\frac{\theta}{2}m_f^2}$. There are two noteworthy observations to be made here. First, that the term $e^{-\frac{\theta}{2}m_f^2}$ falls within the interval $(0,1]$, ensuring that it necessarily results in a decrease of all the thermodynamic quantities. Second, that on the limit $m_f \to 0$, the effects of noncommutativity are not perceived.

For the sake comparison, we first write out (6.13)-(6.15) in the $T \to 0$ limit

$$\varepsilon_\theta = \sum_f \frac{\gamma_f\, e^{-\frac{\theta}{2}m_f^2}}{2\pi^2} \frac{1}{8} \Big[p_f E_{p_f} \left(2p_f^2 + m_f^2\right) - m_f^4 \omega_f\Big] + B \ , \tag{6.16}$$

$$\mathscr{P}_\theta = \sum_f \frac{\gamma_f\, e^{-\frac{\theta}{2}m_f^2}}{6\pi^2} \frac{1}{8} \Big[p_f E_{p_f} \left(2p_f^2 - 3m_f^2\right) + 3m_f^4 \omega_f\Big] - B \ , \tag{6.17}$$

$$\rho_\theta = \sum_f \frac{\gamma_f\, e^{-\frac{\theta}{2}m_f^2}}{6\pi^2} \frac{1}{3}\, p_f^3 \ . \tag{6.18}$$

And we see, both in (6.13)-(6.15) and (6.16)-(6.18), that the mass-dependent reductions affect the thermodynamic quantities in equal manner. In other words, while one can indeed expect to observe such decreases as the mass and momentum (energy) scales increase, the relations between the thermodynamic quantities themselves are not expected to change.

Now it is also interesting to also take the massless limit, for which we easily find:

$$\varepsilon_\theta = \sum_f \frac{\gamma_f}{2\pi^2} \frac{1}{4}\, \mu_f^4 + B = \varepsilon_0 \ , \tag{6.19}$$

$$\mathscr{P}_\theta = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{4}\, \mu_f^4 - B = \mathscr{P}_0 \ , \tag{6.20}$$

$$\rho_\theta = \sum_f \frac{\gamma_f}{6\pi^2} \frac{1}{3}\, \mu_f^3 = \rho_0 \ , \tag{6.21}$$

which, as we stated before, simply coincide with the commutative case, irrespective of the choice of the parameter θ. Naturally, this means that (2.89) and (2.90) are identically valid here.

The vanishing of noncommutative effects in the zero mass limit in our on-shell case has an intimate connection with the Lorentz-invariance of the theory, as this is not necessarily the case for the two-dimensional non Lorentz-invariant model [41,42,57], we speculate that the difference in the predictions between these models can be dramatically different in some cases and care should be taken when trying to generalize the findings of one to the other. Still, we must recall that the exponential term, in either case, has regularizing properties for propagators in this massless limit [38,56]. This feature is also in agreement with the general idea that the effects of noncommutativity are suppressed for lower mass-energy scales.

We conclude this section by noting that the noncommutative effects in this model, while offering qualitative insights into the properties of minimal length scales, are not expected to quantitatively contribute significantly to either the MIT Bag Model EoS or the associated mass-radius relation. The uniform rate at which these effects contribute to all quantities tends to obscure their impact when these quantities are expressed as functions of each other. Furthermore, when we consider the experimental upper bounds on the noncommutative parameter θ [59, 60] and the established values for the quark masses m_f [61], we may estimate that $e^{-\frac{\theta}{2}m_f^2} \approx 1$, leading to negligible deviations from conventional theory. Hence, we have opted not to include graphical representations of the outcomes of this model, as they are unlikely to illustrate any significant divergence from well-established results.

6.2 Comparison with the GUP Model

Though the GUP effective formalism and the coherent state coordinate approach both introduce a minimal length scale to space, their foundational principles diverge significantly. The findings detailed in the fifth chapter and the present chapter underscore this distinction, as evidenced by the varying large-scale effects predicted by these theories.

Both modified MIT Bag Model EoSs derived from these minimal length frameworks imply a diminution of thermodynamic quantities with rising energy scales. On one hand, the amplification of minimal length effects at high energies aligns with expectations, given that the influence of quantum gravity is anticipated to become pronounced under such conditions. On the other hand, the observed tendency of these effects to dampen thermodynamic values is consistent with the cutoff behavior produced by the introduction of a minimal length.

