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On the possibility of ultracompact stars in semiclassical gravity
Sobre a possibilidade de estrelas ultracompactas na gravidade semiclássica *

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Abstract

In broad terms, compact stars are astrophysical objects that are stabilized by the pressure of degenerate matter. Driven by the rising gravitational-wave era in astronomy we can find a renewed interest in theoretical scenarios where new classes of highly compact objects could emerge. Recently a generalization of the Tolman-Oppenheimer-Volkoff (TOV) equation has been proposed by analyzing the hydrostatic equilibrium through a framework known as semiclassical gravity, a minimal extension of general relativity that incorporates the quantum behavior of matter but still treats gravity classically. In this work some models for highly dense matter available in the literature are revisited by applying the semiclassical picture for stellar equilibrium. In particular, the goal is to verify the possibility of achieving *ultracompact* solutions, a class of hypothetical compact stars where the stellar radius is less than one and a half times the Schwarzschild radius. The extreme gravity of such bodies provide them a distinctive feature in comparison to regular compact stars, namely, ultracompact stars exhibit circular photon orbits called *photon spheres*. Although the quantum corrections furnished by semiclassical gravity are negligible in most scenarios, for ultracompact configurations where the star is not much larger than its gravitational radius such effects may be relevant. Since any object that undergoes complete gravitational collapse passes through an ultracompact phase, the semiclassical corrections can create a midterm solution between regular compact stars and black holes. Distinct models indicate a considerable increase of the compactness parameter when semiclassical gravity is considered, motivating future studies.

Keywords: Ultracompact Stars, Semiclassical Gravity, General Relativity.

Resumo

Em termos gerais, estrelas compactas são objetos astrofísicos que são estabilizados pela pressão da matéria degenerada. Impulsionado pelo surgimento da era das ondas gravitacionais na astronomia podemos observar um interesse renovado em cenários teóricos nos quais novas classes de objetos altamente compactos podem emergir. Recentemente, uma generalização da equação de Tolman-Oppenheimer-Volkoff (TOV) foi proposta analisando o equilíbrio hidrostático através de um formalismo conhecido como gravitação semiclássica, uma extensão mínima da relatividade geral que incorpora o comportamento quântico da matéria, mas que ainda trata a gravidade classicamente. Neste trabalho alguns modelos para matéria altamente densa disponíveis na literatura são revisitados aplicando a representação semiclássica para o equilíbrio estelar. Em particular, o objetivo é verificar a possibilidade de alcançar soluções *ultracompactas*, uma classe de estrelas compactas hipotéticas onde o raio estelar é menor que uma vez e meia o raio de Schwarzschild. A extrema gravidade de tais corpos fornece uma característica distintiva em comparação com estrelas compactas regulares, a saber, estrelas ultracompactas exibem órbitas circulares de fótons chamadas de *fotoesferas*. Embora as correções quânticas fornecidas pela gravitação semiclássica são desprezíveis na maioria dos cenários, para configurações ultracompactas onde a estrela não é muito maior que seu raio gravitacional tais efeitos podem ser relevantes. Tendo em vista que qualquer objeto que passa por um colapso gravitacional completo passa por uma fase ultracompacta, as correções semiclássicas podem criar uma solução intermediária entre estrelas compactas regulares e buracos negros. Modelos distintos indicam um aumento considerável do parâmetro de compacidade quando a gravitação semiclássica é considerada, motivando estudos futuros.

Palavras-chave: Estrelas Ultracompactas, Gravitação Semiclássica, Relatividade Geral.

Resumo simplificado (Press Release)

Usualmente são considerados três destinos possíveis para estrelas que chegaram ao final da sua evolução: anãs brancas, estrelas de nêutrons e buracos negros. Esses objetos são denominados genericamente como objetos compactos, devido às suas enormes densidades. No entanto, a existência de outras classes de objetos compactos não é proibida pela natureza. A descoberta das ondas gravitacionais renovou o interesse em cenários teóricos nos quais novas classes de objetos compactos poderiam surgir. Consequentemente, diversos modelos têm sido propostos ao longo dos anos como resultados alternativos para o colapso de uma estrela massiva. Nesse contexto, o presente estudo dá enfoque a objetos denominados de ultracompactos, uma classe de objetos com densidades situadas entre estrelas de nêutrons e buracos negros.

A Teoria da Relatividade Geral é o formalismo tradicionalmente adotado para estudar objetos astrofísicos que possuem campos gravitacionais intensos. Interessante observar que, após mais de um século de existência, a teoria segue sem modificações. Apesar de seu enorme sucesso, trata-se de uma formulação completamente clássica. Albert Einstein estava ciente que efeitos quânticos demandariam modificações na sua teoria. Na tese, a busca por configurações ultracompactas é feita através de uma extensão mínima da Relatividade Geral chamada de Gravitação Semiclássica, que incorpora o comportamento quântico da matéria mas que ainda trata a gravidade classicamente. Por consequência, trata-se de uma aproximação. Todavia o colapso de uma estrela parece ser um ambiente propício para desencadear efeitos semiclássicos, pois campos quânticos estão inseridos em um espaço-tempo curvo e espera-se que efeitos quânticos da gravidade sejam desprezíveis.

Vale a pena observar que o vácuo, sendo uma entidade dinâmica, gravita. Explorando esse aspecto, interessantes estudos têm apontado que efeitos associados ao vácuo quântico podem atuar como ingredientes adicionais no colapso gravitacional, desempenhando um papel fundamental no equilíbrio estelar em certos cenários. Na Gravitação Semiclássica, as equações que descrevem o equilíbrio entre forças atrativas e repulsivas em uma estrela são generalizadas, abrindo novas possibilidades. Na tese, alguns modelos para matéria altamente densa disponíveis na literatura são revisitados aplicando a representação semiclássica para o equilíbrio estelar. Tais modelos foram propostos visando investigar se é possível obter soluções semiclássicas, com desvios significativos em relação a Relatividade Geral, capazes de produzir objetos ultracompactos sem horizontes de eventos.

Na primeira abordagem, uma das primeiras avaliações numéricas do equilíbrio estelar semiclássico, a matéria é descrita através de um modelo frequentemente adotado para modelar fases exóticas de matéria bariônica. As soluções semiclássicas são mais compactas que aquelas obtidas adotando a Relatividade Geral, sem alterar o perfil esperado para diversas quantidades físicas como densidade de energia, pressão e massa gravitacional. O segundo modelo é construído a partir da noção de estrela estranha, um tipo de estrela hipotética composta inteiramente por quarks, sendo o cenário mais extremo para matéria de quarks em estrelas compactas. Na proposta, essas estrelas estranhas são acrescidas de um ingrediente extra, com pressão negativa, que é o responsável por carregar os efeitos semiclássicos. Nesse caso foi possível identificar soluções capazes de satisfazer simultaneamente dados observacionais recentes e ainda assim produzir soluções ultracompactas. Visando expandir os cenários explorados na tese, no terceiro modelo não é assumida a igualdade entre as pressões radial e tangencial, isto é, considera-se um objeto composto por um fluido imperfeito. Sistemas dessa categoria podem emergir, por exemplo, em transições de fase exóticas passíveis de ocorrer no colapso gravitacional de configurações altamente densas.

Se estrelas ultracompactas como as estudadas na tese realmente existem na natureza é algo que só pode ser estabelecido mediante observações. Tais modelos podem soar excêntricos, todavia é interessante lembrar que até a década de 60 do século XX, antes da descoberta de quasares e pulsares, essa era visão geral diante de propostas como estrelas de nêutrons e buracos negros. Além disso, as ondas gravitacionais estão fornecendo uma nova janela para olhar para o Universo. A descoberta de um objeto de 2.6 massas solares tem aberto discussões, tendo em vista que *a priori* poderia ser tanto o buraco negro mais leve que temos notícia, quanto a estrela nêutrons mais massiva. Não podemos ignorar, diante dos diversos modelos de estrelas hipotéticas disponíveis, a possibilidade de ser um representante de uma nova classe de estrelas compactas.

Palavras-chave: Estrelas Ultracompactas, Gravitação Semiclássica, Relatividade Geral.

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Per aspera ad astra

Latin saying

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Notation and Units

Throughout the text, unless explicitly specified, gravitational units where $G = c = k = 1$ will be adopted [1]. In order to illustrate these units that can bring some confusion for an unfamiliar reader, consider for example, the solar mass. First we observe that $G = c = 1$ implies

$$
\begin{aligned}
1 &= 2.9979 \times 10^{10}\,\mathrm{cm\,s^{-1}}, \\
1 &= 6.6720 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}s^{-2}}.
\end{aligned}
$$

It follows that

$$1\,\mathrm{g} = 7.4237 \times 10^{-29}\,\mathrm{cm}.$$

Since $1\,\mathrm{M}_\odot = 1.989 \times 10^{33}\,\mathrm{g}$ we obtain:

$$1\,\mathrm{M}_\odot = 1.4766\,\mathrm{km}.$$

Important to notice that the compactness parameter, the ratio between mass and radius, is dimensionless.

However when expressing some numerical results, units frequently adopted in astrophysics are employed. The main conversion factors, pressure and energy density, are given by [1]:

$$
\begin{aligned}
1\ \mathrm{dyne\,cm^{-2}} &= 8.2601 \times 10^{-40}\,\mathrm{km^{-2}}, \\
1\ \mathrm{g\,cm^{-3}} &= 7.4237 \times 10^{-19}\,\mathrm{km^{-2}}.
\end{aligned}
$$

Therefore both pressure and energy density are expressed in inverse squared kilometers. Surprisingly these units can offer some advantages, for instance, in the case of the energy density, when integrating a quantity with this unity over the volume of a star, it expresses the mass in kilometers [1].

Regarding the metric signature, the $(-,+,+,+)$ convention will be adopted simply because it is generally much more convenient than the alternative choice $(+,-,-,-)$ in the sense that it induces a positive definite (rather than negative definite) metric on spacelike hypersurfaces [2].

Following the usual convention, components of contravariant vectors, usually called simply vectors, are written upstairs. Components of covariant vectors are written downstairs [3]. The Einstein summation convention[1] will be used all over the text: in an expression where the same index occurs twice, once up once down, the summation symbol will be suppressed [4]. (In a fraction up in a denominator counts as down in the numerator and vice versa). For example:

$$\sum_{\mu=0}^{3} A_\mu A^\mu = A_\mu A^\mu .$$

For the benefit of the reader we list below some mathematical symbols used in the text:

G	Newton's gravitational constant
c	Speed of light
k	Boltzmann constant
$\hbar$	Reduced Planck constant
ρ_P	Planck density
$\varLambda$	Cosmological constant
$M_\odot$	Solar mass
B	Bag constant
m	Gravitational mass
m_S	Strange quark mass
r	Schwarzschild radial coordinate
R	Stellar radius
M	Stellar gravitational mass
$\mathcal{C}$	Compactness
ℓ_P	"renormalized" Planck length
ε_S	Saturation density of nuclear matter
∇	Covariant derivative
p^μ	Four-momentum components
$\Gamma^\mu_{\alpha\beta}$	Christoffel symbols
ds^2	Line element
$d\Omega^2$	Angular line element on the 2-sphere
c_s	Speed of sound
λ	Tidal deformability
Λ	Dimensionless tidal deformability
R_S	Schwarzschild radius

[1] Einstein said to a friend "I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..." [5].

b	Impact parameter of the photon trajectory
p	Isotropic pressure
p_r	Radial pressure
$p_\perp$	Tangential pressure
ε	Mass-energy density
u^μ	Four-velocity components
μ	Chemical potential
Δ	Gap parameter
$T^{\mu\nu}$	Energy-momentum tensor components
$G_{\alpha\beta}$	Einstein tensor components
$g_{\alpha\beta}$	Metric tensor components
$\sqrt{-g}$	Square root of the determinant of the metric tensor
$\eta_{\alpha\beta}$	Minkowski metric components
$\mathcal{R}$	Curvature scalar
$\mathcal{R}_{\mu\nu}$	Ricci tensor components
$\mathcal{R}_{\mu\nu\alpha\beta}$	Riemann tensor components
$C_{\mu\nu\alpha\beta}$	Weyl tensor components
S	Action
$\mathcal{L}_m$	Lagrangian density for the matter fields
$\langle\hat{T}_{\mu\nu}\rangle$	Expectation value of the energy-momentum tensor components
$\delta_{\alpha\beta}$	Kronecker delta
$\mathcal{D}q(t)$	Measure over all paths $q(t)$
$\wedge$	Wedge product
Θ^a	Basis of 1-forms
ω^a_b	Connection forms
Ω^a_b	Curvature forms
$\Box$	d'Alembert operator
$\mathcal{M}$	Manifold
$\mathcal{M}_p$	Tangent space at point p of a manifold

List of Publications (2018-2022)

1. G. L. Volkmer and D. Hadjimichef, *Semiclassical Effects in Color Flavor Locked Strange Stars*, Braz. J. Phys (forthcoming), 10.1007/s13538-022-01161-0.

2. G. L. Volkmer, D. Hadjimichef, M. Razeira, B. Bodmann, C. A. Z. Vasconcellos, *Ultra-compact objects in semiclassical gravity*, Astron. Nachr. 340, 914 (2019).

3. C. A. Z. Vasconcellos, D. Hadjimichef, M. Razeira, G. Volkmer, B. Bodmann, *Pushing the limits of General Relativity beyond the Big Bang singularity*, Astron. Nachr. 340, 857 (2019).

4. M. Razeira, D. Hadjimichef, M. V. T. Machado, F. Köpp, G. Volkmer, B. Bodmann, G. A. Degrazia, C. A. Z. Vasconcellos, *Effective field theory with genuine many-body forces and tidal effects on neutron stars*, Astron. Nachr. 340, 209 (2019).

5. G. L. Volkmer, M. Razeira, D. Hadjimichef, F. Köpp, C. A. Z. Vasconcellos, B. Bodmann, *Pseudo-complex general relativity and the slow rotation approximation for neutron stars*, Astron. Nachr. 340, 205 (2019).

6. C. A. Z. Vasconcellos, J. E. S. Costa, D. Hadjimichef, M. V. T. Machado, F. Köpp, G. L. Volkmer, M. Razeira, *Compact stars in the pseudo-complex general relativity*, J. Phys. Conf. Ser. 1143, 012002 (2018), DOI: 10.1088/1742-6596/1143/1/012002.

7. F. Köpp, A. Quadros, G. Volkmer, M. Razeira, M. V. T. Machado, D. Hadjimichef, C. A. Z. Vasconcellos, *A comparative study of Compact Objects using 3 models: Walecka Model, PAL Model, and M.I.T. Bag Model,* Contribution to: Hadron Physics 2018; e-Print: 1804.08785 [nucl-th].

8. D. Hadjimichef, G. L. Volkmer, R. O. Gomes, C. A. Z. Vasconcellos, *Dark Matter Compact Stars in Pseudo-Complex General Relativity*, Walter Greiner Memorial Volume, 149-167 (2018).

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Chapter 1

Introduction

The history of compact stars dates back to 1844, when the German astronomer and mathematician Friedrich Wilhelm Bessel discovered that Sirius, the brightest star in the night sky, was describing an elliptical orbit [6]. Bessel thus conjectured that Sirius must be part of a binary system with an unknown companion. Eighteen years later, Alvan Clark found the Sirius companion while testing a new telescope [6]. The brightness of the object was orders of magnitude smaller, but the mass (deduced from the orbital period of Sirius using Kepler's Law) was about a solar mass (or half the mass of Sirius). Since stellar radiation can be reasonably approximated by a black body, the faintness of the companion could be easily be explained if the surface temperature of the companion was less than that of Sirius, in other words, the companion was expected to be a red star, instead of being a white star like Sirius.

