Open Access Library Journal
2023, Volume 10, e10192
ISSN Online: 2333-9721
ISSN Print: 2333-9705

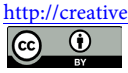
Fundamental Connections in Differential Geometry: Quantum Field Theory, Electromagnetism, Chemistry and Fluid Mechanics

Jorge Zabadal[1], Ederson Staudt[2], Ádnei Marinho[3], Vinícius Ribeiro[1]

[1]Departamento Interdisciplinar do Campus Litoral Norte, Universidade Federal do Rio Grande do Sul (UFRGS), Tramandaí, Brasil
[2]Universidade Federal da Fronteira Sul (UFFS)-Campus Chapecó, Chapecó, Brasil
[3]Universidade Federal Rural da Amazônia (UFRA)-Campus Parauapebas, Parauapebas, Brasil
Email: jorge.zabadal@ufrgs.br, ederson.staudt@uffs.edu.br, admarinho@gmail.com, vinicius.gadis@ufrgs.br

How to cite this paper: Zabadal, J., Staudt, E., Marinho, Á. and Ribeiro, V. (2023) Fundamental Connections in Differential Geometry: Quantum Field Theory, Electromagnetism, Chemistry and Fluid Mechanics. *Open Access Library Journal*, **10**: e10192.
https://doi.org/10.4236/oalib.1110192

Received: April 28, 2023
Accepted: September 4, 2023
Published: September 7, 2023



Abstract

This work presents novel hydrodynamic formulations that reconcile the continuum hypothesis with the emergence of electromagnetic interactions among molecules from fundamental principles. Two models are proposed: a relativistic version of the Navier-Stokes equations derived from commutation relations, and a Helmholtz-like system obtained by applying the Hodge operator to the extended Navier-Stokes equations. Preliminary analysis suggests that the second model, with its nonlinear terms serving as a generalized current, can reproduce microscopic quantum effects. It shows promise for generating self-consistent field equations via Bäcklund transformations, remaining valid across all scales despite the breakdown of the continuum hypothesis.

Subject Areas

Mathematical Analysis, Modern Physics

Keywords

Commutation Relations, Extended Navier-Stokes Equations, Extended Helmholtz Equations, Interaction Terms, Quantum Behavior

1. Introduction

One of the most common questions about mathematical modelling concerns about the reliability of partial differential equations. We usually answer this

question by talking about variational principles, Euler-Lagrange equations and Noether's theorem. We also use them to present a number of continuous symmetries admitted by some equations, which correspond to certain conservation laws, and then assume that these laws are totally reliable.

However, this answer eventually leads to doubts about the circumstances in which we should accept conservation laws. Taking into account that uncertainty principle allows violating some of these laws during small time intervals, and that there exists a scale for which the continuous hypothesis is no longer admissible [1], it seems difficult to find a definitive answer which is not simply based on the agreement with experimental data.

Another common question, concerning about the nature of the potentials, was partially solved in the realm of differential geometry. Nevertheless, when one asks about how forces act over material bodies, we only mention connections, bundles and "minimal" coupling, as we were confident that no further questions could arise from the profound implications of these concepts over the conventional phenomenological point of view.

It occurs that the only way to ensure that a given principle is totally reliable consists ultimately in showing that it comes from an identity. At first, this answer also seems unsatisfactory, because it apparently leads to a trivial conclusion about the underlying dynamics of the equations of motion. However, it is possible to show that new crucial information is generated from commutation relations. In that sense, we can realize that commutation relations are widely used in the formulation of models as well as in the resolution of the resulting equations. In the case where de application of a commutation relation on any field reduces to an identity, we obtain a model that is inherently exact, unlike conservation laws that are subject to experimental verification [2].

More specifically, there are homogeneous commutation relations which produce physical laws in macroscopic scale, when some assumptions about the regularity of the dependent variables are considered. This assumption does not remain valid in microscopic scale, so the resulting inhomogeneous commutation relations must be supplemented by the corresponding anti-commutation ones. Such a procedure generates a self-consistent field model where bosonic and fermionic parts are not only strongly coupled but also undistinguishable from naive traditional considerations about spin and statistics. Therefore, it is essential to construct models that are minimally biased in the pattern identification and the corresponding assignment of concepts to them. This formulation should not consist of a differential equation but an evolution equation involving only differential operators. In order to avoid conceptualization, that is, identifying a field with a particular physical quantity, the basic formulation must be independent of the field on which the operators are applied.