In both of the studied models, we recover the conventional commutative theory when the noncommutative parameter is taken to zero. Accordingly, we may write:

$$\lim_{\beta\to 0} \varepsilon_\beta = \lim_{\theta\to 0} \varepsilon_\theta = \varepsilon_0 \ , \tag{6.22}$$

$$\lim_{\beta\to 0} \mathscr{P}_\beta = \lim_{\theta\to 0} \mathscr{P}_\theta = \mathscr{P}_0 \ , \tag{6.23}$$

$$\lim_{\beta\to 0} \rho_\beta = \lim_{\theta\to 0} \rho_\theta = \rho_0 \ , \tag{6.24}$$

which serves as a consistency check for both frameworks.

Nevertheless, while both models forecast decreases in the thermodynamic quantities with heightened energy scales, the GUP effective formalism exhibits more substantial deviations from conventional theory as compared to the coherent state coordinate approach. In contrast to the uniform and numerically modest alterations produced by the latter, the GUP model induces deformations in the thermodynamic variables that result in modification terms that scale with different powers of the momentum - that is, they are affected at different rates. This is particularly evident from the fact that the damping terms added by the GUP model, unlike the ones added by the coherent states coordinate model, cannot be moved outside of the momentum integrals that originate the thermodynamic quantities.

This distinction in behavior is consistent with the theoretical principles of the two approaches. Specifically, the minimal position uncertainty in the GUP framework is a function of momentum, allowing for the possibility of arbitrarily large delocalization effects as momentum increases. This is in contrast to the coherent state coordinate approach, where delocalization is constrained by the constant parameter θ. This is aligned with the notion that coherent states represent the states of minimal uncertainty on the noncommutative manifold.

7. Final Considerations

In this work, we have conducted an investigation into modifications of the equation of state (EoS) for compact stars under two distinct frameworks that introduce a minimal length scale: the effective model for the Generalized Uncertainty Principle (GUP) and the noncommutative geometry model through coherent state coordinates.

The effective Kempf GUP formalism was for the first time applied to the MIT Bag Model, resulting in a modified EoS that describes the behavior of noninteracting deconfined quark matter in the presence of a minimal length scale and consistently reduces to the conventional theory when the GUP parameter $\beta \to 0$.

We subsequently derived analytical solutions in the zero temperature limit, for which the thermodynamic quantities presented an overall scaling decrease in relation to the momentum variable. Notably, we have found that as the Fermi momentum goes to infinity, the baryon number density converges to a maximum saturation limit. These results are in agreement with the notion that the minimal length scale induced by the GUP imposes an asymptotic cutoff to the possible momentum configurations. We note that we expect to find saturation limits at high momenta for all considered thermodynamic quantities in a full non-approximate analysis, which is currently underway.

We then integrated these new thermodynamic quantities into the Tolman-Oppenheimer-Volkoff (TOV) equations and found that the introduction of the GUP implies a slight increase in the mass-radius relation of compact objects. This means that the GUP-induced minimal length causes greater resistance to gravitational collapse in the stellar structure. We have also further simplified the EoS to the massless case, for which we inferred that any divergences from the conventional theory would be less pronounced.

The findings for the GUP-deformed MIT Bag Model EoS are consistent with and extend upon those from prior studies on GUP-deformed ultra-relativistic degenerate Fermi gases, in the same measure that the MIT Bag Model itself reduces to the ultra-relativistic degenerate Fermi gas [35–37]. This reinforces the validity of our model in the context of existing research and highlights the generality of our approach.

In light of this greater generality, we have briefly outlined the path towards a universalized model that is not confined to Kempf deformations. This allows the integration of various particle species and interactions, offering a more versatile tool for examining a wider array of EoSs.

Next, we turned our attention to the coherent states coordinate model of noncommutative geometry. We again applied it for the first time to the MIT Bag Model and obtained a modified EoS consistent with noncommutative geometries, which coherently returns to the conventional theory when the noncommutativity parameter $\theta \to 0$.