An intriguing moment came in 1914 after spectroscopy observations carried out by Walter Adams showed that the surface temperature of the companion was roughly the same as that of Sirius, implying that the mean density of the companion must be roughly a million grams per cubic centimetre [6]. The confirmation that the Sirius companion was indeed small came through redshift measurements in 1925 [7]. This *white dwarf* seemed to defy comprehension, but sometimes such scenarios can lead to major breakthroughs in theoretical physics, or as Sir Arthur Eddington said: "The white dwarf appears to be the happy hunting ground for the most revolutionary developments of theoretical physics"[8].

The dilemma was solved in 1926 by Ralph Fowler through the brand new Fermi-Dirac statistics[2], in a paper called *dense matter*. Fowler's idea was that, differently from regular stars where stabilization is acquired through thermal pressure induced by nuclear reactions, in a highly dense star such as a white dwarf, gravity is balanced by an electron pressure of quantum mechanical origin called degeneracy pressure. This effect emerges from the Pauli exclusion principle, which dictates that a particle must occupy

[2] Dirac was one of Fowler's students when he derived the Fermi-Dirac distribution [6].

a different quantum state from the others. So as compression increases the electrons are pushed closed to each other and this decreases the de Broglie wavelength and, equivalently, increases the kinetic energy, opposing compression [9]. It is remarkable to notice that the first application of a quantum principle was to solve an astrophysics problem, leading to a major breakthrough in our understanding about the stellar structure. Developing further Fowler's ideas, a maximum mass for white dwarfs due to relativistic effects was found to exist in 1930 by the seminal work of Chandrasekhar [10].

The soviet physicist Lev Landau played a controversial role in the early developments of the theory of dense stars [11, 12]. Landau calculated the maximum mass for white dwarfs independently, but later than Chandrasekhar, in the first part of a concise four page paper. In the second part, Landau speculated about the structure of stars with even higher density where "the laws of ordinary quantum mechanics break down... ". Such dense stars would look like giant atomic nuclei, which in some sense can be seen as an anticipation of the concept of neutron stars albeit prior to the discovery of the neutron [11, 12]. The first explicit neutron star prediction was made by Baade and Zwicky in an extraordinary paper[3], soon after Chadwick's discovery of the neutron in 1932. Once gravity overwhelms electron degeneracy pressure, neutron degeneracy pressure was assumed to be the last hope for averting total gravitational collapse[4], and with that configuration they associated a new type of *compact star*[5], called neutron star, thus extending Fowler's ideas to the neutron [10].

The next vital step was taken independently (although they certainly communicated and discussed their results) by R.C. Tolman from Caltech and by J.R. Oppenheimer and G.M. Volkoff from the University of California, Berkeley. Their papers were received in the Physical Review on the same day and appeared in the same issue [14]. Both papers presented a derivation of the general relativistic equation for the hydrostatic equilibrium of a spherically symmetric star. Tolman proposed eight exact solutions of the new equation (although none of them corresponds to any realistic equation of state) [14]. Oppenheimer and Volkoff on the other hand, in a pioneer work, developed the first numerical neutron star model where the matter was taken to be an ideal degenerate neutron gas [15]. In this study they found a maximum stable gravitational mass of $0.71\,\mathrm{M}_\odot$, which is often called the Oppenheimer-Volkoff mass limit[6]. This result

[3] Besides introducing the notion of a neutron star the same paper also presents another remarkable points: For the first time the existence of supernovae is presented as a distinct class of astronomical objects; It was the first appearance of the name supernovae; It suggests that supernovae represent transitions of ordinary stars into neutron stars; Although the reasoning is incorrect it estimates correctly the total energy released in a supernova; It gives a theoretical scenario for the production of cosmic rays [6].

[4] At some point the electron capture by protons via inverse beta decay is energetically favorable, otherwise relativistic electrons would have a very large electron Fermi energy $\varepsilon_F = p_F$, in comparison with the much smaller value for nucleons $\varepsilon_F = p_F^2/2m_N$ [1, 13].

[5] Throughout the work the term *compact star* refers to white dwarfs, neutron stars and theoretical proposals where a full collapse has not taken place. A more broad term, compact object, also includes black holes. It should be mentioned that depending on the reference the terms compact object and compact star have a different meaning or even are used interchangeably.

[6] Although their mass limit is wrong, when combined with mass measurements of neutron stars it is extremely important

was unpleasant in the sense that it was lower than the Chandrasekhar mass limit of white dwarfs, $1.44\,\mathrm{M}_{\odot}$. This would probably hamper the formation of neutron stars from ordinary stars [14]. Nonetheless, they acknowledged the simplicity of their model based on non-interacting neutrons and discussed a possible repulsive component of neutron-neutron interaction, which could stiffen the equation of state and increase the maximum mass [14]. Neutron stars remained a textbook curiosity until 1967 with the discovery of pulsating radio sources (pulsars in short)[7] by Jocelyn Bell Burnell. Pulsars found no acceptable explanation except as rotation-powered neutron stars, giving life to the idea predicted decades earlier.

Although white dwarfs and neutron stars (the two observationally supported classes of compact stars) are stabilized by the same mechanism, they are fundamentally distinct objects. The gravitational field of most stars are so weak that an astrophysicist can usually ignore general relativity, thus in most cases the Newtonian theory of gravity applies [17]. This is true even for studying the structure of white dwarfs, although general relativity is important to evaluate other issues like their stability and oscillation frequencies [18]. Neutron stars, however, are about 6 orders of magnitude denser than white dwarfs (with no stable stars between them). In this regime the use of general relativity is mandatory[8]. In Einstein's theory the force grows faster than r^{-2} (the term r^{-2} is replaced by $r^{-2}\left(1-\frac{2m}{r}\right)^{-1}$), reducing stability and decreasing the mass limit [20]. Moreover, neutron interactions become important at densities exceeding the nuclear one, which is the kind of environment found in neutron star cores where matter presents densities ranging from a few times ε_S to an order of magnitude higher [21]. Therefore, an additional stabilization against collapse comes from the repulsion forces between nucleons at large densities increasing the mass limit of a neutron star. The second effect appears to be the dominant one and the limiting mass of a neutron star is taken about two times larger than that of a white dwarf [20]. Neutron stars are one of the most interesting and interdisciplinary objects from the physics point of view. Their analysis involve the study of a variety of extreme matter states, such as magnetic fields beyond the quantum electrodynamics vacuum pair-creation limit, supranuclear densities, superfluidity, superconductivity, exotic condensates and deconfined quark matter [20]. To put in numbers [22]:

- Four teaspoons of a neutron star contain as much mass as the moon.
- Their surface gravity is about 100 billion times the Earth's gravity.
- They are also the fastest spinning macroscopic objects. A pulsar, PSR J1748-2446ad in the globular

since it provides direct evidence of strong repulsive interaction in dense matter at supranuclear density [14].

[7] Pulsars are rotating compact stars whose radiation is observed in periodic pulses. The pulsation is a consequence of the radiation's alignment in a beam along the magnetic axis. When the magnetic axis and the rotation axis are distinct, the beam may point towards the earth periodically, similar to a lighthouse when observed from a beach [16].

[8] In terms of the compactness parameter $\mathcal{C}=\frac{M}{R}$ (used to distinguish strong from weak gravity), a quantity that will play a major role throughout this work, generally relativistic corrections start being relevant for objects with $\mathcal{C}\sim 0.1$. For most neutron star models this parameter lies in the range $[0.13-0.25]$ [19].

cluster Terzan 5, has a spin rate of 714 Hz, so that its surface velocity at the equator is about a quarter of the speed of light.

- Huge magnetic fields as high as 10^{15} gauss [9].
- The highest temperature superconductor, with a critical temperature of a few billion K, has been deduced for the superfluid neutron core in the remnant of the Cassiopeia A supernova.
- The highest temperatures known outside the Big Bang, exist at birth or in merging neutron stars, about 700 billion K.
- The pulsar PSR B1508+55 has a spatial velocity in excess[10] of 1100 $\mathrm{km\,s^{-1}}$.
- Neutron stars at birth or in matter from merging neutron stars are the only places in the universe, apart from the Big Bang, where neutrinos become trapped and must diffuse through high density matter to eventually escape.

However, from a phenomenological perspective, the existence of other families of compact objects apart from the standard triad (that is, white dwarfs, neutron stars and black holes) is not forbidden. Therefore a plethora of models have been explored by several authors as possible outcomes that could be formed after the gravitational collapse of a massive star. This class of hypothetical compact stars includes:

- Quark stars: Stars where deconfined quark matter is present. They are usually divided into two possibilities, strange stars which are completely made of quark matter, and hybrid stars where the quark matter is restricted to the core. However, it is unlikely that both types can co-exist in nature [25].
- Boson stars: Hypothetical self-gravitating compact objects resulting from the coupling of a complex scalar field to gravity [26].
- Gravastars[11]: Black hole alternative where an effective phase transition occurs at (or at least near) the position where the event horizon would be formed. The interior is composed of empty space with a strong repulsive vacuum polarization, responsible for preventing further collapses of the matter shell (a suitable outer layer of matter required to ensure stability) [27, 28].

[9] Although most neutron stars have been observed in the form of the classic pulsar, there are other astrophysical objects that are also associated with neutron stars, for instance, magnetars. Magnetars are a class of compact stars characterized by unusually large magnetic fields, up to 10^{15} gauss at the surface and possibly larger at the interior [16, 23].

[10] Phenomenon known as *pulsar kick* in which a newly born neutron star moves with a greater velocity than its progenitor star, possibly due to an asymmetry in the core collapse or the subsequent supernova explosion [24].

[11] The name is a portmanteau of the words “gravitational vacuum star”.

- Black stars: Proposed as "the most compact and quantum mechanical kind of star", it has some similarities with the gravastar model, but the interior is made of extremely dense matter supported by quantum vacuum polarization. A black star could emit Hawking-like radiation, but different from a black hole, the process preserves unitarity. If it could be peeled layer by layer, the result would be a smaller black star, also emitting radiation [27, 29].

- Fuzzballs: Star-like solutions found in string theory with no singularities. The horizon is a transition region between the exterior classical geometry and a quantum interior where the notion of space-time becomes meaningless. Each exterior geometry could have an immense large number of stringy quantum states as its interior [27].

The cornerstone of this work, rather than focusing on a particular type of compact star, is to study the strong gravity regime where the astrophysical objects are called *ultracompact*. The first models date back to the mid-1970s when some speculations were made about compact stars with a proper stellar radius smaller than its photon sphere radius, that is, $R \leqslant 1.5R_S$ [30]. Later studies started to refer to such objects as ultracompact [31]. Even more compact than regular neutron stars, these ultracompact configurations have a compactness parameter $\mathcal{C} = \frac{M}{R}$ larger than $1/3$. In addition, if the compactness parameter satisfies $\mathcal{C} > 1/2.038$, then it is called a *Clean photon sphere object* [32]. As a matter of comparison, the Sun, a star with non relativistic nature, possesses $\mathcal{C} = 2 \times 10^{-6}$ [33]. Ultracompact configurations appear in a wide range of different models. One famous example are *Q-stars*[12] which received some attention in the astrophysical literature as possible candidates for members of binary systems requiring a high mass unseen companion [34, 35, 36, 37].

The ultracompact regime exhibits a variety of interesting visual effects when compared to regular compact stars. A critical distinctive feature is the presence of a photon sphere, in other words, the unstable[13] circular null geodesic of the external Schwarzschild spacetime metric [38]. Unfortunately, the current theoretical status of studies regarding horizonless ultracompact objects is unsatisfactory, as their constructions are often based on assumptions that manifestly deviate from known physics, sometimes making it difficult to verify if other general relativity predictions are not affected [39]. Ultracompact stars can also be seen as an interesting conceptual link between regular neutron stars and black holes, since the first has neither a photon sphere or a event horizon and the second has both, an ultracompact star could be an intermediate step presenting a photon sphere but no event horizon [37].

Initially there was some hope that nuclear forces would always be able to resist gravity, but even

[12] It is worth mentioning that the "Q" in the name does not stand for quark, but for the conserved baryon number charge which stabilizes the matter against strong, weak and electromagnetic decay [34, 35].

[13] Stable circular trajectories are only possible for $r \geqslant 6M$. The surface $r = 6M$ defines what is called the innermost stable circular orbit (ISCO). Some authors adopt the ISCO to provide a precise definition of a compact object, being in this case any object satisfying $\mathcal{C} \geqslant \frac{1}{6}$ [32].

neglecting the uncertainties arising from the equation of state at supranuclear densities, different upper limits can be established from very general considerations [10]. Above such limit is currently believed that the complete collapse is unavoidable, giving rise to a black hole. Nowadays a black hole can informally be defined as "any dark compact object with mass exceeding a threshold of about three solar masses" [40]. However, it has been pointed out by some researches that the phenomena known as *quantum vacuum polarization* may act as an additional ingredient, which under certain circumstances could play a distinctive role in stellar equilibrium [27, 29, 39]. Aiming to incorporate this effect the analysis is developed under the theoretical framework known as semiclassical gravity, which is basically the theory obtained from a self-consistent solution of the geometry and dynamics of the spacetime and the quantum matter field altogether [41].

Classically, matter influences gravity through the stress-energy tensor, which appears as a source term in Einstein's field equations. Aiming to describe the backreaction of the quantum field on the spacetime geometry, instead of the usual Einstein field equations of general relativity, one uses field equations where the expectation value of the stress-energy operator of the quantum matter field act as the source, that is [42, 43, 44]

$$G_{\mu\nu} = 8\pi\langle\Psi|\hat{T}_{\mu\nu}|\Psi\rangle, \tag{1.1}$$

where matter fields are quantized in some appropriate state $|\Psi\rangle$ (Heisenberg-picture) but the gravitational field remains classical [45]. Therefore the effects of quantum vacuum polarizations are represented in the field equations by the expectation value of the stress-energy tensor, an extra contribution present in the semiclassical field equations, which becomes an additional source of gravity. The main conceptual purpose of semiclassical gravity is precisely to include this backreaction, that is, the effects of the quantum fields on the dynamics of the gravitational field itself. For many researchers, especially astrophysicists, the impacts of quantum process, such as vacuum polarization (e.g., trace anomaly) and vacuum fluctuation (e.g., particle creation), on the background geometry is of primary interest [46].

Historically, semiclassical gravity was first applied to investigate cosmological backreaction problems, such as particle creation in cosmological spacetimes [39, 42, 47]. An specific example is the damping of anisotropy in Bianchi universes by the backreaction of particles created from the vacuum [42]. Inflationary cosmology, proposed in the early eighties by Guth and others, is another well known example of semiclassical gravity where the vacuum expectation value of a gauge or Higgs field acts as source in the Einstein field equations. It is not difficult to see that an exponential expansion arises from a constant energy density in a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe, the case of eternal inflation described by a de Sitter universe. Such solution is unpleasant in classical grounds because it corresponds to an unconventional equation of state where $p = -\varepsilon$. A quantum source in the semiclassical framework make this solution not only plausible, but, as later investigations in inflationary cosmology showed, desirable [42].

The basic idea of this work is to study the impacts that semiclassical gravity has upon the ultracompact regime. It should be emphasized that this combined analysis of ultracompact stars and semiclassical gravity is not an arbitrary one. First, considering that any object that undergoes a complete gravitational collapse passes through an ultracompact phase, any effect not taken into account in general relativity could have a prominent role, avoiding the full collapse in certain scenarios [36]. Moreover, since the structure equations for stellar equilibrium in semiclassical gravity differ from those obtained in general relativity, some solutions could be naturally promoted to the ultracompact regime due to different predictions for the final mass and radius. It is also pleasant from the theoretical point of view (although any hypotheses has obviously to be verified by observations) because the physical properties that demarcate the transition between regular compact stars to ultracompact stars, and from the later to black holes, namely, the presence of a photon sphere and an event horizon, could also establish transitions from frameworks. First regular neutron stars (without photon spheres or event horizons) would be described by general relativity. Ultracompact stars (with photon spheres but no event horizon) on other hand, would require semiclassical corrections, and finally black holes (with both photon spheres and event horizons) would demand a complete quantum gravity theory to be fully described.