In that sense, the first model formulated is a relativistic version of the Navier-Stokes equations containing an extra term related to interactions. This system is obtained from commutation relations instead of classical conservation laws. In

this model, the zero[th] component of the velocity vector is the non-dimensional mass density, which can be regarded as the projection of the velocity along the time axis. This definition is consistent with the definitions of the four vectors current in Maxwell and Dirac equations.

The second model is analogous to the classical system of Helmholtz equations. This system is achieved by applying the hodge operator over the exterior derivative of the extended Navier-Stokes equations. Since the hodge of the exterior derivative is a natural extension of the rotational operator to four dimensions, the extended Helmholtz equations are readily obtained from the corresponding Navier-Stokes ones by means of a conventional procedure.

A preliminary analysis of the classical model obtained reveals that it contains sufficient degrees of freedom to emulate quantum behavior. More specifically, the nonlinear terms of the extended Helmholtz equations, which also constitutes natural extensions of advection and vortex-stretching terms, play the role of a generalized current, in the sense that it can reproduce several effects occurring in microscopic scale. This model is a promising starting point to generate self-consistent field (SCF) equations for quantum physics via Bäcklund transformations. Bearing in mind that commutation relations reduce to identities when applied over sufficiently smooth fields, the SCF equations remain valid in any scale, even below a limit one for which the continuous hypothesis breaks down.

In the next section, this idea will be introduced by obtaining Maxwell equations from commutation relations.

2. Commutation Relations, Conservation Laws and Dynamical Equations

This section begins with two examples, which explain the process in which a commutation relation is converted into a physical law. The first example is the commutation relation between time derivative and the curl operator, which can be converted into Faraday's law. In fact, equation

$$\left[\partial_t, \nabla\times\right] = 0, \tag{1}$$

generates an identity when the commutator is applied over a regular field *f*, *i.e.*, the expression

$$\partial_t \nabla\times f - \nabla\times\partial_t f = 0 \tag{2}$$

is automatically satisfied for any smooth vector field *f*. Nevertheless, it can be recognized as the Faraday's law after adopting the following conventions:

$$\nabla\times f = B \tag{3}$$

$$\partial_t f = -E. \tag{4}$$

Hence, this law, given by

$$\partial_t B + \nabla\times E = 0 \tag{5}$$

was obtained from an identity by assigning a specific meaning to certain fields.

Equation (5) can be generalized by including functions belonging to the null space of both operators involved in the commutation relation. In fact, a more general form of Equation (5) can be expressed as

$$\partial_t \left[B + g(x,y,z) \right] + \nabla \times \left(E + \nabla h \right) \tag{6}$$

In this equation, g and h are arbitrary functions. These gauge type terms also appears in another example of differential equation obtained from commutation relation:

$$\left[\partial_t, \nabla \cdot \right] = 0. \tag{7}$$

From this relation, two differential equations can be produced. The first is a conservation law, whose explicit form is obtained by choosing

$$\nabla \cdot f = \rho \tag{8}$$

and

$$\partial_t f = -j. \tag{9}$$

Thus, the identity

$$\partial_t \nabla \cdot f - \nabla \cdot \partial_t f \equiv 0 \tag{10}$$

becomes the continuity equation:

$$\partial_t \rho + \nabla \cdot j = 0. \tag{11}$$

The second equation is a dynamical law, which is generated by reduction of order. Considering that the definition of Equation (8) can be recognized as the Gauss law, given by

$$\nabla \cdot E = \rho, \tag{12}$$

Equation (12) can be written, after using Equation (7), as

$$\nabla \cdot \left(\partial_t E + j \right) = 0. \tag{13}$$

Once the null space of the divergent operator is composed by the sum of a purely rotational field with the gradient of some harmonic function, Equation (13) is equivalent to

$$\partial_t E + j = \nabla \times B + \nabla h \left(\nabla^2 h = 0 \right). \tag{14}$$

This equation can be expressed in a more usual form:

$$\partial_t E - \nabla \times B = \nabla h - j. \tag{15}$$

The former result is a generalization of the Ampère's law, given by

$$\partial_t E - \nabla \times B = -j. \tag{16}$$

because there is an extra term (∇h) which is absent in the original formulation. As will be showed later, this additional field plays the role of a pressure gradient in the corresponding hydrodynamic model.

As in the former case, Equation (15) can be generalized by including gauge terms:

$$\partial_t \left(E + p(x,y,z) \right) - \nabla \times \left(B + \nabla q \right) = -j. \tag{17}$$

It occurs because $\partial_t p(x,y,z)=0$ and $\nabla\times\nabla q=0$. In this equation, the field h was intentionally omitted, in order to preserve the original form of the Ampère's law. Finally, since B was defined as the curl of f, it is possible to establish another equation to complete the model for electromagnetism:

$$\nabla\cdot B=0. \tag{18}$$

Therefore, the Maxwell Equations (6), (12), (17), and (18) are deducible from commutation relations by means of contextualization. In other words, these sets of physical laws were achieved from commutation relations by ascribing a physical meaning to some fields. In this process, first order operators applied over the same field f produce new fields whose meaning is familiar.