The resulting noncommutative-modeled thermodynamic quantities, as in the GUP model, presented an overall decrease in relation to the momentum variable. However, such decreases occur in a linear and uniform manner across all the EoS quantities, meaning that the effects of noncommutativity should be obscured when describing a quantity in terms of another.

Furthermore, we observed that noncommutative influences vanish in the massless limit, a condition indifferent to the choice of θ. We suggest that this phenomenon is intrinsically connected to the fact that the theory preserves Lorentz invariance and does not alter the dispersion relation.

After a brief analysis of the approximate value of the deformation term for the coherent state coordinate approach, we found that, while offering qualitative insights into the properties of minimal length scales, the effects of this model are not expected to quantitatively contribute significantly to either the MIT Bag Model EoS or the associated mass-radius relation.

Though the GUP effective formalism and the coherent state coordinate approach both introduce a minimal length scale to space, their foundational principles diverge significantly. This divergence is evidenced in our dramatically distinct results for each model. The minimal position uncertainty in the GUP framework allows for the possibility of arbitrarily large delocalization effects as the momentum increases. This contrasts with the coherent state coordinate approach, where delocalization is fixed by the constant noncommutativity parameter.

We conclude this work by noting that both models studied provide a mathematically simple representation of the potential influence of quantum gravity on compact stellar structures. Despite their distinct foundational principles and implications, they offer preliminary steps toward a more nuanced understanding of astrophysical phenomena where minimal length scales are expected to play a critical role. It is our aspiration that future research will build upon this work, advancing these models into more complex realms (particularly those involving temperature effects) where quantum gravitational influences may be further amplified.

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A. Derivation of Fermi-Dirac Distributions

We briefly prove relations (2.66). In other words, we derive the Fermi-Dirac distributions from the expectation values of the fermion number operators.

The expectation value of an operator A may be obtained through

$$\langle A \rangle = \mathrm{Tr}(\rho A) \ , \tag{A.1}$$

where ρ is the density matrix. In a grand canonical ensemble (where both energy and particle number may fluctuate, with the system conserving the average of these quantities), the density matrix is given by

$$\rho = \frac{1}{\mathcal{Z}} e^{-\frac{(H-\mu N)}{T}} \ , \tag{A.2}$$

where, from (2.49), we let

$$N = \sum_\lambda N_\lambda = \sum_\lambda c_\lambda^\dagger c_\lambda \tag{A.3}$$

be the particle number operator, H is the Hamiltonian, μ is the chemical potential, T the temperature. Here, $\mathcal{Z}$ is the partition function ensuring normalization

$$\mathcal{Z} = \mathrm{Tr}\left(e^{-\frac{(H-\mu N)}{T}} \right) \ . \tag{A.4}$$

We may therefore write the expectation value of N (recalling the linearity of the trace) as

$$\begin{aligned} \langle N_\lambda \rangle &= \mathrm{Tr}(\rho N_\lambda) \\ &= \frac{1}{\mathcal{Z}} \mathrm{Tr}\left(e^{-\frac{(H-\mu N)}{T}} c_\lambda^\dagger c_\lambda \right) \ . \end{aligned} \tag{A.5}$$

In order to explicitly calculate (A.5), we first identically rewrite

$$\langle N_\lambda \rangle = \frac{1}{\mathcal{Z}} \mathrm{Tr}\left(e^{-\frac{(H-\mu N)}{T}} c_\lambda^\dagger e^{\frac{(H-\mu N)}{T}} e^{-\frac{(H-\mu N)}{T}} c_\lambda \right) \ . \tag{A.6}$$

We recall that, for given operators A and B in a Lie algebra, the Baker-Campbell-Hausdorff formula yields the following lemma

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[(A)^n, B \right] \ , \tag{A.7}$$

where

$$[(A)^n, B] = \underbrace{[A, ...[A, [A}_{n \text{ times}}, B]]...] \ . \tag{A.8}$$

In the special case, for a scalar α, of

$$[A, B] = \alpha B \ , \tag{A.9}$$

relation (A.7) reduces to

$$\begin{aligned} e^{A} B e^{-A} &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} B \\ &= e^{\alpha} B \ . \end{aligned} \tag{A.10}$$