In the next chapter the standard theory of gravitation, general relativity, is briefly discussed with emphasis on the subject of compact stars, highlighting its differences with respect to Newtonian physics. It covers key points like relativistic fluids and the structure equations for relativistic stars made of perfect and imperfect fluids. It also discusses the difficulties in describing the behavior of matter at supranuclear densities, as well as physical properties like the sound speed and the tidal deformability, which are essential topics in compact star physics. Chapter 3 introduces, via photon trajectories in the Schwarzschild metric, the notion of an ultracompact star by discussing some optical effects that arise as compactness increases. Two sections are dedicated to gravitational waves and the impact that this new kind of astronomy may have in future discoveries. After this, it is argued why horizonless ultracompact objects may play a prominent role in this flourishing gravitational wave astronomy era. Chapter 4 presents the formalism which will be applied to study ultracompact stars, semiclassical gravity. It explores its purposes, field equations and its subsequent picture for hydrostatic equilibrium. The chapter finishes with a discussion on how quantum vacuum polarizations may act as a new stabilizing ingredient for relativistic stars. The work reaches its main goal in Chapter 5, where three independent models are proposed. In all cases ultracompact configurations are pursued. The conceptual theoretical motivation for this study is to consider a transition of frameworks to describe different classes of compact stars motivated by physical properties. It should be mentioned, however, that this line of thought is strongly inspired by previous works done by Nemiroff [36, 37]. The first model relies on the linear equation of state, a well known relation frequently used to describe exotic states of matter, which under discussed conditions maximizes compactness. Some of its content can be found at [48]. The second model takes advantage of the fact that there are two ways of expressing the pressure gradient in semiclassical gravity. Therefore the model is constructed attaching to each solution an appropriate equation of state. The resulting object is a color flavor locked strange star where semiclassical effects are added through a negative pressure component. These first two models are constructed upon isotropic sources. In order to diversify this aspect, the last model consists in a primitive attempt to create anisotropic semiclassical solutions. Chapter 6 closes the main text with the final remarks.

The work also counts with five appendices aiming to further elucidate the ideas developed throughout the previous chapters. Appendix A presents a derivation of the structure equations of Schwarzschild stars (static and spherically symmetric stars in general relativity). Appendix B heuristically demonstrates the line element in which the solution presented in Appendix A is based. Appendix C discusses the semiclassical hydrostatic equilibrium equations, in other words, it deals with the generalization of what was presented in Appendix A by incorporating quantum vacuum polarization effects. Appendix D provides some mathematical supplementary material that was chosen to be presented separately to avoid unnecessary interruptions in the main text, although it might be useful for some readers. Appendix E briefly discusses the Einstein field equations, complementing what was discussed in Chapter 2.

For clarity, it is important to explicitly mention what is original in this text. The content of the first four chapters, as well as the appendices, are predominantly reviews. The semiclassical picture for hydrostatic equilibrium was developed in Ref. [39]. However this work dealt only with exact solutions, ignoring explicit descriptions for the material composition. So the idea is to make advances by reinterpreting some stellar configurations produced by different equations of state available in the literature using semiclassical gravity. Therefore Chapter 5, viewed as a semiclassical study of the equations of state considered, is the original piece in this work.

Twenty years ago, Norman Glendenning ended the preface of the second edition of his traditional book about compact stars with the sentence: *Great challenges await those who hear the siren song* [1]. On that occasion he was referring to the discovery of new pulsars, especially those that are accreting matter from a companion. But now, with the emergence of gravitational wave astronomy, as well as other initiatives like the NICER experiment[14], this truly seems to be the case more than ever. The LIGO Scientific and Virgo Collaborations, using ground based laser interferometers, have detected gravitational wave signals of binary black holes and binary neutron stars coalescences, opening a new window to look at gravitational physics, particularly in the strong field regime [49]. More recently the LIGO collaboration has announced the discovery of an object in the so-called mass gap, a mass region that separates the lightest known black hole from the heaviest known neutron star, with 2.6 solar masses [51]. It's not clear yet if what they observe is the lightest black hole or the heaviest neutron star that we know of. Maybe the future reveals that it is actually neither and in that case we would be facing exciting times with new classes of compact objects to be explored.

[14] The Neutron star Interior Composition Explorer (NICER) probes interior composition of neutron stars through stellar radius and mass measurements [50].

Chapter 2

Relativistic Gravitation

Gravity is a fundamental interaction of nature known since antiquity, noticeable by everyone everywhere. The most remarkable property of the gravitational field is its universality, in the sense that all particles, independently of their masses, feel the gravitational attraction equally. In other words, particles with different masses experience a gravitational field in such a way that all of them acquire the same acceleration and, given the same initial conditions, will follow the same path [52]. Newtonian gravity offers no natural explanation for such a fact, it is instead assumed that the inertial mass is exactly equal to the gravitational mass for all bodies [53].

Forces equally felt by all bodies were known since long. They are called inertial forces, which are experienced in non-inertial frames. Common examples found on Earth (not an inertial system) are the centrifugal force and the Coriolis force. Actually, this universality regarding inertial forces has been the first hint towards general relativity [52]. The universality of response is the most distinguished feature of the gravitational interaction. It is an exclusive property of gravity, no other fundamental interaction of nature has it [52]. Due precisely to such universality of response, the gravitational interaction accepts a unique description, which is completely indifferent to the concept of a gravitational force.

In general relativity, the standard theory for the gravitational interaction, Einstein proposed that the geometry of spacetime is a new physical entity with degrees of freedom and dynamics of its own, hence the responsibility of describing the gravitational interaction is transferred to the spacetime geometry [52]. In this sense the theory is also sometimes called *geometrodynamics*, a term coined by John Wheeler [54]. In this new description of gravity the universal character is encapsulated from the beginning, since the relative acceleration of particles is not viewed as a reaction to gravitational forces, but results from the curvature of spacetime in which the particles are moving [10]. It is an outstanding conceptual insight that can only reinforce Einstein's geniality.

The early development of general relativity was slow because the theory seemed to be poor in applications. Despite of its beauty and richness, it is a sterile theory without the stimulus that can only be accomplished through experiments and observations. Today it's a rich field with much more to offer than its three famous tests: the gravitational red shift, the bending of light by the sun, and the precession of the perihelion of Mercury around the sun [54]. Einstein's theory is also responsible for the predictions of exciting new phenomena, such as the expansion of the Universe, black holes, gravitational lenses and gravitational waves, all of them currently supported by observations.

The theory has been frequently praised as one of greatest intellectual achievements of all times by many figures of the physics pantheon. According to Lev Landau general relativity is *the most beautiful of the existing physical theories* [55]. If Landau was exaggerating he was certainly not alone, Paul Dirac said: *probably the greatest scientific discovery* [56]. Let us also quote Max Born [57]:

> *(The general theory of relativity) seemed and still seems to me at present to be the greatest accomplishment of human thought about nature; it is a most remarkable combination of philosophical depth, physical intuition and mathematical ingenuity. I admire it as a work of art.*

and finally Einstein himself:

> *In the light of present knowledge, these achievements seem to be almost obvious, and every intelligent student grasps them without much trouble. Yet the years of anxious searching in the dark, with their intense longing, their alternations of confidence and exhaustion and the final emergence into the light—only those who have experienced this can understand it.*

There are many excellent textbooks on the subject and the reader should consult them for systematic and complete discussions [2, 3, 4, 17, 54, 58]. The goal here is to discuss only the aspects relevant for the subsequent chapters. Aiming to avoid successive interruptions on the main text, the definitions of some mathematical notions used in what follows are presented in Appendix D.

2.1 Geometry of Spacetime Before General Relativity

Perhaps the most significant obstacle to comprehend the theories of special and general relativity resides in the difficulty in realizing that a number of previously held basic assumptions about the nature of space and time, although intuitive, are simply wrong [2]. Despite the fact that prerelativity physics was developed without explicitly mentioning this idea, the concept of spacetime can be perfectly incorporated to it [59]. The spacetime in Newtonian physics has a particular structure that implements the notions of space and time as absolute independent entities, completely unperturbed by matter and events that may happen in them. Therefore the Newtonian spacetime can be understood as a sequence of "pictures" of the absolute space taken at all the successive instants of the absolute time [59]. In mathematical terms, since space and time are entirely distinct objects, the associated spacetime is simply the direct product of its underlying structures and its invariance group is called the Galileo group. The variation of time is represented by $\mathbb{R}$ and space itself is taken to be an Euclidean space $\mathbb{E}_3$, in other words, $\mathbb{R}^3$ equipped with an Euclidean metric, which in Cartesian coordinates assumes the form [3]:

$$ds^2 = \sum_i \left(dx^i\right)^2. \tag{2.1}$$

This mathematical model is associated with observations by identifying points of this Euclidean space with observed objects [3].

This framework was seriously challenged for the first time in the theory of special relativity, whose invariance group is the Poincaré group, and the underlying spacetime is called the Minkowski spacetime $\mathbb{M}_4$, which replaces the Newtonian absolute space $\mathbb{E}_3$ and the absolute time $\mathbb{R}$ [3]. The Minkowski spacetime is represented by $\mathbb{R}^3 \times \mathbb{R}$ endowed with a Lorentzian[15] flat metric, which in inertial coordinates is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.2}$$

with $\eta_{\mu\nu} = diag\{-1, 1, 1, 1\}$. Observe that in both cases spacetime is represented by a fixed four-dimensional affine space[16]. Affine spaces are sufficient to treat electromagnetism, hydrodynamics or even relativistic quantum field theory. However in order to incorporate gravitation into relativity in a satisfactory manner, the general notion of a manifold, not simply reduced to an affine space, has to be invoked [59]. The reason to abandon the Minkowski metric can be exemplified in very practical terms, for example, by the fact that it cannot account for the gravitational redshift, a well tested physical effect predicted by any relativistic theory of gravitation based on the equivalence principle [17].

[15] Positive definite metrics are called Riemannian. On the other hand, metrics with signatures like those on spacetime (one minus and the remainder plus) are called Lorentzian [2]. A standard well known result from linear algebra, the "Sylvester's law of inertia ", guarantees that the signature is basis-independent [4].

[16] The definition is presented in Appendix D, despite of this it may be helpful to mention, for instance, that an one dimensional affine space is a straight line and a two dimensional affine space is a plane [59].

[17] It can be demonstrated that if gravitation obeys the equivalence principle, then the proper time no longer satisfies the Minkowski metric [60].

2.2 General Relativity

The two frameworks mentioned in the previous section, namely, Newtonian physics and special relativity are incompatible and therefore it is impossible to construct a relativistic theory of gravitation by simply merging them. The reason for this is rooted in the fact that Newtonian gravitation is built upon the notion of action at distance, more specifically, the Newtonian expression for the gravitational force relates the force between two bodies in a given time with their distance at the same instant of time. This absolute notion of simultaneity is meaningless in special relativity. However it is worth mentioning that general relativity coincides with special relativity whenever gravity can be neglected, and Einstenian gravitation nearly coincides with Newtonian gravitation in the physical domain where the latter has proved to be successful, that is, when the velocities of the gravitating bodies are slow (in comparison with the speed of light) and the gravitational fields are relatively weak [3].

A brief introduction to the main points of the theory can be made by outlining its four key principles [10]. As a starting point consider the notion of a field. This concept provides the best procedure to describe interactions consistent with special relativity [52]. In fact all known forces are mediated by fields on spacetime, which is the natural arena where physical phenomena takes place. Now, if gravitation is to be represented by a field, it makes sense, having in mind the considerations made earlier, to use an universal field. A natural possibility is to change spacetime itself. As Hawking and Israel stated: *it was Einstein's greatest stroke of genius to realize that it could be given a dynamical role* [61]. Among all fields present in a space, the first fundamental form, or metric, seemed to be the most suitable one [52]. Therefore the simplest way to modify spacetime would be to change its metric. This motivates the following premise:

1. Spacetime is mathematically described by a four-dimensional manifold $\mathcal{M}$ endowed with a global symmetric metric field g.

Another important structure on the manifold is the connection, which essentially provides a prescription to "connect" neighboring points on the spacetime in order to calculate derivatives. Consequently, the connection and its associated covariant differentiation operator are frequently treated as synonyms [62].

A connection also defines the notion of parallel transport of a vector along a curve, allowing the comparison between vectors belonging to different tangent spaces. A vector with components v^α is said to be parallelly transported along some curve with tangent vector t^α if, and only if [53]

$$t^\alpha \nabla_\alpha v^\beta = 0. \tag{2.3}$$

This expression is simply a generalization to manifolds of the notion of "keeping a vector constant" in ordinary vector spaces [53].

Many connections can be defined on the same metric spacetime and in the general case it can be decomposed into a symmetric and a antisymmetric part [18] ,

$$\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{(\alpha\beta)} + \Gamma^{\lambda}_{[\alpha\beta]}. \tag{2.4}$$

In order to continue our discussion, consider the following theorem [4]:

▲ **Theorem 1.** *If a manifold possesses a metric* g *then there is a unique symmetric connection, the Levi-Civita connection or metric connection such that*[19]

$$\nabla \mathbf{g} = 0. \tag{2.5}$$

. In general relativity only the Levi-Civita connection is considered. The connection coefficients are then called Christoffel symbols and satisfy

$$\Gamma^{\lambda}_{[\alpha\beta]} = 0 \implies \Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{(\alpha\beta)}. \tag{2.6}$$

In the language of differential geometry, the property $\Gamma^{\lambda}_{[\alpha\beta]} = 0$ has a deep meaning which brings us to our second premise:

2. There is no torsion associated with gravity[20].

Therefore when restricted to the case of general relativity only the zero-torsion connection is present. This allows the interpretation that the presence of a gravitational field induces a curvature in spacetime itself, but no other kind of deformation. This deformation preserves the Lorentzian character of the flat Minkowski spacetime associated with the absence of gravitation.

In flat spacetime, a free particle follows a straight line, that is, a curve keeping a constant direction. However, in curved spacetime the gravitational field prevents massive free bodies from moving in straight lines with constant velocity with respect to inertial frames. So the trajectory of a particle submitted exclusively to gravity will follow a geodesic[21] of the deformed spacetime, which is a curve whose tangent vector is parallelly transported along itself satisfying [53]

$$t^{\alpha}\nabla_{\alpha}t^{\beta} = 0. \tag{2.7}$$

[18] Symmetrizing of indices is denoted by parentheses, $A_{(\alpha\beta)} := \frac{(A_{\alpha\beta}+A_{\beta\alpha})}{2}$, and antisymmetrization by brackets, $A_{[\alpha\beta]} := \frac{(A_{\alpha\beta}-A_{\beta\alpha})}{2}$.

[19] In the general case $\nabla_{\mu}g_{\alpha\beta} = K_{\mu\alpha\beta}$, where $\mathbf{K}$ is called the non-metricity tensor that measures the incompatibility between the metric and the connection [63].

[20] Curvature and torsion are properties of Lorentz connections. Nevertheless, in general relativity the gravitational interaction is described only by curvature. It should be mentioned that alternative approaches may take different paths. For example, in teleparallel gravity the gravitational field is represented by torsion, not curvature. This framework uses a special linear connection, the Weitzenböck connection [52].