3. A New Formulation for Fluid Mechanics

The former argument induces to ask whether one can apply the same strategy to obtain a mathematical model for other areas of physics. For the specific case of fluid mechanics, the commutation relation employed to obtain Navier-Stokes type equations, namely $[\partial_t,\nabla\cdot]=0$, also generates the Ampère's law.

However, before obtaining the equations of motion for fluid mechanics, it is important to establish a correspondence between the dependent variables in both models. For instance, the field f, identified with the Maxwell vector potential A_μ, in the electromagnetic model, corresponds to the velocity vector in hydrodynamics. This correspondence arises after differentiating Equation (17) respect to the time variable:

$$\partial_{tt}\left(E+p(x,y,z)\right)-\partial_t\nabla\times\left(B+\nabla q\right)=-\partial_t j. \tag{19}$$

Once $[\partial_t,\nabla\times]=0$ and function p does not depend on time, it yields

$$\partial_{tt}E-\nabla\times\partial_t\left(B+\nabla q\right)=-\partial_t j. \tag{20}$$

In this equation, it is possible to eliminate the time derivative of the magnetic induction. Substituting Equation (6) in Equation (20), a Klein-Gordon type equation is obtained:

$$\partial_{tt}E+\nabla\times\nabla\times E=-\partial_t j. \tag{21}$$

It occurs because

$$\nabla\times\nabla\times E=\nabla\nabla\cdot E-\nabla\cdot\nabla E. \tag{22}$$

However, due to (12), the first term in the right hand side of (22) results

$$\nabla\nabla\cdot E=\nabla\rho, \tag{23}$$

and the second term is the Laplacian of the electric field:

$$\nabla\cdot\nabla E=\nabla^2 E. \tag{24}$$

Hence, Equation (21) becomes

$$\partial_{tt}E-\nabla^2 E=\nabla\rho-\partial_t j. \tag{25}$$

In an analogous way, another version of the Klein-Gordon equation is obtained by differentiating either Equation (5) or Equation (6) respect to t. For simplicity,

Equation (5) is chosen to be differentiated, resulting

$$\partial_{tt} B + \nabla \times \partial_t E = 0. \tag{26}$$

Replacing Equation (15) in Equation (26), it yields

$$\partial_{tt} B + \nabla \times \left(\nabla \times B - j \right) = 0. \tag{27}$$

Since

$$\nabla \times \nabla \times B = \nabla \nabla \cdot B - \nabla \cdot \nabla B = 0 - \nabla^2 B. \tag{28}$$

Equation (27) can be written in the form

$$\partial_{tt} B - \nabla^2 B = \nabla \times j. \tag{29}$$

The structure of the source terms in Equations (25) and (29) leads to conclude that the Maxwell potential also must obey an inhomogeneous Klein-Gordon equation, whose source is the current *j*. Indeed, this result is a well-known model [3]:

$$\partial_{tt} A^{\mu} - \nabla^2 A^{\mu} = -j. \tag{30}$$

For classical systems, the current is defined as

$$j = \rho v, \tag{31}$$

where v denotes the velocity vector. The source in Equation (30) accounts for few point charges travelling with a certain velocity. Therefore, this equation describes a flow of a very low-density gas, composed by few charged particles, which do not interact with each other, but only with the external potential *A*. Hence, each particle travels along independent paths whose trajectories can be described by parametric equations. For dense gases and liquids, the current corresponds to the inertial terms of the Navier-Stokes equation. In this case, the fluid is regarded as a continuous medium, so the material derivative of the velocity field accounts for the parametric equations in an implicit way.