We now wish to evaluate

$$\left[H - \mu N, c_\sigma^\dagger\right] = \left[H, c_\sigma^\dagger\right] - \mu \left[N, c_\sigma^\dagger\right] \ . \tag{A.11}$$

We first recall the commutator identity

$$[AB, C] = A \{B, C\} - \{A, C\} B \ , \tag{A.12}$$

so that we may write the second commutation relation in the left-hand side of (A.11) as

$$\begin{aligned} \left[N, c_\sigma^\dagger\right] &= \sum_\lambda \left[c_\lambda^\dagger c_\lambda, c_\sigma^\dagger\right] \\ &= \sum_\lambda \left(c_\lambda^\dagger \underbrace{\left\{c_\lambda, c_\sigma^\dagger\right\}}_{=\delta_{\lambda\sigma}} - \underbrace{\left\{c_\lambda^\dagger, c_\sigma^\dagger\right\}}_{=0} c_\lambda \right) \\ &= c_\sigma^\dagger \ . \end{aligned} \tag{A.13}$$

On the other hand, we have for the first term in the left-hand side of (A.11)

$$\begin{aligned} \left[H, c_\sigma^\dagger\right] &= \left[E_p(N + \overline{N}), c_\sigma^\dagger\right] \\ &= E_p \underbrace{\left[N, c_\sigma^\dagger\right]}_{=c_\sigma^\dagger} + E_p \underbrace{\left[\overline{N}, c_\sigma^\dagger\right]}_{=0} \\ &= E_p c_\sigma^\dagger \ , \end{aligned} \tag{A.14}$$

where we have used the definition of the Hamiltonian in terms of the energy E_p and the result (A.13).

Inserting (A.13) and (A.14) in (A.11), we find

$$\left[H - \mu N, c_\sigma^\dagger\right] = (E_p - \mu) c_\sigma^\dagger \ . \tag{A.15}$$

We note that (A.15) is simply relation (A.9), as such, one may apply (A.10) to (A.6), whence we obtain

$$\begin{aligned}
\langle N_\lambda \rangle &= \frac{1}{\mathcal{Z}} \operatorname{Tr}\left(\underbrace{e^{-\frac{(H-\mu N)}{T}} c_\lambda^\dagger e^{\frac{(H-\mu N)}{T}}}_{(\mathrm{A.10})} e^{-\frac{(H-\mu N)}{T}} c_\lambda \right) \\
&= \frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\frac{(E_p-\mu)}{T}} c_\lambda^\dagger e^{-\frac{(H-\mu N)}{T}} c_\lambda \right) \\
&= e^{-\frac{(E_p-\mu)}{T}} \frac{1}{\mathcal{Z}} \operatorname{Tr}\left(c_\lambda^\dagger e^{-\frac{(H-\mu N)}{T}} c_\lambda \right) \, .
\end{aligned} \tag{A.16}$$

We now recall the trace cyclic property

$$\operatorname{Tr}(ABC) = \operatorname{Tr}(BCA) = \operatorname{Tr}(CAB) \tag{A.17}$$

and write

$$\langle N_\lambda \rangle = e^{-\frac{(E_p-\mu)}{T}} \frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\frac{(H-\mu N)}{T}} c_\lambda c_\lambda^\dagger \right) \, . \tag{A.18}$$

We rearrange $c_\lambda c_\lambda^\dagger$ and multiply both sides of (A.18) by $e^{\frac{E_p-\mu}{T}}$ to get

$$\begin{aligned}
\langle N_\lambda \rangle e^{\frac{E_p-\mu}{T}} &= \frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\frac{(H-\mu N)}{T}} \left(-c_\lambda^\dagger c_\lambda + 1 \right) \right) \\
&= \frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\frac{(H-\mu N)}{T}} \left(-N_\lambda + 1 \right) \right) \\
&= - \underbrace{\frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\frac{(H-\mu N)}{T}} N_\lambda \right)}_{=\langle N_\lambda \rangle} + \underbrace{\frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\frac{(H-\mu N)}{T}} \right)}_{=1} \\
&= - \langle N_\lambda \rangle + 1 \, .
\end{aligned} \tag{A.19}$$

Finally, we isolate $\langle N_\lambda \rangle$ in (A.19) and write

$$\langle N_\lambda \rangle = \left[e^{\frac{E_p-\mu}{T}} + 1 \right]^{-1} \, . \tag{A.20}$$

We notice, however, that none of the quantities in the right-hand side of (A.20) depend on λ. In other words, we have

$$\langle N_\lambda \rangle = \langle N_{\lambda'} \rangle \, . \tag{A.21}$$

As such, the sum over λ (for given γ_f terms) results in

$$\sum_\lambda \langle N_\lambda \rangle = \gamma_f \, \langle N_\lambda \rangle \, , \tag{A.22}$$

where in our case we sum $\gamma_f = 6 = 2_{\text{spin}} \times 3_{\text{color}}$ terms.