Obviously physics is much more than free particles, so it is legitimate to ask what happens to the rest of physics (for instance Maxwell's equations) under the influence of gravitational fields. The answer is expressed in the next premise, namely

3. Any non-gravitational physical interaction behaves in a local inertial frame as if gravitation were absent.

This means that locally spacetime preserves the same structure found in special relativity, and all its experimentally observed effects are predicted [61]. This statement is sometimes called the Einstein equivalence principle, which deeply guided him while developing his theory of gravitation. As a matter of completeness it is important to enunciate two other versions of the equivalence principle. First the weak equivalence principle, which states that all test bodies follow the same trajectories when submitted to the same initial position and velocity. There is also a strong version which says that the effects of an external gravitational field can always be removed by choosing a local inertial frame in which all laws of physics (including gravity) assume the same form as in the absence of this external gravitational field [53]. The weak equivalence principle can be seen as a corollary of Einstein equivalence principle, which itself is a corollary of the strong equivalence principle. However, none of the converse implications are necessarily true [53]. Even though, Schiff's conjecture states that any acceptable theory of gravity that obeys the weak version of the equivalence principle must also obey the Einstein equivalence principle. Among all metric theories of gravity, general relativity is one of the few that obeys the strong equivalence principle, although the weak version is respected in all of them [53].

The last topic regards the field equations. If no other geometrical structure is added to spacetime apart from the metric itself, and if one requires that the field equations should not contain derivatives higher than the second, it can be shown that one is led uniquely to Einstein field equations [61]

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.8}$$

which is the mathematical manifestation of our last premise:

4. The Einstein tensor couples to the matter-energy content of the Universe.

Having in mind that the components of the Einstein tensor are defined by the relation $G_{\mu\nu} := \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}$, the Einstein field equations can be expressed in a alternative way by taking the trace of equation (2.8), namely

$$\mathcal{R} - 2\mathcal{R} = 8\pi T. \tag{2.9}$$

[21] In other words, a geodesic is basically a curve keeping a constant direction in curved spacetime or as Wald describes, geodesics are the "straightest possible lines" [2, 52].

Substituting this result in (2.8) one obtains

$$\mathcal{R}_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{2.10}$$

In particular, in vacuum (that is, when $T_{\mu\nu} = 0$) the components of both Einstein and Ricci tensors vanish. In this case spacetime is said to be *Ricci flat*.

The only quantity responsible to describe the properties of matter, as well as its influence on the metric tensor, is the *stress-energy-momentum tensor*[22]. General Relativity does not provide the possible forms of this tensor corresponding to different types of matter, so the theory is void of physical content without information prescribed by other branches of physics. This shortcoming implies that general relativity is not a closed theory. This point has been regarded, particularly by Einstein, as an inconvenience of the theory [64].

The interactions between the spacetime geometry and the matter distribution can be summarized by the words of Misner, Thorne and Wheeler [54]: *Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve*[23]. This sentence beautifully expresses the non-linear character of the theory, since geometry is dynamically determined by its matter content. Due to this complexity, Einstein thought that it would be impossible to find exact solutions[24][7].

The field equations can be theoretically justified and experimentally verified, however they can not be derived from some fundamental principle. Despite of this, the Einstein field equations can be obtained by applying the action principle (see Appendix E) [58]

$$\delta S = 0, \tag{2.11}$$

to the following action

$$S = \int \left(\frac{1}{16\pi} \mathcal{R} + \mathcal{L}_M \right) \sqrt{-g} d^4x, \tag{2.12}$$

where the variation is made with respect to the metric[25].

[22] Following the standard practice, throughout the text the stress-energy-momentum tensor will be referred either as stress-energy tensor or energy-momentum tensor.

[23] In general relativity "matter" refers to anything else which is not the gravitational field. For example, not only atomic nuclei and electrons, but also the electromagnetic field [65].

[24] Contrary to Einstein's expectations, the first solution was proposed by Karl Schwarzschild only two months after the publication of the theory. This solution remains one of the few with real astrophysical interest, describing the exterior geometry of non-rotating black holes and stars [7].

[25] The first variation of an operator $P : u \mapsto P(u)$ between open sets of two vector spaces E_1 and E_2, at some point $u \in E_1$, is a linear operator acting on vectors $\delta u \in E_2$ given by the (Frechet) derivative P'_u of P at u, namely $\delta P := P'_u \delta u$ with $P\left(u + \delta u\right) - P(u) = P'_u(u)\delta u + \mathcal{O}\left(|\delta u|\right)$ [3].

In general relativity the energy-momentum tensor obeys the equation [61]

$$\nabla \cdot \mathbf{T} = 0 \tag{2.13}$$

as a consequence of the field equations (because the covariant divergence of the Einstein tensor vanishes, as shown in Appendix D). This relation expresses the local conservation of energy and momentum. It should be emphasized however that the curvature of the metric prevents integration in order to obtain global conservation laws. The physical reason for this is rather simple once one observes that $\mathbf{T}$ represents the energy and momentum of the matter field only, not including a contribution from the gravitational field. There is no locally defined quantity capable of measuring the energy and momentum of the gravitational field since the field can always be transformed to zero locally by choosing inertial coordinates. This non-localizability of the gravitational energy was a source of much confusion in the early days of general relativity [61].

To summarize, general relativity is the most successful gravitational theory so far proposed. The gravitational field is represented by a mathematical object called spacetime and is formalized as the pair $(\mathcal{M}, \mathrm{g})$, where $\mathcal{M}$ is a differentiable manifold and g is a metric of Lorentzian signature. The restrictions on the metric are imposed by the field equations (together with boundary and other initial conditions) and there are ten second order partial differential equations for the metric [3, 66].

2.3 Compact Stars in General Relativity

When, due to a gravitational instability initiated by some external factor or temperature drop, a cloud of interstellar gas starts to contract due to its own gravitational attraction, the formation of a star begins. During the contraction there is a conversion of the gravitational potential energy into thermal energy, which raises the temperature of the gas. The pressure of the gas tries to stop the collapse and if it did not lose energy, this pressure would prevent the collapse at this stage. However the gas loses energy in the form of electromagnetic radiation and as a result the cloud is unable to maintain the necessary pressure and the collapse continues, getting hotter and hotter. After a slow collapse the center of the cloud becomes hot enough for nuclear reactions to occur and stability is achieved. The radiated energy is balanced by the generation of nuclear energy so that the cloud does not need to follow the collapse to obtain the thermal energy necessary to maintain the pressure required to achieve stability. A star has been born and if its temperature lies above that of its surroundings, it will continuously lose energy (and hence mass), mostly in the form of radiation [65].

Frequently this interplay between gravitational force, thermal pressure and outgoing radiation provides a relatively stable state [65]. However, eventually the star exhausts all the available sources of nuclear fusion energy. At this point the star can no longer sustain itself against the gravitational force, and its matter must resume the gravitational collapse upon itself [67]. For massive stars, the gravitational collapse is one of the most severe deviations of general relativity with respect to Newtonian gravity. Classically speaking, nothing can halt this collapse. Nonetheless, quantum matter obeys quantum statistics. Driven by Fermi-Dirac statistics, if the mass of the star is not too great, and it has cooled sufficiently, a new stable configuration is possible, a white dwarf star held by the quantum degeneracy pressure of its electrons. In other cases, the collapse produces an even more compact configuration, in which the electrons and protons are forced under high pressure to become neutrons, giving rise to a neutron star, also sustained against further collapse by the quantum degeneracy pressure mechanism. If the mass of the stellar remnant core exceeds a certain value, intimately related with the equation of state of dense matter, which is not very accurately known, not even the degeneracy pressure achieved by neutron star matter is sufficient to prevent the final and inexorable collapse due to gravity.

Regarding the gravitational description of stars, conventional stars with less than 100 solar masses do not require general relativity, at least not while such stars are in states normally observed by astronomers. Good agreement with the observational data is found by combining Newtonian gravitation with thermodynamics, nuclear physics and plasma physics [68]. However, for neutron stars, the Newtonian mass predictions differ from Einstein predictions by 10 to 100 percent [54]. Therefore neutron stars are fully relativistic objects and their extreme conditions (impossible to reproduce in terrestrial laboratories) require General Relativity[26].

2.3.1 Relativistic Fluids

Generally speaking, a fluid can be viewed as a special kind of continuum, which in turn can be seen as some collection of particles sufficiently numerous that its dynamics can be well represented by average or bulk quantities such as energy density or pressure [69]. In the theory of relativity the word fluid may be used to describe not only ordinary fluids, but also gases, radiation and even vacuum energy [70].

This notion is useful when dealing with stellar configurations, since their properties do not depend upon where any of its individual particles happens to be. In fact, most matter in the Universe can be approximated as a fluid and in several occasions it undergoes relativistic motion (for instance in high-energy particle beams or in supernova explosions) [71].

In particular, since the Schwarzschild's interior solution of 1916, the perfect fluid approximation

[26] Although dense objects like neutron stars could in principle exist in Newton's theory, they would be completely different objects. All degenerate stars have a maximum possible mass, but in the Newtonian description such limit is attained asymptotically when all fermions whose pressure supports the star are ultrarelativistic [1]. Under such conditions stars populated with heavy quarks would exist. Such unphysical stars do not occur in general relativity.

is a common assumption to describe matter distributions in self-gravitating systems. In this approximation, the macroscopic distribution of mass can reasonably be described by a continuous energy density associated with a large assembly of particles and it is assumed that those particles collide frequently enough that their mean free path is short when compared with scale on which the density changes, allowing a local thermodynamic equilibrium [72]. It can also be seen as a medium in which the pressure is isotropic (equal principal stresses)[27] in the rest frame of each fluid element, and shear stresses and heat transport are absent. The perfect fluid can be mathematically represented by a stress-energy tensor with components given by[28]:

$$T^{\mu\nu} = (\varepsilon + p)\, u^{\mu} u^{\nu} + p g^{\mu\nu}, \tag{2.14}$$

where the four-velocity is defined so that [17]

$$u^{\mu} u_{\mu} = -1. \tag{2.15}$$

Regarding its motion, observe that the energy-momentum conservation $\nabla_{\mu} T^{\mu\nu} = 0$ for a perfect fluid yields [7]

$$(\varepsilon + p)\, u^{\mu} \nabla_{\mu} u^{\nu} + u^{\nu} \nabla_{\mu} \left[(\varepsilon + p)\, u^{\mu} \right] + g^{\mu\nu} \nabla_{\mu} p = 0. \tag{2.16}$$

It is useful to decompose the above expression into two components, one aligned with the flow and an orthogonal part. The component along the four-velocity can be obtained by contracting (2.16) with u_{ν}, that is

$$u^{\mu} \nabla_{\mu} \varepsilon + (\varepsilon + p)\, \nabla_{\mu} u^{\mu} = 0. \tag{2.17}$$

Assuming that the fluid is composed of single-particle species, the chemical potential (basically the energy cost to add a single particle to the system) is given by [7]

$$\mu = \frac{d\varepsilon}{dn}, \tag{2.18}$$

where n is the particle number density. The associated thermodynamic relation is then [7]

$$\varepsilon + p = n\mu. \tag{2.19}$$

Using the above expression equation (2.17) reduces to

$$\nabla_{\mu} (n u^{\mu}) = 0, \tag{2.20}$$

which expresses the particle flux conservation, not accounting for particle creation or destruction [7].

The orthogonal part is given by

$$(\varepsilon + p)\, \dot{u}_{\alpha} + \varsigma^{\mu}_{\alpha} \nabla_{\mu} p = 0, \tag{2.21}$$

where $\varsigma_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu}$ are the components of the projection tensor[29] and $\dot{u}^{\mu} = u^{\nu} \nabla_{\nu} u^{\mu}$ is the four-

[27] Fluids in which the pressure is the same in every direction are called Pascalian [68].

[28] The principle of general covariance assures that the stress-energy tensor of a perfect fluid has the same form familiar from special relativity, just replacing the Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$. Even so, the energy-momentum tensor of a perfect fluid is verified in Appendix E.

[29] The contraction of a tangent vector $\mathbf{V}$ with ς projects $\mathbf{V}$ into the 3-surface orthogonal to the four-velocity $\mathbf{u}$ [54].

acceleration. Expression (2.21) is the relativistic version of the Euler equations, showing that the fluid deviates from geodesic motion due to pressure gradients [7].

In equilibrium, a neutron star can be accurately approximated as a self-gravitating perfect fluid[30], but in general, the properties of the matter in a compact star will depend on several parameters: fluid and magnetic stresses, composition, entropy gradients, heat flow and neutrino emission [72]. Nevertheless, it is often assumed a one-parameter equation of state to describe a compact star, because neutrino emission cools the object within a short time after formation to $10^{10}K \approx 1$ MeV. This value is negligible in comparison to the Fermi energy in the interior, where the density is greater than the nuclear one, which is approximately 60 MeV. In this sense a neutron star is cold, and since nuclear reaction times are shorter than the cooling time, a zero-temperature equation of state with barotropic flow is suitable to describe the matter, that is [72]:

$$\varepsilon = \varepsilon(p). \tag{2.22}$$

It is noteworthy that in the strong gravity regime, pressure and stresses are normally so large that incompressibility can not be assumed. Differently from usual plasmas, where the stress-energy tensors are dominated by their rest-mass density, the pressure contributions to the stress-tensor can be of the same order as those from the energy density [10].

Deviations from local isotropy may take place under particular circumstances, an idea perhaps first explored by Lemaître while studying cosmological problems [75]. Anisotropic pressure means that the radial pressure, p_r, differs from the angular components, $p_\theta = p_\phi = p_\perp$ (while the equality between the angular components is a direct consequence of spherical symmetry). Spherical symmetry also implies that both p_r and $p_\perp$ must be functions of the radial coordinate [26].

Anisotropic fluid spheres have been invoked in many compact star models, giving more freedom to the equation of state while maintaining the spherical symmetry, although it is not known to which extent anisotropy may contribute in realistic models. A scalar field with non-zero spatial gradient is an example of a physical system where the pressure is anisotropic. This anisotropic character of a scalar field occurs already at the level of special relativity [26]. Another example can be found in the internal structure of gravastars, in which the use of anisotropic pressure is unavoidable, otherwise the gravastar swells up to infinite size or even will form undesirable structures like horizons or naked singularities [76].

For anisotropic fluids the energy-momentum tensor assumes the form [77]:

$$T_{\mu\nu} = (\varepsilon + p_\perp)\, u_\mu u_\nu + p_\perp g_{\mu\nu} + (p_r - p_\perp)\, k_\mu k_\nu, \tag{2.23}$$

[30] In particular, deviations from perfect fluid equilibrium due to a solid crust are expected to be smaller than 10^{-3}. This corresponds to the maximum strain that an electromagnetic lattice can support, which is verified by the observations of pulsar glitches, being consistent with departures from a perfect fluid equilibrium of order 10^{-5} [72, 73, 74].

where p_r and $p_\perp$ are respectively the radial and tangential pressure components, and k^μ are the components of a space-like vector (satisfying $k^\mu u_\mu = 0$ and $k^\mu l_\mu = 0$, where l_μ represent the components of a null-vector). Clearly, when $p_r(r) = p_\perp(r) = p(r)$ the above equation recovers the isotropic expression (2.14).

2.3.2 Structure of Relativistic Stars

Every physical model, aiming to describe certain aspects of reality, relies upon some assumptions. Stellar structure models often assume a static and spherically symmetric distribution of matter, and as a consequence all physical quantities depend only on the radial coordinate r. Spherical symmetry implies that every point of spacetime is on a two-surface which is a two-sphere [69]. This can be represented by the line element:

$$ds^2 = f(r^*, t)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.24}$$

where $f(r^*, t)$ is some function of the two other coordinates of the manifold. The associated area of each sphere is $4\pi f(r^*, t)$. The radial coordinate can be defined as $f(r^*, t) := r^2$, representing a coordinate transformation from (r^*, t) to (r, t) [69]. In this case, any surface with r and t constant is a two-sphere of area $4\pi r^2$ and circumference $2\pi r$. This coordinate r is called *curvature coordinated* or *area coordinate*, since it defines the area and the radius of curvature of these spheres [69]. It is important to emphasize that no *a priori* relation can be established between r and the proper distance from the center of the sphere to its surface. In fact, what could in principle be called their centers (at $r = 0$ in flat space) are not points on the spheres themselves, which implies that spherical symmetry does not even require the existence of a point at the center [69].