Once established a qualitative correspondence between the current *j* and the advection term $v \cdot \nabla v$, it is necessary to replace the charge density in Maxwell equations by a non-dimensional mass density for the hydrodynamic model. The immediate consequence of this assumption is that v must be a four vector whose zeroth component is the non-dimensional density. Hence, the hydrodynamic model consists in four nonlinear Klein-Gordon equations given by

$$\partial_{tt} v - \nabla^2 v = -v \cdot \nabla v. \tag{32}$$

As in the former examples, this model can be generalized by including extra terms, such as the pressure gradient, which generalizes Equation (15) in the context of electromagnetic theory (see Equation (15)). In short, once the Maxwell potential is substituted by the four vectors defined as

$$v = \left(\rho, u, v, w \right), \tag{33}$$

the respective current is obtained by taking the four dimensional divergence over the tensor product

$$v^{\mu\nu} = v \otimes v = \left[\rho, u, v, w\right]^{\mathrm{T}} \left[\rho, u, v, w\right] = \begin{bmatrix} \rho^2 & \cdots & \rho w \\ \vdots & \ddots & \vdots \\ w\rho & \cdots & w^2 \end{bmatrix} = v^{\mu} v^{\nu}. \tag{34}$$

In fact,

$$j = \partial_{\mu} v^{\mu\nu} = \partial_{\mu} \left(v^{\mu} v^{\nu} \right) = \left(\partial_{\mu} v^{\mu} \right) v^{\nu} + v^{\mu} \partial_{\mu} v^{\nu}, \tag{35}$$

which is equivalent to

$$j = \left(\nabla \cdot v \right) v + \left(v \cdot \nabla \right) v = \left(\partial_{t} \rho \right) v + \left(v \cdot \nabla \right) v. \tag{36}$$

Thus, even for incompressible fluids, the hydrodynamic model contains additional terms, which comes from a natural extension of the Gauss and Ampère's Law:

$$\partial_{tt} v - \nabla^2 v = -\nabla \cdot \left(v \otimes v \right) + \nabla h. \tag{37}$$

Once v_0 is the density function, the two last terms cancels each other due to the continuity equation. Moreover, the scalar function is readily identified with the pressure field, *i.e.*,

$$h = \frac{-p}{\rho}. \tag{38}$$

Therefore, equation (36) reduces to

$$\partial_{tt} v - \nabla^2 v = -\nabla \cdot \left(v \otimes v \right) - \frac{\nabla p}{\rho}, \tag{39}$$

or

$$\partial_{tt} v + \left(v \cdot \nabla \right) \cdot v = \nabla^2 v - \frac{\nabla p}{\rho}. \tag{40}$$

In cartesian coordinates, this model is written as

$$\frac{\partial^2 \rho}{\partial t^2} + \rho \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = \nabla^2 \rho - \frac{1}{\rho} \frac{\partial p}{\partial t}. \tag{41}$$

$$\frac{\partial^2 u}{\partial t^2} + \rho \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nabla^2 u - \frac{1}{\rho} \frac{\partial p}{\partial x}. \tag{42}$$

$$\frac{\partial^2 v}{\partial t^2} + \rho \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \nabla^2 v - \frac{1}{\rho} \frac{\partial p}{\partial y}. \tag{43}$$

$$\frac{\partial^2 w}{\partial t^2} + \rho \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \nabla^2 w - \frac{1}{\rho} \frac{\partial p}{\partial z}. \tag{44}$$

Aside from the second order time derivatives, which are responsible to a memory effect, this model also accounts for electromagnetic interactions. This mechanism is not contemplated in the original form of the Navier-Stokes equations. Notice that the zero$^{\text{th}}$ component of the velocity field acts as an interaction term. This term ensures that even in the limit of vanishing density, two parallel streamlines can eventually generate a vortex if the corresponding velocities are different. This effect resembles two particles travelling at different speed,

which interact with each other at relatively small distances. This scattering effect is a noticeable feature whose consequences will be discussed in the next sections.

4. The Generalized Vorticity

Equations (41) to (44) constitute a generalization of the Navier-Stokes system, which can be easily converted into a new version of the Helmholtz equations. Applying the exterior derivative over the system, it results a new one containing six nonlinear Klein-Gordon equations, whose current is defined as the difference between two nonlinear terms, and the vorticity arises as a tensor. This structure also appears in an extension of the Newton's law. This point of view is useful to confirm the correspondence between the velocity field and the Maxwell potential. The starting point to obtain the vorticity tensor from the classical law

$$F = ma, \tag{45}$$

is to extend it to four dimensions as

$$-\nabla V = m\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2}, \tag{46}$$

where V stands for the Coulomb potential and the second order time derivative can be written as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) = \frac{\mathrm{d}v^\mu}{\mathrm{d}t} = \partial_0 v^\mu. \tag{47}$$