We have therefore shown that

$$\sum_{\lambda} \langle N_\lambda \rangle = \gamma_f \left[e^{\frac{E_p - \mu}{T}} + 1 \right]^{-1} \ . \tag{A.23}$$

Evidently, an analogous calculation yields the result for antiparticles

$$\sum_{\lambda} \langle \overline{N}_\lambda \rangle = \gamma_f \left[e^{\frac{E_p + \mu}{T}} + 1 \right]^{-1} \ . \tag{A.24}$$

This proves relations (2.66).

B. Proof of the Jacobian Identity

We prove the Jacobian identity (3.43) up to the three-dimensional case. In what follows we reproduce the demonstration made in [30]. We denote the canonical variables associated with the non-deformed Heisenberg algebra by the capitalized letters X_i and P_i and the ones associated with a generally deformed algebra by the lower case letters x_i and p_i.

We begin by denoting $x_i = A_{2i-1}$, $p_i = A_{2i}$. The derivative of A_j with respect to X_i is denoted as $A_{j,2i-1}$ and the one with respect to P_i, by $A_{j,2i}$. We then have

$$\{A_i, A_j\} = \sum_{k=1}^{D}(A_{i,2k-1}A_{j,2k} - A_{i,2k}A_{j,2k-1}) \ . \tag{B.1}$$

Let us first prove the following identity

$$J = \frac{\partial\,(x_1, p_1, \ldots, x_D, p_D)}{\partial\,(X_1, P_1, \ldots, X_D, P_D)} = \frac{1}{2^D D!}\sum_{i_1,\ldots i_{2D}=1}^{2D} \varepsilon_{i_1\ldots i_{2D}}\{A_{i_1}, A_{i_2}\}\ldots\{A_{i_{2D-1}}, A_{i_{2D}}\} \ , \tag{B.2}$$

where $\varepsilon_{i_1\ldots i_{2D}}$ is the Levi-Civita symbol. The right-hand side of (B.2) is equal to

$$\begin{aligned}
J &= \frac{1}{2^D D!}\sum_{i_1,\ldots i_{2D}=1}^{2D} \varepsilon_{i_1\ldots i_{2D}}\{A_{i_1}, A_{i_2}\}\cdot\cdots\cdot\{A_{i_{2D-1}}, A_{i_{2D}}\} \\
&= \frac{1}{2^D D!}\sum_{i_1,\ldots i_{2D}=1}^{2D} \varepsilon_{i_1\ldots i_{2D}}\sum_{j_1=1}^{D}(A_{i_1,2j_1-1}A_{i_2,2j_1} - A_{i_1,2j_1}A_{i_2,2j_1-1})\ldots \\
&\times\sum_{j_D=1}^{D}(A_{i_{2D-1},2j_D-1}A_{i_{2D},2j_D} - A_{i_{2D-1},2j_D}A_{i_{2D},2j_D-1}) \\
&= \frac{1}{D!}\sum_{j_1,\ldots j_D}\sum_{i_1,\ldots i_{2D}} \varepsilon_{i_1\ldots i_{2D}}A_{i_1,2j_1-1}A_{i_2,2j_1}\ldots A_{i_{2D-1},2j_D-1}A_{i_{2D},2j_D}.
\end{aligned} \tag{B.3}$$

The Levi-Civita symbol is antisymmetric with respect to any indexes permutation, thus

$$\sum_{i_1,i_2}\varepsilon_{i_1\ldots i_{2D}}A_{i_1,2j_1-1}A_{i_2,2j_1} = -\sum_{i_1,i_2}\varepsilon_{i_1\ldots i_{2D}}A_{i_1,2j_1}A_{i_2,2j_1-1} \ . \tag{B.4}$$