On the other hand, static spacetime is characterized by a time coordinate t that satisfies two properties, namely, all metric components are independent of t and the geometry must remain unchanged by time reversal $t \to -t$ [69]. The first condition does not guarantee the latter, a situation that happens, for example in a rotating star and in this case the spacetime is said to be stationary.

The structure equations of relativistic stars can be better understood when contrasted with the Newtonian standpoint, so it is pertinent to quickly review the Newtonian equations for hydrostatic equilibrium. In this case consider a static and spherically symmetric star with pressure p and mass density ρ. The total mass interior to the radius r is given by

$$m(r) = \int_0^r 4\pi u^2 \rho(u) du, \tag{2.25}$$

or as a differential equation:

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2, \tag{2.26}$$

subjected to the initial condition $m(0) = 0$. A thin spherical shell of radius r and thickness dr has its mass given by $dm = 4\pi r^2 \rho dr$, therefore it feels a gravitational force given by [78]

$$F = -\frac{4\pi\rho(r) r^2 m(r) dr}{r^2} = -4\pi\rho(r) m(r) dr. \tag{2.27}$$

Besides the inward gravitational force, the shell also experiences an outward buoyant force F_b, which is equal to the pressure force on the inner surface of the shell minus the pressure force on its outer surface, namely [78]

$$F_b = 4\pi r^2 \left[p(r) - p(r+dr)\right] = -4\pi r^2 p'(r) dr. \tag{2.28}$$

Hydrostatic equilibrium is achieved when these two forces are equal, yielding

$$\frac{dp(r)}{dr} = -\frac{m\rho}{r^2}. \tag{2.29}$$

Moreover, the gravitational potential obeys the relation

$$\frac{d\Phi(r)}{dr} = \frac{m(r)}{r^2}. \tag{2.30}$$

The above equation satisfies the condition that $-\Phi'(r)$ must be equal to the ratio between the Newtonian gravitational force and the mass $-m(r)/r^2$. An arbitrary constant of integration is assumed so that $\Phi(r) \to 0$ as $r \to \infty$.

Now let is turn to the general relativistic picture of static and spherically symmetric stars, also called Schwarzschild stars [1]. Schwarzschild stars are mathematically represented by the following line element

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.31}$$

By solving the Einstein field equations for the above line element, treating matter as a perfect fluid, one is led to a set of four equations which completely determines the structure of a compact star:

$$\frac{dm(r)}{dr} = 4\pi\varepsilon(r) r^2; \tag{2.32}$$

$$\frac{dp(r)}{dr} = -\frac{m(r)}{r^2}\left[\varepsilon(r) + p(r)\right]\left[1 + \frac{4\pi r^3 p(r)}{m(r)}\right]\left[1 - \frac{2m(r)}{r}\right]^{-1}; \tag{2.33}$$

$$e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r}; \tag{2.34}$$

$$\frac{d\Phi(r)}{dr} = \frac{1}{1 - \frac{2m(r)}{r}}\left(\frac{m(r)}{r^2} + 4\pi p(r) r\right). \tag{2.35}$$

These equations are called the Tolman–Oppenheimer–Volkoff (hereafter abbreviated by TOV) equations[31] [15]. This set of equations replaces the Newtonian expressions for hydrostatic equilibrium. It is worth

[31] The derivation of the TOV equations can be found at Appendix A.

noticing by looking at the equation (2.29), the fundamental equation of Newtonian astrophysics, and comparing with equation (2.33) that general relativity imposes four modifications with respect to the Newtonian case and three additional correction terms [10, 17]:

1. The mass density ρ has to be replaced by the total mass-energy density ε.
2. The inertial mass density is given by $\varepsilon + p$ (first factor on the right hand side).
3. Pressure acts as an active volume correction (second factor).
4. The last factor represents three-space metric contributions essential to determine stability properties of the solutions. The surface of the object always has to be far outside the Schwarzschild surface.

So we recover the Newtonian equations for stellar structure when $p \ll \varepsilon$ (roughly speaking for sound velocities much less than the speed of light), $\frac{2m(r)}{r} \ll 1$ (low compactness) and low pressure-mass $4\pi p(r) r^3 \ll m(r)$ [10]. It is also interesting to notice the dual role of pressure in (2.33). In Newtonian gravity the pressure within a star has only the role of opposing self-gravity, and its gradient is responsible to prevent the star from collapsing [79]. In general relativity on the other hand, all sources of energy and momentum contribute to gravity. Therefore pressure influences the stress-energy tensor and acts as a source of gravity, not only opposing but also enhancing it. Consequently matter in a relativistic star has to withstand much larger internal forces to maintain hydrostatic equilibrium. This role of pressure becomes more pronounced with increasing compactness, the ratio between mass and radius, which can be seen as a measure of the strength of its gravity. The assertion that neutron stars have a maximum mass beyond which the collapse is inevitable, independently of the model proposed, is intimately connected to this fact [18, 79]. This maximum mass has no Newtonian analogue and it is fundamentally different from the Stoner-Landau-Chandrasekhar limit in Newtonian gravity, attained asymptotically for ultra-relativistic fermions [79]. The critical mass for white dwarfs is very close to the Stoner-Landau-Chandrasekhar limit, which in some sense reinforces the Newtonian character of its structure [79]. In the case of neutron stars the Oppenheimer-Volkoff limit is about a few times smaller than the limiting mass that would be obtained using Newtonian gravity.

Despite its simple appearance, the highly non-linear character of the Einstein field equations, as well as the fact that spacetime and matter act upon each other, implies that an analytic solution of the TOV equations is impossible for general problems. It follows that numerical computation is a fundamental aspect of this research field. Numerical relativity is a vast subject on its own and there are many excellent books dedicated to it [19, 60, 80]. With respect to the TOV equations, the system can be integrated for a given equation of state with the conditions $m(0) = 0$, $\varepsilon(0) = \varepsilon_c$ (an arbitrary value for the central energy density) and $\Phi(0)$ arbitrary[32]. The condition $m(0) = 0$ can be justified as follows. Imagine a tiny sphere with

[32] The arbitrariness in the initial value for $\Phi(r)$ is removed by matching the solution at the surface of the star with the analytic

radius ϵ. Its circumference is given by $2\pi\epsilon$ and its proper radius $|g_{rr}|^{1/2}\epsilon$. Consequently the ratio between the circumference and the radius is $2\pi|g_{rr}|^{-1/2}$. On the other hand, the manifold structure demands that the spacetime around $r = 0$ must be locally flat (just as any other point), then the referred ratio must be 2π, implying $g_{rr}(0) = 1$. Therefore as r approaches zero, $m(r)$ also goes to zero, in fact even faster than r [69].

Since $m(0) = 0$, one might think that the gravity in the center is weak and general relativistic effects could be negligible in its neighborhood. Actually this line of reasoning carries a Newtonian imprint in thinking about gravity. Although it is true that the Newtonian acceleration is small near the center, the relativistic correction is significant due to the contribution of pressure to gravity [79]. The integration is performed from the center until the pressure drops to zero (because zero pressure can support no overlying matter against the gravitational attraction) at some point $r = R$, which is subsequently interpreted as the radius of the particular star with central energy density ε_c and mass given by $m\,(r = R) = M$.

2.3.3 Matter at Supranuclear Densities

It is usually assumed that general relativity is the ultimate theory of gravity for describing compact stars. Fixing the gravitational side in this way, properties like the mass-radius relation of a family of compact stars is uniquely determined by the equation of state. Although many advances have been made since the free neutron-gas model of Oppenheimer and Volkoff, the search for the equation of state at supranuclear densities keeps puzzling both the theorists and observers [15, 81].

The problem is that quantum chromodynamics (the fundamental theory of strong interaction on which nuclear and hadronic physics is based) is notoriously hard to solve for temperatures and densities relevant for compact stars. In other words, the associated ground state of the strongly interacting matter at this regime is largely unknown, and since terrestrial experiments are not able to sufficiently constrain the equation of state, some uncertainty about the matter composition of such objects is inevitable [16, 79].

So in practice compact stars act as astrophysical laboratories for nuclear and particle physics and their measurements are crucial to distinguish between different theoretical models for the equation of state [21]. However, most of the electromagnetic observations available provide weak constraints for properties as mass, spin, and gravitational redshift [82]. Observations in which mass and radius are measured simultaneously could in principle provide stronger constraints. The problem is that these measurements deeply rely on the details regarding radiation mechanisms at the neutron star surface and absorption in the inter-

exterior solution which is an asymptotically flat vacuum solution. According to Birkhoff's theorem, the Schwarzschild solution, which describes a static, spherically symmetric vacuum spacetime, is the only spherically symmetric, asymptotically flat solution to the Einstein vacuum field equations [10, 72].

stellar medium, being subject to systematic uncertainties [82]. Until very recently, the main outcome of this scenario was a wide range of possible astrophysical properties for neutron stars, most notably perhaps was an almost 50% variation on their predicted size [83]. However, several studies using NICER and XMM data are changing this situation [84, 85, 86]. For instance, the radius of the most massive pulsar known, PSR J0740+6620, has been set in $R = 12.39^{+1.30}_{-0.98}$ km. The radius of a $1.4\,M_{\odot}$ neutron star was also updated to $R_{1.4} = 12.18^{+0.56}_{-0.79}$ km [85]. Moreover, the measurements of massive neutron stars ($M \approx 2\mathrm{M}_{\odot}$) suggests that the high density equation of state is stiff[33] [79]. Currently, three pulsars fall under this regime, namely, PSR 1614–2230 ($M = 1.908 \pm 0.016\mathrm{M}_{\odot}$), PSR J0348+0432 ($M = 2.01 \pm 0.04\mathrm{M}_{\odot}$) and PSR J0740+6620 ($M = 2.08 \pm 0.07\mathrm{M}_{\odot}$) [86]. NICER data have also been used to study configurations at the threshold which marks the ultracompact regime (defined by $\mathcal{C} > 1/3$). Although such stars are not currently an observational reality, they may become important in the near future [87].

In fact, our degree of uncertainty about the behavior of matter above the nuclear density is great enough to consider that beyond some threshold density a neutron star core may contain deconfined quarks. This is motivated by the so-called strange matter hypothesis, developed independently by Bodmer and Witten, which asserts that the true ground state of the strong interaction is strange quark matter, composed of an approximately equal proportion of up, down and strange quarks[34] [88, 89]. Ultimately one may argue that if strange-quark matter is the most stable form of matter at supranuclear densities as the ones found in compact stars, then all observed neutron stars could actually be strange stars. In addition, a recent study has suggested that the existence of quark cores in massive neutron stars should be considered the standard scenario, and not an exotic alternative [90]. More cautiously though, it could be said at least that the existence of strange stars is theoretically well founded, and if confirmed it would provide important constraints on the QCD parameters [7]. It is another intricate problem to distinguish between strange and neutron stars through observations, since for the canonical mass of $1.4\mathrm{M}_{\odot}$ gravity dominates the strong interaction, and the resulting strange stars are similar in size when compared with regular neutron stars [7].

Despite all challenges around measuring compact star properties, the future looks promising with regard to new constraints on dense matter and strong gravity. For instance, the Neutron Star Interior Composition Explorer (NICER), successfully installed on the International Space Station in 2017, is devoted to achieve precise mass and radius measurements of neutron stars through soft X-ray timing. NICER covers a energy range from $0.2 - 12$ keV, offers a high effective area ($> 2000\,\mathrm{cm}^2$ at 1.5 keV and $600\,\mathrm{cm}^2$ at 6 keV) and an unprecedented timing precision (absolute time-tagging resolution of < 300 ns) [91]. The Large Observatory For X-ray Timing (LOFT) (proposed to The European Space Agency) is a concept mission that

[33] Stiff means that for a given energy density the pressure is higher when compared with a softer equation of state. The stiffest equation of state compatible with causality is the one for which the speed of sound and the speed of light are the same [1].

[34] In the quark phase the difference between quark masses is significantly smaller than their respective Fermi energies, therefore the equilibrium configuration should involve an equal fraction of these three flavors, with a strangeness fraction per baryon of almost unity [7].

aims to study the properties of dense matter and strong gravity by X-ray observations of accreting neutron stars and black holes, with a very large effective area up to 30 keV [92]. Together with other proposed instruments (for example: FAST, ATHENA and SKA), these observations may provide an unprecedented amount of data on compact stars, covering the electromagnetic spectrum from radio to gamma rays [91].

Another possibility for obtaining information about compact star composition is due to gravitational radiation. Gravitational waves will be better discussed in Chapter 3, however, it is interesting to mention a recent landmark, the observation made in August 2017 of two coalescing neutron stars on the event GW170817. This observation placed constraints in the tidal effects and consequently on the equation of state.

2.3.4 Sound Speed

The sound speed is a key physical property in characterizing dense matter, directly related to the equation of state. For a general medium, the effective sound speed is the propagation speed of acoustic scalar fluctuations in the rest frame, given by [93]

$$c^2_{s\,eff} = \frac{\delta p}{\delta \varepsilon}. \tag{2.36}$$

In the general case the pressure perturbation δp is composed of adiabatic and non-adiabatic parts:

$$\delta p = c_s^2 \delta\varepsilon + \delta p_{nad}, \tag{2.37}$$

where c_s is the adiabatic sound speed defined by

$$c_s^2 := \frac{p'}{\varepsilon'} = \omega + \frac{\varepsilon}{\varepsilon'}\omega'. \tag{2.38}$$

For barotropic fluids (or any adiabatic medium) $\delta p_{nad} = 0$ and $c_s = c_{s\,eff}$, so the sound speed is given simply by

$$c_s^2 = \frac{dp}{d\varepsilon} \tag{2.39}$$

Causality demands that the sound speed must be less than the speed of light. Non-relativistic models under appropriate densities predict $c_s \ll 1$, while gases of ultrarelativistic massless particles may present c_s as high as $1/3$. It is sometimes conjectured that the sound speed in any medium should be smaller than the velocity of light in vacuum divided by $\sqrt{3}$. However it has been argued that there is a tension between this view and the existence of neutron stars with masses $2\,\mathrm{M}_\odot$ for all reasonable low density equations of state [94]. Besides that, local mechanical stability imposes that the sound speed must be real, otherwise if it is imaginary it will tend to collapse the matter instead of producing a wave [93].

2.3.5 Tidal Deformability

The tidal response is the latest observable macroscopic property added to the set of astrophysical constraints regarding compact stars [83]. This effect is strongly tied to the equation of state, as first observed by Flanagan and Hinderer [95]. Since relativistic stars are not point-like structures but rather finite size objects, they are tidally deformed when exposed to the influence of an external tidal field, an effect expressed through an induced quadrupole moment. The coalescence between compact binaries, detectable with gravitational waves, sets a natural stage for this interaction, in which one member is subject to the gravitational field of its companion. The induced quadrupole moment affects the binding energy of the system, increasing the rate of emission of gravitational waves [83]. This deformation leaves a measurable imprint, quantified through a single parameter termed the tidal deformability, on the form of the gravitational wave during the inspiral phase, that is, even before the two neutron stars touch each other [23]. The desire of constraining the equation of state of compact stars by measuring the tidal deformability via gravitational waves became a reality in 2017, with the detection of the inspiral signal from the binary neutron star system GW170817 [96]. It initially set the dimensionless tidal deformability of a $1.4\,\mathrm{M}_\odot$ star at $\Lambda_{1.4} = 800$ for low spin priors, but later this result was improved for $\Lambda_{1.4} = 190^{+390}_{-120}$ [96]. After several sophisticated analysis the overall finding is that the event GW170817 constrains the radius of a $1.4\,\mathrm{M}_\odot$ star $R \leqslant 13.2 - 13.7\,\mathrm{km}$ [23].