However, the Coulomb potential is the zero$^{\text{th}}$ component of the Maxwell vector field, hence,

$$\nabla V = \nabla A^0 = \partial_\mu A^0. \tag{48}$$

Substituting (47) and (46) in (45), it yields

$$\partial_\mu A^0 = m\partial_0 v^\mu. \tag{49}$$

This equation can suffer another generalization by swapping the fixed index from 0 to 3:

$$\partial_\mu A^\nu = m\partial_\nu v^\mu. \tag{50}$$

The skew symmetric form associated to Equation (50), namely

$$\partial_\mu A^\nu - \partial_\nu A^\mu = m\left(\partial_\nu u^\mu - \partial_\mu u^\nu\right), \tag{51}$$

contains two exterior derivatives: the Maxwell tensor and the generalized vorticity one, which consists in an extension of the vorticity vector which accounts for viscous effects and electromagnetic interactions. In a complete analogy with electromagnetism, the Maxwell tensor, defined as

$$F^{\nu\mu} = \partial_\mu A^\nu - \partial_\nu A^\mu = \partial A, \tag{52}$$

becomes

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{53}$$

when expressed in terms of the electric Field and the magnetic induction. Analogously, the vorticity tensor, defined as

$$\omega^{\nu\mu} = \partial_\mu v^\nu - \partial_\nu v^\mu, \tag{54}$$

is written as

$$\omega^{\mu\nu} = \begin{bmatrix} 0 & -\varepsilon_x & -\varepsilon_y & -\varepsilon_z \\ \varepsilon_x & 0 & \omega_z & -\omega_y \\ \varepsilon_y & -\omega_z & 0 & \omega_x \\ \varepsilon_z & \omega_y & -\omega_x & 0 \end{bmatrix}. \tag{55}$$

When expressed in terms of the classical velocity vector and an additional field which accounts for compression and decelerating processes. These processes are characteristic of interaction potentials, whose connection with relativistic mechanisms will be explained as follows.

Imagine a two dimensional element suffering space-time shear stress in the *x*-*t* plane. In other words, after reaching a "solid interface" whose orientation is parallel to the time axis, the element is deformed by deviation of the mainstream from x to the time direction.

In this case, applying the classical no slip condition at this "wall" is equivalent to prescribe an interacting field. It must be taken into account that no distinction can be clearly verified between the actions of a "real boundary" or a short ranged electromagnetic field over the main flow. In fact, even in mathematical formulations there is no significant difference between advection terms and scalar potential ones. Although advection terms contain first derivatives and potentials arise in zeroth order terms, this difference is only apparent. More specifically, relativistic models can be formulated in the following tensor form:

$$\partial_\mu \Psi_\nu = U_\mu \Psi_\nu + T_{\mu\nu}. \tag{56}$$

This system constitutes Bäcklund transformations for Klein-Gordon type equations. Here Ψ represents the four vector wave function, U is a vector field and T a tensor one, which accounts for gauge potentials. In fact, applying the divergent operator over Equation (56), it yields

$$\partial^\mu \partial_\mu \Psi_\nu = \left(\partial^\mu U_\mu\right)\Psi_\nu + U_\mu \partial^\mu \Psi_\nu + \partial^\mu T_{\mu\nu}. \tag{57}$$

The second term in the right hand side of Equation (57) is clearly inertial. In this case, the components of the corresponding velocity vector are easily identified as U_μ. However, Equation (57) can be rewritten in such a way that the first derivatives of the wave functions no longer appear in the resulting system. Replacing Equation (56) in Equation (57) and regrouping terms, the following

auxiliary model is obtained:

$$\partial^{\mu}\partial_{\mu}\Psi_{\nu}=\left(\partial^{\mu}U_{\mu}+U_{\mu}U^{\mu}+U^{\mu}T_{\mu\nu}\right)\Psi_{\nu}+\partial^{\mu}T_{\mu\nu}. \tag{58}$$

This is a typical Klein-Gordon equation, which contains only second derivatives and zeroth order terms. These terms accounts for mass and scalar potentials, which are responsible to the scattering effect. Hence, advection terms are hidden in this structure, so there is no reason to distinguish between a vector potential field and a velocity one.

5. The Extended Advection Term and the Space-Time Vortex Stretching

After justifying the correspondence between velocity fields and interaction potentials, it is convenient to introduce the concepts of extended advection and space-time vortex stretching, whose relevance is crucial to understand interactions from a different point of view. The tensor form of the Helmholtz equation can be contracted by applying the hodge operator, resulting in a set of four equations in which the usual vorticity becomes a four vector. This procedure generates a new set of Klein-Gordon equations:

$$\partial_{tt}\omega-\nabla^{2}\omega=v\cdot\nabla\omega-\omega\cdot\nabla v. \tag{59}$$

Following the terminology of classical fluid mechanics, we define the terms in the right hand side as the extended advection $(v\cdot\omega)$ and the space-time vortex stretching ($\omega\cdot v$).