Taking into account the fact that

$$\sum_{i_1,\dots i_{2D}} \varepsilon_{i_1\dots i_{2D}} A_{i_1,2j_1-1} A_{i_2,2j_1} \dots A_{i_{2D-1},2j_D-1} A_{i_{2D},2j_D} =$$
$$\det \begin{pmatrix} A_{1,2j_1-1} & A_{1,2j_1} & \dots & A_{1,2j_D-1} & A_{1,2j_D} \\ A_{2,2j_1-1} & A_{2,2j_1} & \dots & A_{2,2j_D-1} & A_{2,2j_D} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{2D,2j_1-1} & A_{2D,2j_1} & \dots & A_{2D,2j_D-1} & A_{2D,2j_D} \end{pmatrix}$$

and using the determinant properties, it then becomes evident that in (B.3) only the terms with different j contribute to the final result. All terms with $j_1 \neq j_2 \neq \dots \neq j_{2D}$ are equal. The number of these terms is $D!$. Hence,

$$\begin{aligned} J &= \frac{1}{D!} \sum_{j_1,\dots j_D} \sum_{i_1,\dots i_{2D}} \varepsilon_{i_1\dots i_{2D}} A_{i_1,2j_1-1} A_{i_2,2j_1} \dots A_{i_{2D-1},2j_D 1} A_{i_{2D},2j_D} \\ &= \sum_{i_1,\dots i_{2D}} \varepsilon_{i_1\dots i_{2D}} A_{i_1,1} A_{i_2,2} \dots A_{i_{2D-1},2D-1} A_{i_{2D},2D} \\ &= \det(A_{ij}) \ , \end{aligned} \tag{B.5}$$

which proves identity (B.2).

The right-hand side of (B.2) contains $(2D)!$ terms, each of them is a product of D Poisson brackets. Due to the skew-symmetry of the Poisson bracket, some of the terms are equal and the total number of terms can be reduced to $(2D-1)!!$. In what follows, we explicitly list the Jacobian terms for the cases of one, two, and three dimensions.

For the one-dimensional case, we have the trivial expression

$$\frac{\partial\,(x,p)}{\partial\,(X,P)} = \{x,p\} \ . \tag{B.6}$$

For the two-dimensional case, we have

$$\frac{\partial\,(x_1,p_1,x_2,p_2)}{\partial\,(X_1,P_1,X_2,P_2)} = \{x_1,p_1\}\{x_2,p_2\} - \{x_1,x_2\}\{p_1,p_2\} - \{x_1,p_2\}\{x_2,p_1\} \ . \tag{B.7}$$

And for the three-dimensional case, we obtain

$$\begin{aligned}
&\frac{\partial\left(x_1, p_1, x_2, p_2, x_3, p_3\right)}{\partial\left(X_1, P_1, X_2, P_2, X_3, P_3\right)}=\left\{x_1, p_1\right\}\left\{x_2, p_2\right\}\left\{x_3, p_3\right\} \\
&-\left\{x_1, p_3\right\}\left\{p_1, p_2\right\}\left\{x_2, x_3\right\}-\left\{x_1, p_2\right\}\left\{x_2, p_1\right\}\left\{x_3, p_3\right\} \\
&-\left\{x_1, p_3\right\}\left\{x_2, p_2\right\}\left\{x_3, p_1\right\}-\left\{x_1, p_1\right\}\left\{x_2, p_3\right\}\left\{x_3, p_2\right\} \\
&+\left\{x_1, x_2\right\}\left\{p_1, p_3\right\}\left\{x_3, p_2\right\}+\left\{x_1, p_3\right\}\left\{x_2, p_1\right\}\left\{x_3, p_2\right\} \\
&-\left\{x_1, x_2\right\}\left\{p_2, p_3\right\}\left\{x_3, p_1\right\}+\left\{x_1, p_2\right\}\left\{x_2, x_3\right\}\left\{p_1, p_3\right\} \\
&-\left\{x_1, x_3\right\}\left\{p_1, p_3\right\}\left\{x_2, p_2\right\}+\left\{x_1, x_3\right\}\left\{x_2, p_1\right\}\left\{p_2, p_3\right\} \\
&+\left\{x_1, x_3\right\}\left\{p_1, p_2\right\}\left\{x_2, p_3\right\}-\left\{x_1, x_2\right\}\left\{p_1, p_2\right\}\left\{x_3, p_3\right\} \\
&-\left\{x_1, p_1\right\}\left\{x_2, x_3\right\}\left\{p_2, p_3\right\}+\left\{x_1, p_2\right\}\left\{x_2, p_3\right\}\left\{x_3, p_1\right\} \ .
\end{aligned} \tag{B.8}$$