In order to discuss tidal effects a little further, consider a static and spherically symmetric star placed in a static external quadrupolar tidal field ε_{ij}. To linear order, the tidal deformability λ is defined as the proportionality coefficient between the external quadrupolar tidal field and the induced quadrupole moment Q_{ij} of the compact star [23]

$$Q_{ij} = -\lambda\varepsilon_{ij}. \tag{2.40}$$

Both ε_{ij} and Q_{ij} are symmetric and traceless, and defined to be coefficients in an asymptotic expansion of the total metric at large distances from the star [82]. It is worth mentioning that the above expression is quite general and valid irrespective of a Newtonian or relativistic approach [97].

The tidal deformability is related to the coefficient for the tidal quadrupole moment k_2 (called second Love number[35]) by

$$\lambda = \frac{2}{3}k_2R^5 \tag{2.41}$$

It is customary to define a dimensionless tidal deformability Λ by dividing both sides of the above expression by M^5, yielding

$$\Lambda = \frac{\lambda}{M^5} = \frac{2}{3}k_2\mathcal{C}^{-5}, \tag{2.42}$$

For a fixed equation of state the value of Λ can vary three orders of magnitude from $\approx 1\mathrm{M}_\odot$ stars to the maximum mass configuration (neglecting other effects such as rotation and magnetic fields), therefore highly

[35] The name is after Augustus Edward Hough Love who studied the deformability of Earth and the Moon in Newtonian theory.

sensitive to the star's composition [97]. Thus even with one event the tidal deformability can potentially provide robust information about the equation of state.

Before providing the expression from which the second Love number is calculated, consider a perturbation around the spacetime metric associated with the equilibrium solution of the TOV equation [23],

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{2.43}$$

where $h_{\mu\nu}$ represents the metric perturbation. Its ansatz adopts spherical harmonics in such way that the perturbed metric can be expressed as

$$h_{\mu\nu} = diag\left[-e^{\alpha(r)}H_0(r), e^{\beta(r)}H_2(r), r^2K(r), r^2\sin^2(\theta)K(r)\right]Y_{20}\left(\theta,\phi\right), \tag{2.44}$$

where both $\alpha(r)$ and $\beta(r)$ are metric functions from the equilibrium solution [23]. The perturbations of the energy-momentum tensor have components given by

$$\delta T_0{}^0 = -\delta\varepsilon Y_{20}\left(\theta,\phi\right); \quad \delta T_i{}^i = \delta p Y_{20}\left(\theta,\phi\right). \tag{2.45}$$

By solving the perturbed field equations, namely

$$\delta G_\mu{}^\nu = 8\pi\delta T_\mu{}^\nu, \tag{2.46}$$

using equation (2.44) and the perturbed energy-momentum tensor one is lead to the relations:

$$-H_2(r) = H_0(r) = H(r); \quad K'(r) = H'(r) + 2H(r)\alpha'(r). \tag{2.47}$$

The second Love number is better expressed in terms of a radial function $y(r)$, given by

$$y(r) = \frac{rH'(r)}{H(r)}, \tag{2.48}$$

which obeys the differential equation

$$ry' + y^2 + yF(r) + r^2Q(r) = 0. \tag{2.49}$$

Equation (2.49) has to be solved as part of a coupled system containing the hydrostatic equilibrium equations, alongside with the boundary condition $y(0) = 2$ [97, 23]. The auxiliary radial functions $F(r)$ and $Q(r)$ are respectively given by

$$F(r) = \frac{1 - 4\pi r^2\left(\varepsilon(r) - p(r)\right)}{1 - \frac{2m(r)}{r}}, \tag{2.50}$$

$$Q(r) = 4\pi\left(1 - \frac{2m(r)}{r}\right)^{-1}\left[5\varepsilon(r) + 9p(r) + \frac{\varepsilon(r) + p(r)}{c_S^2} - \frac{6}{4\pi r^2}\right] - 4\left[\frac{m(r) + 4\pi r^3 p(r)}{r\left(r - 2m(r)\right)}\right]^2. \tag{2.51}$$

Finally, the Love number k_2 can be computed using[36]

$$
\begin{aligned}
k_2 =& \frac{8\mathcal{C}^5}{5}(1-2\mathcal{C})^2\left[2+2\mathcal{C}(y_R-1)-y_R\right]\{2\mathcal{C}\left[6-3y_R+3\mathcal{C}(5y_R-8)\right] \\
&+4\mathcal{C}^3\left[13-11y_R+\mathcal{C}(3y_R-2)+2\mathcal{C}^2(1+y_R)\right] \\
&+3(1-2\mathcal{C})^2\left[2-y_R+2\mathcal{C}(y_R-1)\right]\ln(1-2\mathcal{C})\}^{-1}
\end{aligned} \tag{2.52}
$$

where $y_R = y(r=R)$[37].

The Love number vanishes for $\mathcal{C} = 1/2$, therefore for a Schwarzschild black hole $k_2 = 0$. The incompressible fluid sets the upper limit for the value of k_2, since by definition it can not be compressed, it is squished and flows in reaction to the external quadrupole potential [23]. On the other hand, for a highly compressive fluid a high density core can be formed with a low density mantle. By increasing the energy density in the core the fluid can then react to an external potential and only a small quadrupole moment is induced [23]. So the Love number provides information about the compressibility of the fluid, in other words, the softness or stiffness of the equation of state. Regarding size, a larger compact star is more likely to be deformed by its companion, since the gravitational bound is relatively weaker when compared to a smaller compact star with the same mass [96].

[36] The Love number in general relativity, as expressed in equation 2.52, will be smaller than the newtonian case, given by $k_2^{Newton} = \frac{1}{2}\left(\frac{2-y_R}{y_R+3}\right)$, due to the correction terms from the compactness of the star [23].

[37] In the case of self-bound configurations, like strange stars, y_R needs a correction to account for the energy discontinuity between the star's surface and its outside, namely, $y_R \to y_R - \frac{4\pi R^3 \varepsilon_S}{M}$, in which ε_S is the energy density on the surface [97].

Chapter 3

Ultracompact Stars

This chapter aims to explore the ultracompact regime for relativistic stars by studying some of its interesting physical properties, as well as its perspectives regarding gravitational wave astronomy. A pleasing starting point is to investigate the rich visual features that emerge as compactness increases, taking a typical neutron star as a basis. If a regular neutron star could be somehow compressed, as the compactness grows more and more surface would be visible, due to gravitational lens effects. Through this process some interesting effects would appear, for instance, at $R \simeq 1.76R_S$ $\left(R_S \equiv 2M \simeq 2.953\frac{M}{\mathrm{M}_\odot}\,\mathrm{km}\right)$ the whole surface of the star becomes visible [36]. The opposite point on the neutron star's surface with respect to the observer is now mapped onto a circle centered on the neutron star. This circle is called an Einstein ring[38], or to be more precise, the "first surface Einstein ring". If compactness is increased even further, another Einstein surface ring is possible, which is associated with light from the point just below the observer that would be visible after circling the neutron star once again. An ultracompact star is the limit case because it exhibits infinitely many surface Einstein rings to any observer. With enough resolution all surface features of an ultracompact star would be visible, even the back [36].

Another interesting optical consequence of the ultracompact regime is that any star with proper radius $R \leqslant 1.5R_S$ actually resolved telescopically would appear to have the same radius of a $R = 1.5R_S$ star [36]. However, Einstein himself tried to refute the possibility of compact objects whose radius were less than 1.5 times the critical radius [99]. The idea was to study the circular orbits of massive particles in the Schwarzschild spacetime. As the orbits get smaller, the velocity of the particle increases, until the

[38] Rudi Mandl, a Czech engineer, approached many scientists in 1936 (including Einstein) to share his idea that a foreground star may act as a "gravitational lens" for light corning from a background star. Einstein wrote an article on the subject, which said that at perfect alignment a background star undergoing lensing by a foreground one will appear as a very bright ring, hence the name "Einstein ring". However, Einstein concluded that the ring would not be resolvable, being pessimistic about a possible detection. Fortunately, history ended up favoring Mandl [98].

speed of light is reached at $R = 1.5R_S$ and no object could, according to his conclusion, get smaller than this. Although the reasoning is correct, Einstein neglected radial motions which are inevitable for all particles orbiting below 1.5 times the critical radius [99]. In addition, the aforementioned uncertainty underlying the equation of state leaves space for interesting theoretical scenarios. One cannot rule out *a priori* the existence of ultracompact neutron stars that are smaller than their photon spheres[39]. However, in reality this possibility seems unlikely at first sight since it cannot be realized by viable equations of state for neutron stars [38].

3.1 Photons in the Schwarzschild Metric

According to the physicist Vladimir Fock the basic idea of general relativity is that the spacetime surrounding massive bodies is non-Euclidean [56]. With this spirit in what follows photon orbits in the Schwarzschild metric are considered to illustrate some visual properties of ultracompact stars [100]. The Schwarzschild solution is the metric in vacuum outside a spherical mass distribution, in our case represented by a relativistic star. It depends on a single parameter, the Schwarzschild radius of the central mass distribution M, defined by $R_S = 2M$. In the usual Schwarzschild coordinates (t, r, θ, ϕ) the metric is given by

$$g_{\mu\nu} = diag\left(-\left(1 - \frac{R_S}{r}\right), \left(1 - \frac{R_S}{r}\right)^{-1}, r^2, r^2 \sin^2(\theta)\right). \tag{3.1}$$

Spherical symmetry implies that orbits are confined to a single plane. One can choose the polar coordinates θ and ϕ in such a way that this plane is the equatorial plane, then $\theta = \frac{\pi}{2}$ is constant along the orbit, which implies $p^\theta = \frac{d\theta}{d\lambda} = 0$. Photon orbits are null geodesics, in other words, the photon four-momentum satisfies the geodesic equation and is null [100]

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\kappa} p^\nu p^\kappa = 0; \quad g_{\mu\nu} p^\mu p^\nu = 0. \tag{3.2}$$

It follows that the geodesic equations for the momentum components are

$$\frac{dp^t}{d\lambda} + \frac{R_S}{r^2\left(1 - \frac{R_S}{r}\right)} p^t p^r = 0; \tag{3.3}$$

$$\frac{dp^r}{d\lambda} + \frac{R_S\left(1 - \frac{R_S}{r}\right)}{2r^2}\left(p^t\right)^2 - \frac{R_S}{2r^2\left(1 - \frac{R_S}{r}\right)}\left(p^r\right)^2 - r\left(1 - \frac{R_S}{r}\right)\left(p^\phi\right)^2 = 0; \tag{3.4}$$

$$\frac{dp^\phi}{d\lambda} + \frac{2}{r} p^\phi p^r = 0. \tag{3.5}$$

[39] Photons traveling at the photon sphere are not in a stable orbit. Any small perturbation will cause them to spiral either in or out [37].

The above equations can be integrated analytically, which yields

$$p^t = \frac{k^t}{\left(1 - \frac{R_S}{r}\right)}; \tag{3.6}$$

$$p^r = \pm\sqrt{(k^t)^2 - \left(k^\phi\right)^2 \frac{\left(1 - \frac{R_S}{r}\right)}{r^2} + k^r\left(1 - \frac{R_S}{r}\right)}; \tag{3.7}$$

$$p^\phi = \frac{k^\phi}{r^2}; \tag{3.8}$$

where k^t, k^r and k^ϕ are constants of integration, which parameterize the photon orbits [100]. Since p^μ is a null vector it follows that

$$0 = g_{\mu\nu} p^\mu p^\nu = k^r. \tag{3.9}$$

Now consider a measurement of the photon energy by a local inertial observer momentarily at rest. Its four-velocity is given by

$$u^\mu = \left(\frac{1}{\sqrt{\left(1 - \frac{R_S}{r}\right)}}, 0, 0, 0\right). \tag{3.10}$$

Using equations (3.6), (3.7) and (3.8) it follows that

$$E = -g_{\mu\nu} u^\mu p^\nu = \frac{k^t}{\sqrt{\left(1 - \frac{R_S}{r}\right)}}. \tag{3.11}$$

When $r \to \infty$ the above equation reduces to $E_\infty = k^t$, therefore the constant k^t can be seen as the photon energy at infinity. The third constant of integration, k^ϕ, can be more easily understood by looking at the ratio $b := \frac{k^\phi}{k^t}$, which is called the impact parameter of the photon trajectory [100]. This fact becomes more evident at the limit of vanishing central mass $R_S \to 0$. The photon orbit depends on three functions: $t(\lambda)$, $r(\lambda)$ and $\phi(\lambda)$. If one is not interested in the lapse of the time coordinates along the orbit, the trajectory $\phi(r)$ is sufficient, that is, the set of all points (r, ϕ) that the photon passes through, which can be expressed by

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \frac{p^\phi}{p^r}. \tag{3.12}$$

Using the expressions for p^ϕ and p^r and setting $R_S = 0$ yields

$$\frac{d\phi}{dr} = \pm\frac{b}{r^2\sqrt{1 - \frac{b^2}{r^2}}}. \tag{3.13}$$

The above equation can be integrated, which gives

$$\sin\left(\phi - \phi_0\right) = \pm\frac{b}{r}, \tag{3.14}$$

which is the equation for a straight line in polar coordinates. The impact parameter can be visualized by assuming that if the photon is far from gravitating objects it travels following a straight line, in that case the impact parameter is the distance between the closest approach of the continuation of this straight line and the center of the gravitating body [36].

The photon trajectory is defined by

$$\frac{d\phi}{dr} = \frac{p^\phi}{p^r} = \pm\frac{b}{r^2\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{R_S}{r}\right)}} \implies \tag{3.15}$$

$$\phi(r) = \phi_0 \pm \int_{r_0}^{r} \frac{bdr}{r^2\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{R_S}{r}\right)}}, \tag{3.16}$$

which is an elliptic integral of the first kind with no analytic solutions, but numerical integration is straightforward. Since in our case the central mass is a star, then only those parts of the orbits that are outside the star are relevant [100]. An important case in (3.16) is found when $\Delta\phi$, that is, $(\phi - \phi_0)$ diverges to infinity. In this situation the photon will circle the massive star in a region called *photon sphere*, located at $r = 1.5R_S$ [37].

Photons emitted from infinity with impact parameters slightly greater than a certain critical value, $b_c = 3\sqrt{3}R_S/2$, will spiral around the compact star near the photon sphere and then spiral out [37]. On the other hand, photons emitted from infinity with impact parameters slightly less than the critical value will spiral around the compact star near the photon sphere and then spiral in, eventually colliding with the compact star. The photon can also be emitted from the compact star surface, orbit near the photon sphere and then spiral back in again impacting the surface. In other words the three cases of photon orbits near a gravitational body can be classified as [37]: “always outside the photon sphere”, “crossing the photon sphere” and “always inside the photon sphere”. The first case is the case of a photon passing a compact object, reaching some critical radius R_c and then escaping towards infinity. In this case the photon does not reach or cross the photon sphere. Its distance from the object decreases monotonically until R_c and monotonically increases thereafter. The second case is that of a photon continuing to come toward the compact object until it impacts the surface and the distance decreases monotonically. The third case is that of a photon emitted from the surface of the compact object, reaching a critical radius R_c and then falling back and returning to its surface. As a matter of curiosity this critical radius is given by

$$R_c = \frac{2b}{\sqrt{3}}\cos\left[\frac{1}{3}\arccos\left(-\frac{\sqrt{27}R_S}{2b}\right) + \frac{2n\pi}{3}\right], \tag{3.17}$$

where $n = 0$ corresponds to the first case and $n = 2$ to the third case [37].