In a certain sense, these terms constitute a generalization of the classical current. It comes out that Maxwell equations represent the bosonic system of a complete set whose fermionic counterpart are the Dirac equations or any other system obtained from the Klein-Gordon model via factorization [4]. The current j in electromagnetism is defined in terms of a bilinear form involving the Dirac wave functions. In this bilinear form, the wave functions play the role of components of a velocity vector and the corresponding conjugates may be regarded as the vorticity field in a coarse-grained medium. It is important to keep in mind that in a scale for which the continuous hypothesis breaks down, it makes no sense to attribute identity to particles, so the internal and external degrees of freedom become undistinguishable. Therefore, the advection term and the vortex stretching work as discrete translational parcels whose nature is essentially analogous to a trembling motion. Roughly speaking, "zitterbewegung" would be emulated by the extended Helmholtz equation, provided that a four dimensional complex version of the original model can be deduced by symmetries.

At this point, it becomes convenient to summarize the former arguments in a wider principle. For each quantum model, which stands for the fermionic part of a system describing the behaviour of a physical phenomenon in microscopic scale, there exists a supplementary set of equations, which accounts for the respective bosonic part. The link between these models is identified by the potentials in the fermionic system and currents in the respective bosonic part. Al-

though the isolated models seem linear, the coupling between fermions and bosons is hidden in a unique nonlinear self-consistent system of equations. This system can be written in terms of either the wave functions only or exclusively in terms of the corresponding potentials. In the next section, this principle will be exploited in order to obtain a self-consistent model from the Klein-Gordon equation.

6. A Self Consistent Model for the Klein-Gordon Equation

A four dimensional system of equations for fluid mechanics seems to be a promising starting point to formulate self-consistent models for microscopic scale. In other words, the hydrodynamic model contains no currents, but only extended velocity fields, so it can be regarded as a "macroscopic self-consistent projection" of a complete set of equations in quantum field theory. A simple way to extend such a classical model to the microscopic scale consists in applying a generalized "analytic continuation", based on symmetries of quantum models or Bäcklund transformations. The possibility of simulating quantum behavior using classical models was implicitly pointed out by Giese [5], whose classical interpretation of the "zitterbewegung" apparently solved one of the most interesting puzzles of quantum mechanics: elucidating the underlying dynamics of the Dirac equations [4] [6] [7] [8] [9]. Besides, it was demonstrated that classical diffusion equations exhibit quantum behavior when the corresponding diffusion coefficient is an imaginary parameter. For practical purposes, applying an analytic continuation over a classical model constitutes a more convenient way to obtain self-consistent formulations and solutions than employing iterative procedures, such as "*ab initio*" calculations. These schemes require a high computational effort even to simulate the evolution of small quantum systems, such as in chemical reactions between inorganic molecules. Finding mappings between nonlinear classical models and microscopic ones seems to be a straightforward way to perform the simulation of huge molecular systems. This is the main purpose of the proposed formulation.

6.1. Bäcklund Transformations for the One Dimensional Klein-Gordon Equation

So far, the models obtained from commutation relations are closely related with Klein-Gordon type equations. This subsection is dedicated to exploit some non-trivial consequences of this fact, as well as to establish a nonlinear Klein-Gordon equation via Bäcklund transformations. This model is expressed in terms of a scalar potential, and represents a self-consistent version of the original Klein-Gordon equation [10] [11]. The corresponding solutions, given by Weierstrass p functions, present a quasi-periodic behaviour which allows representing some realistic multiparticle states using a single potential term.

The one dimensional Klein-Gordon equation, expressed exclusively in terms of an interaction potential, is given by

$$\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \frac{V^2}{2} \tag{60}$$

From this equation it is possible to obtain a system of nonlinear PDEs via split and reduction of order:

$$\frac{\partial V}{\partial x} = \Psi \tag{61}$$

$$\frac{\partial^2 V}{\partial t^2} - \frac{\partial \Psi}{\partial x} = \frac{V^2}{2} \tag{62}$$

Indeed, differentiating Equation (61) respect to x and adding to Equation (62) restores Equation (60). Imposing the consistency between two possible definitions of the third order derivative V_{xtt}, namely

$$\frac{\partial^2 V}{\partial x \partial t^2} = \frac{\partial^2 \Psi}{\partial x^2} + V \frac{\partial V}{\partial x} \tag{63}$$

and

$$\frac{\partial^2 V}{\partial x \partial t^2} = \frac{\partial^2 \Psi}{\partial t^2} \tag{64}$$

it yields

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial x^2} + V \frac{\partial V}{\partial x}. \tag{65}$$

Finally, using Equation (61) in order to eliminate V_x, the original Klein-Gordon arises:

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial x^2} + V\Psi. \tag{66}$$

Although Bäcklund transformations do not constitutes a simple way to isolate fermionic and bosonic parts of a quantum model, this approach is particularly useful to reinterpret and unify the underlying dynamics of some differential equations in classical and quantum physics. For example, the same procedure applied over a nonlinear version of the Schrödinger equation produces a KdV-type model by differentiation respect to x. This subject will be discussed in future works.