It is not difficult to see that in order to obtain such an expression, one needs to start from the term $\{x_1, p_1\} \cdot \cdots \cdot \{x_D, p_D\}$ and add to it all possible permutations, factor multiplying each term by either $+1$ for even permutations or by -1 for odd permutations. In the above expression each Poisson bracket x_i is placed before p_j, with x_i before x_j if $j > i$, and p_i before p_j if $j > i$.

In the particular case of Poisson brackets (3.37)-(3.39) induced by the Kempf deformed algebra, the Jacobian (B.8) simply yields

$$\frac{\partial\left(x_1, p_1, x_2, p_2, x_3, p_3\right)}{\partial\left(X_1, P_1, X_2, P_2, X_3, P_3\right)}=\left(1+\beta p^2\right)^3 \ . \tag{B.9}$$

One can easily see that, in cases for generic (3.34)-(3.36) where only the first term $\{x_1, p_1\} \cdot \cdots \cdot \{x_D, p_D\}$ contributes to the computation of the Jacobian (as is the case for the Kempf-induced deformation), one obtains

$$J=\prod_{i=1}^{D} f_{ii}(x, p) \ . \tag{B.10}$$

This, of course, implies relation (3.43), where $f_{ii}(x, p) = (1 + \beta p^2)$.

C. 3+1-Dimensional Coherent States Approach

We concisely derive the momentum expansions of bosonic fields in the coherent states coordinate approach of noncommutativity for the case of a four-dimensional Minkowski spacetime. This is a reduction of the general even-dimensional case studied in the fourth chapter, specifically, we follow the general ideas of [56]. The results obtained here naturally extend to the fermionic fields; this justifies relations (6.1) and (6.2).

We first define the commutation relation

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \ , \tag{C.1}$$

where

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 \\ 0 & 0 & -\theta_2 & 0 \end{pmatrix} \ . \tag{C.2}$$

We then construct the ladder operators:

$$A_1 \equiv \frac{1}{\sqrt{2}}\left(X^0 + iX^1\right) \ , \quad A_2 \equiv \frac{1}{\sqrt{2}}\left(X^2 + iX^3\right) , \tag{C.3}$$

$$A_1^\dagger \equiv \frac{1}{\sqrt{2}}\left(X^0 - iX^1\right) \ , \quad A_2^\dagger \equiv \frac{1}{\sqrt{2}}\left(X^2 - iX^3\right) \ , \tag{C.4}$$

which obey the commutation relation

$$\left[A_j, A_k^\dagger\right] = \delta_{jk}\,\theta_j \ . \tag{C.5}$$

Given the coherent state defined by

$$|\alpha\rangle \equiv \prod_j \exp\left(\frac{\overline{\alpha}_j A_j - \alpha_j A_j^\dagger}{\theta_j}\right)|0\rangle \ , \tag{C.6}$$

we may apply our definitions to the general plane wave relation (4.36), which reduces to (after we write the noncommutative spacetime coordinates as functions of the ladder operators):

$$\langle\alpha|e^{p_\mu X^\mu}|\alpha\rangle = \exp\left[-\frac{1}{4}\left[\theta_1({p_0}^2 + {p_1}^2) + \theta_2({p_2}^2 + {p_3}^2)\right] + ip_\mu x^\mu\right] \ , \tag{C.7}$$

where $x^\mu \equiv \langle\alpha|\, X^\mu \,|\alpha\rangle$ is the usual commutative spacetime coordinate. We then apply the Lorentz invariance constraint $\theta_j = \theta$ and find

$$\langle\alpha|e^{p_\mu X^\mu}|\alpha\rangle = \exp\left[-\frac{\theta}{4}p_\mu p^\mu + ip_\mu x^\mu\right] \ . \tag{C.8}$$