Figure (3.1) depicts the photon orbits coming from a compact star surface and reaching an observer [100]. The impact parameter varies from $b = 0$ (radial emission) to some b_{max}. In the picture, the compact star is larger than its photon sphere, in this case the impact parameter is maximum when emitted in the tangential direction. Substituting $p_r = 0$ at $r = R$ in equation (3.7) it follows that $b_{max} = R\left(1 - \frac{R_S}{R}\right)^{-1/2}$. If the compact star is smaller than its photon sphere, a photon emitted tangentially to the stellar surface remains confined within the photon sphere and does not reach the observer. The photon will only leave the photon sphere if its impact parameter is smaller than the critical parameter $b_c = 1.5\sqrt{3}R_S$.

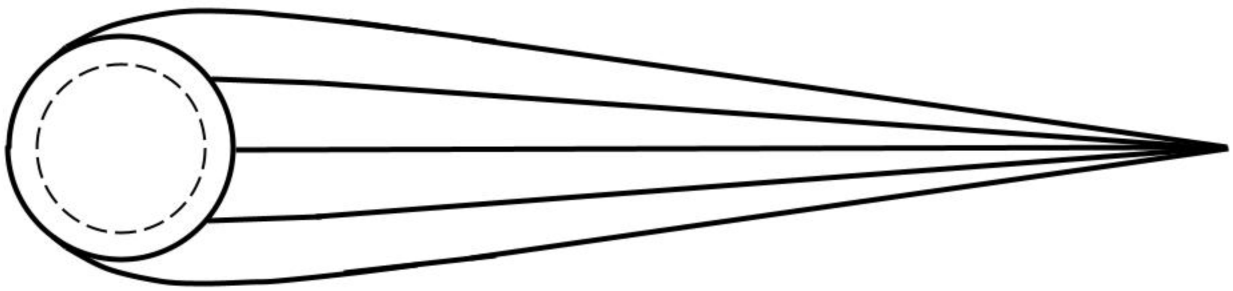

Figure 3.1: *Orbits of photons reaching an observer from a compact star surface. The dashed line marks the photon sphere.*

Light deflection is particularly important for objects whose radius is not too large compared to its Schwarzschild radius, which is precisely the case found in ultracompact stars. Two of the main consequences of light deflection are enhanced surface visibility and increased angular size. Consider, for example, an observer that is close enough to resolve some compact star but at the same time many compact star radii away [100]. If $R \gg R_S$ light deflection can be neglected and only the near side is visible. The far side is hidden from view. If this is not the case, light deflection means that photons emitted on the far side may be accessible to the observer, hence a larger part of the surface is visible. Table 3.1 depicts the surface visibility for different values for the ratio R/R_S, where the last case, $R/R_S = 1.5$, corresponds to an ultracompact star [100]. Note that compact stars with $R/R_S \leqslant 1.75$ can not eclipse anything at

R/R_S	∞	3	2	1.75	1.509	1.5
Visible part of the surface	50%	74%	94%	100%	200%	∞

Table 3.1: *Surface visibility for different values of R/R_S.*

all. Consider now the same observer, but now at an intermediate distance. The angular size of the star as determined by the outermost photon orbit, $\alpha = b_{max}/d$, where d is the observer's distance can be summarized as in Table 3.2 [100]. For stars smaller than its photon sphere, the angular size is a function of mass only and completely independent of the geometric size.

R/R_S	∞	> 1.5	< 1.5
b_{max}	R	$R/\sqrt{1 - R_S/R}$	$1.5\sqrt{3}R_S$
α	$\alpha = \alpha(R)$	$\alpha = \alpha(M, R)$	$\alpha = \alpha(M)$

Table 3.2: *Angular size of the star as determined by the outermost photon orbit.*

The effects discussed so far are explored through information carried by electromagnetic waves. In practice, testing the nature of compact objects through electromagnetic radiation has always been challenging. Complications arise, for example, due to the incoherent nature of electromagnetic radiation in astrophysics (as well as the modeling uncertainties associated with such emissions). Another important example is the absorption by interstellar medium [32]. With this in mind, the rest of the chapter discusses why gravitational waves are ideal for probing the strong gravity regime, and how they are renewing the interest in theoretical scenarios where new classes of highly compact objects could emerge.

3.2 Gravitational Waves

As previously observed, in Newtonian theory gravitation acts instantaneously, a feature that excludes the possibility of gravitational waves. Even though the mathematician and physicist Pierre Simon de Laplace studied the consequences of a gravitational force that does not travel instantaneously, but rather at a finite speed [101]. Laplace was dealing with the onerous problem of describing the Moon's orbit, which differs from planetary orbits since it is subject to many additional influences [101]. In the end Laplace concluded that the Newtonian description was sufficient, asserting that the speed of propagation of the gravitational force had to be at least 100 million times the speed of light. This idea came back later with Poincaré in a work entitled "Sur la dynamique d' l'électron.", which summarizes his views on Relativity. He applied his notion of a *onde gravifique* to tackle the problem of the anomaly in the perihelion shift of Mercury, although he deduced that the effect was not sufficient to explain the anomaly [101, 102].

The modern view on gravitational waves begins just a few months after the publication of general relativity, when Albert Einstein predicted the existence of curvature perturbations on a flat and empty spacetime, called gravitational waves [103]. The notion of a gravitational wave in general relativity is, at least conceptually, intuitive. It reserves some analogy with electromagnetic wave theory, introduced when the Coulomb theory of electrostatics was replaced by the theory of electrodynamics, predicting that waves transport information through space about the dynamics of charged systems. Similarly, mass-energy distributions changing in time should provide information about its dynamics in the form of waves [104]. However, differently from electromagnetic waves, gravitational waves are metric waves [104]. When a gravitational wave propagates it perturbates the geometry, as well as the proper distance between spacetime points. Therefore gravitational waves do not propagate in spacetime, instead they are precisely waves of spacetime [104].

Einstein's gravitational wave solutions were obtained via a linearized gravitational wave theory, treating weak waves as perturbations of a flat background. In addition, general relativity is consistent with special relativity regarding the fact that nothing travels faster than light. Therefore these changes in the gravitational field can not be felt everywhere instantaneously, actually they propagate at exactly the same speed as vacuum electromagnetic waves, the speed of light. Some physicists had serious doubts about their existence, similar to what happened when electromagnetic waves were first predicted[40]. Arthur Eddington, for instance, thought that these weak-field solutions in general relativity were just coordinate changes which were *propagating with the speed of thought* [103]. Surprisingly, even Einstein himself for some time nurtured doubts about his own proposal, as can be observed in a letter wrote to Max Born [105]:

> *Together with a young collaborator [Rosen], I arrive at the interesting result that gravitational waves do not exist, though they have been assumed a certainty to the first approximation.*

Eventually Einstein became fully convinced of the existence of gravitational waves, although his collaborator on the subject, Nathan Rosen, always thought that they were just a formal mathematical construct, without any physical meaning.

The linearized theory is an approximation since it is not valid for sources where gravitational self-energy is not negligible. A gravitational wave description valid for self-gravitating systems with slowly moving bodies was provided by Landau and Lifshitz in 1941. Moreover, as previously observed, General relativity is a nonlinear theory and in general a sharp distinction between the parts of the metric that represent the waves and the rest is not possible. Only in specific situations is it possible to clearly define gravitational radiation [103].

The skepticism about the reality of gravitational waves strongly persisted until 1957 when Herman Bondi did a *gedanken* experiment showing that gravitational waves indeed carry energy [103]. Imagine that a system is composed of two beads sliding on a stick and consider that the friction opposing their motion is small. If a plane gravitational wave reaches this system, the beads will move back and forth because the proper distance changes due to the metric perturbation. This change obeys the geodesic equation and the proper displacement between the two beads is a function of the gravitational wave metric perturbation. So the friction between the beads and the stick heats the system, increasing its temperature. Since there was an energy transfer from the gravitational waves to the system in the form of a temperature increase,

[40] Lord Kelvin said: "The so-called "electromagnetic theory of light" has not helped us hitherto . . . it seems to me that it is rather a backward step . . . the one thing about it that seems intelligible to me, I do not think is admissible . . . that there should be an electric displacement perpendicular to the line of propagation" [103].

the experiment successfully demonstrates that gravitational waves indeed carry energy, thus being a real physical entity [103].

Nevertheless there was little hope for a real detection of the phenomenon, and in the 1960s Joseph Weber started the experimental side of this research field, essentially alone [103]. Weber even claimed to have detected gravitational waves by measuring the energy they transfer to large metal cylinders, but there is a consensus that it was a false detection [78]. In face of this experimental status Steven Weinberg wrote [17]: *gravitational radiation would be interesting even if there were no chance of ever detecting any, for the theory of gravitational radiation provides a crucial link between general relativity and the microscopic frontiers of physics*. The first strong, although indirect, evidence for the existence of gravitational waves came from the observation of the energy loss from the binary pulsar system PSR 1913+16, discovered in data from the Arecibo radio telescope by Hulse and Taylor in 1974. The observed orbital decay was in full agreement with Einstein's General Theory of Relativity. For this discovery they earned the Nobel prize in physics in 1993. More recently another compact binary system, PSR J0737-3039, provided additional indirect evidence, also in agreement with general relativity [7].

The Laser Interferometer Gravitational-Wave Observatory (LIGO) project was initiated in the early 1990s with the goal of opening the field of gravitational-wave astrophysics through a direct detection. LIGO's first version operated from 2002 to 2010 and reported no gravitational waves [78]. Then the project was succeeded by an advanced version. In the first observing run of Advanced LIGO, almost a century after Einstein's prediction, two LIGO detectors picked a signal which matched the predictions from numerical simulations of the merger of a pair of black holes with $36\mathrm{M}_{\odot}$ and $29\mathrm{M}_{\odot}$, forming another black hole with $62\mathrm{M}_{\odot}$ [7]. The missing mass was radiated in the form of gravitational waves. The so expected detection was made on the 14th of September 2015 and was announced to the world in February 2016. It was the first verification of general relativity in a dynamical strong-field system, allowing the identification of more massive black holes than so far found in X-ray binaries [7]. Prior to the discovery it was expected that the first gravitational wave signal at Advanced LIGO would be from coalescing neutron stars, however, this type of detection had to wait until August 2017. The analysis showed that the signal GW170817 was compatible with coalescence of orbiting bodies with individual masses (1.36–1.60)$\mathrm{M}_{\odot}$ and (1.17–1.36)$\mathrm{M}_{\odot}$, and total mass $2.74^{+0.04}_{-0.01}\,\mathrm{M}_{\odot}$. These direct detections of gravitational waves mark a pathway for a new kind of astronomy, supplementing the traditional electromagnetic, cosmic ray and neutrino observations. The rate of discoveries has increased to the extent that one could soon expect about a hundred compact object coalescences per year [106].

3.3 A New Kind of Astronomy

Most of our understanding about the Universe is ultimately connected to observations made through electromagnetic signals. Therefore, even though from the theoretical point of view electromagnetic and gravitational waves share some similarities (for example, both types of waves are multipolar and oscillate transverse to the direction of propagation), it is natural to compare them in order to comprehend how the later can provide a new window to observe the Universe in a complementary direction, opening the modern period of multi-messenger astronomy [7, 53, 106].

First let is describe some basic properties of electromagnetic waves. They are oscillations of electric and magnetic fields propagating in a given spacetime, created when individual particles are accelerated. The electromagnetic radiation generated by the motion of a large number of microscopic charges gives rise to an incoherent superposition of waves with a dipolar structure in the wave zone, useful to infer, for instance, the thermodynamics of the source [7, 53]. The typical wavelengths of electromagnetic waves are much smaller than the size of their sources in coherent motion. In this case light can be approximated as a null particle following geodesics on a stationary background and the associated information is normally used to produce images. Also, electromagnetic waves interact strongly with matter and in general they are scattered many times as they propagate away from the sources before they reach our telescopes. This intense interaction guarantees that the power in the field, which decays proportionally to the inverse distance squared to the source, can be easily detected [53]. Particularly, observations of compact objects are typically performed in situations where spacetime fluctuations are negligible, either due to the timescales involved or because the backreaction of the environment on the object is irrelevant [32]. In electromagnetic observations one can find sources that are so strong for each waveband, so they can be detected without a deep understanding about them [106]. In other words, one does not need to understand nuclear fusion to see the Sun.

On the other hand, gravitational waves are oscillations of the spacetime itself, varying on a length scale which is much smaller than that of the 'background' curvature (gravity as experienced on a daily basis) [7]. In this case spacetime fluctuations are therefore relevant. Gravitational waves are generated by asymmetric bulk motion of macroscopic masses (for example, the motion of neutron stars in a binary), with the most prominent gravitational waves coming from very dense sources. Gravitational radiation produces a coherent superposition of waves with a quadrupolar structure in the wave zone. Regarding their coherence, gravitational waves are similar to laser light. Differently from electromagnetic waves, their gravitational counterparts are usually larger than the size of their sources and the associated information are more similar to stereo sound. In this case the geodesic approximation is not appropriate, although it can be used as a guide [32]. A fundamental property of gravitational waves is that they barely interact with matter and their propagation is practically absorption free, therefore they provide the cleanest signatures of the nature of compact objects, complementing the information furnished by telescopes and particle detectors. Apart from compact objects, the weak nature of gravitational waves are also expected to improve our understanding of other phenomena difficult to study by traditional means, for instance, the internal dynamics of supernova explosions and the quantum fluctuations in the very early Universe. The weakness of gravitational radiation also requires sophisticated statistical techniques. This involves a matching between templates of expected waveforms and the observed data stream. So, differently from the electromagnetic case, gravitational radiation demand some previous understanding about the associated waves and the sources that produce them [106]. Therefore, at the same time that gravitational waves carry key information that would otherwise remain hidden, it also makes its detection so challenging. Nevertheless the typical frequency is sufficiently low so the wave's amplitude, decaying like the inverse distance, can be tracked in time [53]. Fortunately, gravitational waves are subjected to less modeling uncertainties because they depend on fewer parameters than electromagnetic probes [40].

Even with such differences, gravitational and electromagnetic astronomy will establish a synergistic relationship. In electromagnetic astronomy one usually obtains a large amount of information about sources on a small region on the sky [107]. Gravitational wave astronomy has the potential to cover almost all the sky. An obvious drawback is that sources are poorly localized in comparison with traditional astronomical standards, but on the other hand more sources will be detectable, not only those sources to which the detectors are pointing. So electromagnetic and gravitational astronomy present some similarity with sight and hearing, respectively, enhancing their complementary character [107]. Besides that, sophisticated tools for data analysis used in gravitational wave observations are expected to be beneficial for X-ray and radio astronomy [7]. For example, gravitational wave inspired techniques are being adapted to detect gamma-ray pulsars in Fermi data [7].

3.4 Gravitational Wave Echoes and Ultracompact Stars

Fortunately, nowadays gravitational waves are a solid reality and their historical detection made by LIGO has opened exciting possibilities of testing the strong gravitational fields with unprecedented accuracy [40]. This brings hope for testing many models of compact objects in a regime in which they are expected to exhibit different predictions than its general relativistic counterparts. However, the gravitational wave signal is accurately known only for some special configurations, under very specific assumptions on the matter content [32]. For this reason a very rich phenomenon known as gravitational wave echoes, which is largely influenced by the underlying theory and the dynamics of the object, is gaining ground and different techniques have been proposed to model it [32].

The preferred systems for gravitational wave detection are compact binaries. The gravitational wave signal from compact binaries can be naturally divided in three stages, corresponding to different cycles in their evolution. The first is the inspiral stage corresponding to large separations, which can be approximated by post-Newtonian theory. The merger phase comes when the two objects coalesce. This stage can only be accurately described by sophisticated numerical simulations. The last stage is the ringdown phase when the merger end-product relaxes, becoming a stationary equilibrium solution of the field equations [40]. All three stages are important in the sense that they provide independent, unique tests of gravity and of compact gravitational wave sources. Here the last stage is particularly important, since ringdown waveforms may contain "echoes" that encode new physics in the strong gravity regime [49].