6.2. Bäcklund Transformations for the One Dimensional Klein-Gordon Equation

The nonlinear model obtained is readily solved in two steps: reduction to an ODE and mapping into a first order PDE. It occurs that all autonomous and homogeneous PDEs can be converted into an ODE by writing the derivatives respect to a new argument defined as a linear combination of the original independent variables. Defining a new independent variable as

$$u = c_0 + c_1 x + c_2 t \tag{67}$$

and rewriting the derivatives in terms of this argument, it results

$$\left(c_2^2 - c_1^2\right)\frac{\mathrm{d}^2 V}{\mathrm{d}u^2} = \frac{V^2}{2}. \tag{68}$$

The solutions of this nonlinear ODE are the Weierstrass p functions:

$$V = W_p\left(k, 0, c_0\right). \tag{69}$$

Notice that this solitonic solution can be easily extended to three dimensions, since $u(x,t)$ can be expressed as a linear combination of all space-time coordinates. The resulting travelling wave solution will be employed to reinterpret some relativistic effects whose consequences leads to elucidate certain interaction processes.

A closer look at the solutions given by Equation (69) reveals a new point of view about symmetries and conservation laws. Notice that some particular cases of these solutions resembles the classical Lennard-Jones potential.

7. Conclusions

This section begins with a summary of the main conclusions obtained so far, in order to emphasize the basic assumptions which will be employed as a starting point to a more profound discussion.

1) Commutation relations are more reliable than conservation laws, because it remains valid even when the second ones are violated.

2) Physical models arise naturally as a consequence of ascribing a particular meaning to the vector fields over which the operators involved in commutation relations are applied.

3) This procedure allows identify some extra terms which are absent in the original formulations.

4) Such a generalization always produces Klein-Gordon type equations.

5) The Klein-Gordon equation can be mapped into a nonlinear model containing only a scalar potential.

6) This nonlinear model can be readily converted into an ODE whose argument is defined as a linear combination of the original independent variables.

7) The explicit solutions of the resulting ODE are the Weierstrass p functions (W_p), whose local qualitative behaviour resembles the classical Lennard-Jones potential.

8) A single W_p function is capable to represent the physical behaviour of a multiparticle potential, so the computational effort required to proceed a simulation of many body quantum systems can be significantly reduced.

9) While the explicit solutions exhibit soliton behaviour, the corresponding space of implicit solutions represents realistic skyrmions.

These preliminary conclusions encourage us to step forward in the following direction: obtaining exact solutions to Yang-Mills type equations which represent not only instantons, but more comprehensive structures in space-time. The viability of the proposed study is justified by the fact that the classical vector identity, given by

$$\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla \cdot \nabla \tag{70}$$

can be immediately converted into a more general commutation relation in four

dimensions, namely

$$\left[\nabla,\nabla\cdot\right]=d*d \tag{71}$$

where $*$ is the Hodge operator and d stands for the exterior derivative. This inhomogeneous commutation relation can be converted into a Yang-Mills type model, after choosing a particular vector field over which the operators in both sides of Equation (71) could be applied. Choosing $f=A$ it results

$$\left[\nabla,\nabla\cdot\right]A=d*dA, \tag{72}$$

This equation has some interesting features whose similarity with the extended Helmholtz equation is noticeable. A preliminary analysis of the nonlinear terms in the right hand side of the hydrodynamic model furnishes strong evidence that some crucial features of microscopic scale can be reproduced by the extended Helmholtz equations. The term $v\cdot\omega$ is responsible not only by the transport of the vortices and dissipating them by generating high frequency perturbations in the velocity field, such as in the Kolmogorov cascade. It is also responsible by attractive forces between these coherent structures, which maintains the integrity of the vortex wake as it were a stable resonant molecular structure. The term $\omega\cdot v$ performs a kind of "parallel transport" of velocity vectors along "wiggly" space-time hyper surfaces. The combination of these two effects resembles some features of quantum field dynamics which have no counterpart in classical models, such as vacuum fluctuations, asymptotic freedom, particle creation and annihilation, as well as skyrmionic structures.