Now, we write the Fourier momentum representation of a bosonic field in four-dimensional noncommutative spacetime as (following the definition for conventional QFT [18]):

$$\phi\left(X\right) = \frac{1}{\left(2\pi\right)^{3/2}} \int d^4p \; \delta(p^\mu p_\mu - m^2)\, H(p^0) \left[a(p^\mu) e^{-ip_\mu X^\mu} + b^\dagger(p^\mu) e^{ip_\mu X^\mu}\right] \;, \tag{C.9}$$

where $H(p^0)$ is the well-known step function.

We may now take the mean value $\langle\alpha|\, \phi(X) \,|\alpha\rangle = \phi(x)$ and apply (C.8) to find

$$\phi(x) = \frac{1}{\left(2\pi\right)^{3/2}} \int d^4p \; \delta(p^\mu p_\mu - m^2)\, H(p^0) \left[a(p^\mu) e^{-ip_\mu x^\mu} + b^\dagger(p^\mu) e^{ip_\mu x^\mu}\right] e^{-\frac{\theta}{4} p^\mu p_\mu} \;. \tag{C.10}$$

We recall that, for real numbers x and a, with $a > 0$, we have

$$\delta(x^2 - a^2) = \frac{1}{2a} \left[\delta(x - a) + \delta(x + a)\right] \;. \tag{C.11}$$

Since

$$p^\mu p_\mu - m^2 = (p^0)^2 - (E_{\mathbf{p}})^2 \;, \tag{C.12}$$

and

$$d^4p = dp^0\, dp^1\, dp^2\, dp^3 = dp^0\, d\mathbf{p} \;, \tag{C.13}$$

we may rewrite (C.10) as

$$\begin{aligned} \phi(x) = {} & \frac{1}{\left(2\pi\right)^{3/2}} \int dp^0\, d\mathbf{p}\; \frac{1}{2E_{\mathbf{p}}} \left[\delta(p^0 - E_{\mathbf{p}}) + \delta(p^0 + E_{\mathbf{p}})\right] \; H(p^0) \\ & \times \left[a(p^\mu) e^{-ip_\mu x^\mu} + b^\dagger(p^\mu) e^{ip_\mu x^\mu}\right] e^{-\frac{\theta}{4} p^\mu p_\mu} \;. \end{aligned} \tag{C.14}$$

It is convenient now to renormalize the bosonic ladder operators as [18]

$$a(\mathbf{p}) = \frac{a(p^\mu)}{\sqrt{2k^0}} \;, \tag{C.15}$$

$$b^\dagger(\mathbf{p}) = \frac{b^\dagger(p^\mu)}{\sqrt{2k^0}} \;. \tag{C.16}$$

And since we have that, for a function $f(p^0)$,

$$\left\{ \begin{array}{l} \int dp^0\, \delta(p^0 - E_{\mathbf{p}}) H(p^0) f(p^0) = f(E_{\mathbf{p}}) \\ \int dp^0\, \delta(p^0 + E_{\mathbf{p}}) H(p^0) f(p^0) = 0 \end{array} \right. \;, \tag{C.17}$$

we integrate (C.14) over p^0 and obtain

$$\phi(x) = \frac{1}{\left(2\pi\right)^{3/2}} \int d\mathbf{p} \; \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a(\mathbf{p}) e^{-ip_\mu x^\mu} + b^\dagger(\mathbf{p}) e^{ip_\mu x^\mu}\right] e^{-\frac{\theta}{4} p^\mu p_\mu} \;. \tag{C.18}$$

Similarly, we may find

$$\phi^{\dagger}(x) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p} \, \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a^{\dagger}(\mathbf{p}) e^{ip_{\mu}x^{\mu}} + b(\mathbf{p}) e^{-ip_{\mu}x^{\mu}} \right] e^{-\frac{\theta}{4} p^{\mu} p_{\mu}} \ . \tag{C.19}$$

Equations (C.18) and (C.19) correspond to the momentum representations of bosonic fields in four-dimensional Minkowski spacetime.

It is easy to see that an analogous procedure leads to the fermionic Fourier expansions given by (6.1) and (6.2).