Even the most trivial feature associated with the advection term of the extended Helmholtz equation, e.g., the "scattering between two adjacent molecules", produces remarkable effects. One of the immediate consequences of the scattering effect is that turbulence may arise in almost any scale below a given upper limit, provided that coherent structures of different sizes can be produced and destroyed from "almost uniform" flows, depending on the magnitude of the free average velocity. In this point of view, viscosity appears as a measure of the average molecular weight of the clusters, which constitute the focuses of coherent motion. When a given cluster of molecules collides with other one or even with a solid wall, the coherent structure may be decomposed, producing smaller "oligomers", which are associated with regions of lower viscosity. This argument corroborates the need to redefine a local Reynolds number based on more than one scale [12]. Once fluctuations are generated by deviations, which ultimately consists in scattering, a local Reynolds number would represent a more adequate measure of the turbulence than the corresponding usual definition, which depends only on a macroscopic characteristic length. It is also possible to infer that the drag crisis arises as a consequence of an intense fragmentation of the clusters near the wall, a process which is similar to a local reversible phase transition. In this case, the clusters reach a minimum size, corresponding to individual molecules, which offers a very low resistance to the main flow.

These qualitative ideas will be fully exploited in future works, when the results

obtained via numerical and analytical solutions will furnish subsidies to show the fundamental differences between the original and extended Helmholtz equations in a more detailed way. Although some interesting scenarios in fluid mechanics were yet obtained, we also intend to simulate events in microscopic scale, in order to verify the limitations of the proposed formulation.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Bodmann, B.E.J., Vilhena, M.T., Zabadal, J.R.S., Luna, J.P. and Schuck, A. (2011) Scale Invariance and Some Limits in Transport Phenomenology: Existence of a Spontaneous Scale. In: Constanda, C., Bodmann, B. and Velho, H. (Eds.), *Integral Methods in Science and Engineering*: *Progress in Numerical and Analytical Techniques*, Birkhäuser, Springer, New York, 57-64. https://doi.org/10.1007/978-1-4614-7828-7_4 https://link.springer.com/chapter/10.1007/978-1-4614-7828-7_4

[2] Olver, P. (2000) Applications of Lie Groups to Differential Equations. Springer-Verlag, New York.

[3] Felsager, B. (1998) Geometry, Particles and Fields. Springer, New York. https://doi.org/10.1007/978-1-4612-0631-6

[4] Hestenes, D. (2003) Mysteries and Insights of Dirac Theory. *Annales de la Fondation Louis de Broglie*, **28**, 390-408.

[5] Giese, A. (2014) The Seeming Mystery of Electron. http://www.ag-physics.org/electron

[6] Hestenes, D. (1990) The Zitterbewegung Interpretation of Quantum Mechanics. *Foundations of Physics*, **20**, 1213-1232. https://doi.org/10.1007/BF01889466

[7] Pandey, S.K. and Chakravarti, R.S. (2009) The Dirac Equation: An Approach through Geometric Algebra. *Annales de la Fondation Louis de Broglie*, **34**, 223-228.

[8] Hestenes, D. (2003) Spacetime Physics with Geometric Algebra. *American Journal Physics*, **71**, 691-704. https://doi.org/10.1119/1.1571836

[9] Kiessling, M.K.-H. and Tahvildar-Zadeh, A.S. (2016) A Novel Quantum-Mechanical Interpretation of the Dirac Equation. *Journal of Physics A*: *Theoretical and Mathematical*, **49**, Article ID: 135301. https://doi.org/10.1088/1751-8113/49/13/135301

[10] Zabadal, J., Borges, V., Santos, M. and Ribeiro, V. (2015) Analytical Solutions for Dirac and Klein-Gordon Equations Using Bäcklund Transformations—ENFIR 2015 (National meeting of Nuclear Reactor Physics)—Rio de Janeiro, Brasil.

[11] Zabadal, J., Staudt, E., Ribeiro, V., Petersen, C.Z. and Schramm, M. (2022) A Quantum Toy-Model for Inelastic Scattering and Catalysis Based on Bäcklund Transformations. *Open Access Library Journal*, **9**, e8659. https://doi.org/10.4236/oalib.1108659

[12] Bodmann, B.E.J., Vilhena, M.T., Zabadal, J.R. and Beck, D. (2011) On a New Definition of the Reynolds Number from the Interplay of Macroscopic and Microscopic Phenomenology. In: Constanda, C. and Harris, P.J. (Eds.), *Integral Methods in Science and Engineering*: *Computational and Analytic Aspects*, Springer, London, 7-14. https://doi.org/10.1007/978-0-8176-8238-5_2 https://link.springer.com/chapter/10.1007/978-0-8176-8238-5_2