Black holes, bizarre objects predicted by general relativity, are the ones responsible for the most intense gravitational fields. A detailed knowledge of their gravitational radiation emitted as a response to a perturbation will improve our understanding about the properties of the horizon, as well as their mass and spin [108]. Although black holes are indeed the most extreme astrophysical objects, relativistic stars are comparable to black holes in the sense that their compactness has the same order. Naturally black holes and relativistic stars are fundamentally distinct objects because the second has a surface and an interior structure which is not causally disconnected from the exterior spacetime [108]. Similar to black holes, relativistic stars can respond to perturbations and emit gravitational waves, but their gravitational radiation is extremely rich in details, with vital physical information. Imprinted in these gravitational waves is possible to find a detailed map of their internal structure, which can be used to deduce the properties of matter at conditions not attainable in terrestrial laboratories, therefore this information will be very useful in order to investigate the equation of state of matter at high density [108].

At the dawn of this fascinating gravitational wave era in astronomy, ultracompact stars may play an important role, in particular for the characterization of the signal emitted in the final state of a compact binary merger [38]. It has been argued that the gravitational echoes mentioned earlier (secondary pulses of gravitational radiation after the main burst of radiation related to the post-merger ringdown waveform) are not, as commonly assumed, a unique prerogative of deviations from general relativity at the horizon scale. Similar signals may arise from a large class of horizonless ultracompact stars, thus featuring a photon sphere [109, 110]. Echoes would be a powerful physical property to distinguish different compact objects, possibly revealing a new branch of stable configurations [31]. First, echoes imply that the remnant is more compact than a neutron star with an ordinary equation of state [109]. In addition to that, although the post-merger ringdown waveform of an ultracompact star is initially identical to that of a black hole, the echoes in the late-time ringdown would present significant differences [38, 40]. Besides being a possible smoking-gun signature of exotic compact objects, from the theoretical perspective echoes are a rich phenomenon to study quantum corrections at this scale [111]. Recently some evidence for echoes have been reported, with controversial results [109]. In particular, a tentative detection of echoes in the post-merger signal of the neutron-star binary coalescence GW170817 has been claimed at a frequency of about 72 Hz with 4.2σ significance level. The typical echo time can be described as the light time from the center of the star to the photon sphere, namely [112]

$$\tau_{echo} = \int_0^{3M} \frac{dr}{\sqrt{e^{2\Phi}\left(1-\frac{2m}{r}\right)}}, \tag{3.18}$$

and the respective echo frequency can be approximated by $f_{echo} \approx \pi/\tau_{echo}$. Low frequency echoes like the one associated with the event GW170817 would require a compact star far deep in the ultracompact regime [109].

This scenario regarding gravitational echoes has motivated several studies, most of them aiming to predict low frequency signals (see for instance Refs. [38, 40, 109, 110, 111, 113]). However, it is a well known fact that viable equations of state for neutron stars are not generally compatible with echoes, and since the first study most analyses have considered constant-density stars as a toy model [32, 38]. For constant-density stars it is possible to find stable configurations arbitrarily close to the Buchdahl limit[41]. It is trivial to see such stars are unphysical because an incompressible fluid violates causality (the speed of sound is infinite as in any other ideal incompressible fluid or medium) [38].

Urbano and Veermäe tried to study echoes under what they considered to be more proper physical conditions. Aiming to circumvent undesired aspects that plague previous models, Urbano proposed to restrict the analysis to perfect fluids obeying the following physical assumptions [38]:

1. Matter must satisfy the weak energy condition

$$\varepsilon \geqslant 0; \quad \varepsilon + p \geqslant 0. \tag{3.19}$$

Energy violations are a common side effect of many models of ultracompact stars that traces back their early studies [36, 37]. Besides that, this condition is also a way to exclude stars made of exotic matter, like traversable wormholes [38].

2. It is also assumed that matter is microscopically stable, which can be mathematically stated as[42]

$$p \geqslant 0; \quad \frac{dp}{d\varepsilon} \geqslant 0. \tag{3.20}$$

Negative pressure energetically favors the collapse, and if $p \geqslant 0$ but $\frac{dp}{d\varepsilon} < 0$ the system is unstable with respect to volume fluctuations since a contraction would induce a decrease in pressure and consequently generate a further contraction. Note that $\varepsilon + p \geqslant 0$ and $p \geqslant 0$ are equivalent when $\frac{dp}{d\varepsilon} \geqslant 0$ and $\varepsilon \geqslant 0$ are assumed. This second condition excludes, for example, gravastars which require a negative pressure

[41] Maximum compactness of static fluid spheres of arbitrary density profile as long the density does not increase outwards [80].

anisotropy in order to avoid the presence of a pressure discontinuity at the junction between the inner core and the crust [38].

3. The speed of sound of the fluid, that is the speed at which pressure disturbances travel in the fluid, have to respect the causality constraint

$$c_s = \left(\frac{dp}{d\varepsilon}\right)^{1/2} \leqslant 1. \tag{3.21}$$

Causality, together with general relativity, set important constraints to the compactness of relativistic stars [22].

Under these three conditions, the Linear equation of state (LinEos hereafter) given by

$$p = \omega\left(\varepsilon - \varepsilon_0\right), \tag{3.22}$$

is the stiffest possible equation of state since the speed of sound takes the maximal value throughout the star, and also is the one able to achieve the highest compactness [38]. In equation (3.22), the parameter ω is related to the speed of sound through $c_S^2 = \omega$ and ε_0 is a positive nonzero constant. We emphasize that the case $\varepsilon_0 = 0$ is excluded because the subsequent equation of state is not able to support stable stars [38]. If we set $\omega = 1/3$ and $\varepsilon_0 = 4B$, the MIT bag model equation of state is obtained, which was invented to try to account for hadronic masses in terms of their quark constituents [1, 114]. Considering a bag constant of $B = 145 - 235$MeVfm^{-3}, general relativity predicts maximum masses of $M_{max} = 2.01 - 1.57$M$_\odot$ and radii of $R = 10.9 - 8.56$km, respectively [23]. Such values are similar to the ones produced by neutron star models using nucleonic matter. Focusing on the mass limit of 2.01M$_\odot$ for $B = 145$MeVfm^{-3} (for larger values of B, the maximum mass gets smaller), the pulsar mass measurement PSRJ0740+6620 sets a mass limit of $2.14^{+0.10}_{-0.09}$M$_\odot$. This implies that B must be smaller than 145MeVfm^{-3}. On the other hand, there are reasonable constraints that demand $B > 145$MeVfm^{-3} in order to absolutely stable strange quark matter to exist [23]. Therefore the MIT bag model, at least in general relativity, does not provide stellar solutions compatible with modern pulsar mass measurements.

The LinEos, differently from constant-density models, is not able to support an infinitely high pressure, in other words, when pressure is increased above some critical value the system becomes unstable. The maximal compactness of constant-density models and the LinEos are obtained by different mechanisms: for the LinEos it is set by the star's stability and for the constant-density models it follows from the positivity of the g_{tt} component of the metric. However it was concluded that the LinEos is not able to generate gravitational echoes like those that characterize the relaxation phase of a black hole mimicker. Under general relativity, the LinEos is only able to generate significant gravitational echoes considering values

[42] The condition $\frac{dp}{d\varepsilon} \geqslant 0$ is also known as the Le Chatelier's principle [1].

of ω which are unphysical according to the aforementioned premises of the work [38]. Since gravitational echoes may be related with quantum properties in the strong gravity regime, a semiclassical analysis may provide a different and more accurate picture. The linear equation of state will be the first equation of state employed in the numerical analysis presented in Chapter 5. This choice may sound an oversimplification if compared with the highly sophisticated models applied in the equations of state for neutron stars, but since it is the one that maximizes compactness it is actually desirable. In fact, the LinEos is also used to model very exotic phases of dense baryonic matter like abnormal matter and Q-matter [81].

Chapter 4

Semiclassical Gravity

Classical physics is an accurate approximation in almost all situations, working so well that quantum physics was not discovered until the 20th century. Nevertheless, it is now definitely clear that the microscopic domain of atomic, nuclear and particle physics obeys quantum principles, and the standard model of strong and electroweak interactions is a fully quantum theory of nature. However, gravitational phenomena are still described in general relativity in a completely classical fashion [67]. In fact, Einstein himself was aware that quantum effects would demand modifications in his theory [115]. This formal gap between gravitational and quantum principles is not restricted to their respective theories, as significant as this is the fact that most of our intuitions about gravity remain essentially classical, particularly in the macroscopic domain [67]. Although it is a general consensus that at the fundamental microscopic Planck scale a theory of gravitational interactions must be in harmony with the quantum aspects of matter, it is often assumed that quantum effects should amount only to negligible, subdominant contributions in the macroscopic scale, at astrophysical or cosmological distances, where general relativity is presumed to hold unquestionably. Such view seemed to be endorsed by the seminal results obtained over the last decades in the context of renormalization of quantum fields in curved spacetimes [116].

Therefore it appears that the reason why quantum phenomena are difficult to observe is that they are solely relevant on microscopically small scales, that is, when only a relatively small number of quanta is involved. This does not always need to be the case. If we take a look at non-gravitational physics, it is possible to find large systems which can not be described classically, revealing that quantum effects can permeate a wide range of scales in a variety of ways, some of them evident, others more subtle and less appreciated, at least at a first glance. Macroscopic quantum phenomena can be found in systems with a large number of states or in a quantum state occupied by a large number of particles. Semiconductors, superfluids, superconductors and atomic Bose-Einstein condensates are undeniably macroscopic manifestations of the underlying quantum world. These are examples of quantum objects that can maintain quantum coherence at macroscopic scales [117]. The aforementioned degeneracy pressure of fermions at first thought as an esoteric feature of quantum statistics, is now fully accepted as the basis for stability of observed macroscopic objects like white dwarfs and neutron stars, therefore fundamental at the astrophysical scale [67].

Among the direct consequences of quantum statistics we can cite the periodic table, hence the foundations of chemistry itself and also, as a byproduct, the basis of biological processes. In this realm it is possible to find examples in ordinary experiences and structures that seem far less exotic than compact stars or superfluidity. Nonetheless chemical bonding, the structure and function of hemoglobin and DNA in the human body, as well as the overall stability of matter itself at ordinary temperatures and densities are a consequence of quantum principles justly as a sample of superfluid helium-4 climbing up the walls of its dewar [67].

Even looking from a historical perspective, it was not only in the microscopic world of the atom but experiments on macroscopic matter and the tough challenges they generated for classical mechanics, such as the ultraviolet problem of blackbody radiation and the specific heat of solids, that eventually led to the development of quantum mechanics. Whereas quantum effects play a crucial role in the properties of bulk matter of macroscopic phenomena in most every other area of physics, there is no reason why gravity, which couples to the energy content of quantum matter at all scales, should be immune from quantum effects on macroscopic scales [67]. If true, quantum-gravitational states could remain coherent at astrophysical, galactic and cosmological scales, playing a fundamental role in understanding macroscopic gravitational phenomena [117]. In fact there are reasons to believe that quantum-gravitational effects may soon be accessible. For instance, some works have shown that the quantization of a black hole's area can predict non-negligible effects already at the classical level [32].

4.1 Merging Two Worlds

In the early days of quantum theory, before a full theory of quantum electrodynamics was available, many problems were approached considering a classical electromagnetic field interacting with quantized matter [46]. Some results of this semiclassical approximation were later checked to be in complete accordance with quantum electrodynamics. One could ask, as a full theory of quantum gravity is not available yet, if it is possible to apply a similar reasoning to study the influence of the gravitational field on quantum phenomena. In order to properly explore those physical systems where the quantum nature of the fields and the effects of gravitation are both important, one has to consider a fusion between two pillars of modern physics. At one hand quantum field theory describes the microscopic constituents of matter and the other, general relativity, deals with the large scale structure of spacetime [42, 44]. The merger of these two theories is called *quantum field theory in curved spacetime* and remains valid as long as the quantum nature of gravity itself does not play a crucial role. Quantum field theory in curved spacetime can be considered now a well defined theory for both free and interacting fields[43] [118].

However, one must not infer that this combination is trivial. For instance, standard treatments of quantum field theory in Minkowski spacetime rely extensively on Poincaré symmetry[44] (usually entering the analysis implicitly via plane-wave expansions). In this case, Poincaré symmetry is used to pick out a preferred representation, which is mathematically equivalent to a selection of a preferred "vacuum state ". This, in turn, is mathematically equivalent to the selection of a preferred definition for the notion of "particles " in the theory. Neither Poincaré (or other) symmetry nor a useful notion of particle exists in a general, curved spacetime. Therefore some familiar tools and concepts of field theory must be "unlearned" in order to grasp quantum field theory in curved spacetime [44]. In the past, much effort has been devoted to the issue of how to generalize the notion of "particles" to curved spacetime. However one should remember that quantum field theory is a quantum theory of fields, not particles[45]. Although in appropriate circumstances a particle interpretation of the theory may be available, this notion plays no fundamental role in the formulation or interpretation of quantum field theory in curved spacetime [44].

In quantum field theory in curved spacetime, a quantum matter field treated as a test field, propagates in a specified classical background spacetime, as in general relativity. Thus, the spacetime behavior is still described by a manifold $\mathcal{M}$, on which is defined a classical, Lorentzian metric, $g_{\mu\nu}$ [44]. Given the above assumptions, quantum field theory is expected to have a limited range of validity. As mentioned earlier is generally believed that it should break down when spacetime curvatures reach Planck scales. In this case the theory must be replaced by a quantum theory of gravitation coupled to matter [44]. However it is worth noticing that the precise criteria for the validity of quantum field theory in curved spacetime can only be obtained when a quantum theory of gravity is available [44]. Despite of this, the theory is expected to embrace a wide range of interesting phenomena, such as quantum field process in the early universe, Casimir effect of quantum fields in spacetimes with boundaries, and the well-known Hawking and Unruh effects [44, 42].

The formulation of quantum field theory in curved spacetime assumes that the spacetime geom-

[43] A significant simplification comes from considering quantum fields interacting only with classical backgrounds, but not with other quantum fields. These fields are also called free fields, although they are coupled to the background [119].

[44] The Poincaré group is in fact the group of isometries of a flat Lorentzian 4-manifold, the Minkowski spacetime. This isometry group plays an important role in the analysis of the behavior of physical fields on Minkowski spacetime, particularly in the proof of conservation laws [2, 3].

[45] However, when dealing with the interaction between some quantum field with other systems to which it is coupled, an interpretation in terms of "particles" naturally arises. This is essential in order for quantum field theory to be described by observed phenomena, which commonly have a "particle-like" behavior. Experiments designed to explore quantum field phenomena are usually called "particle physics " experiments [44].

etry is given, therefore the theory deals with quantum fields propagating in a fixed background spacetime [44]. Nevertheless, it is evident on physical grounds that the quantum field must have a *backreaction* effect upon the spacetime geometry. In the absence of a full quantum gravity theory, the next structural level towards understanding the interaction of quantum fields with gravity is precisely to develop an approximation capable of including the effect of backreaction, that is, the effects of quantum matter fields exerted on the spacetime, therefore impacting on both its structure and dynamics [42, 116].

The resulting theory is called semiclassical gravity, a framework that provides quantum corrections into general relativity in a "minimal" way, in which matter (and other interaction) fields are quantized in some appropriate state on a classical strong external field called the background [27, 119]. In fact this construction is very useful since there is no absolute certainty that the metric itself should be quantized because, after all, it is different from all other fields [120]. However it may be argued that, due to the character of its foundations, it is unlikely that such a program reflects an exact description of nature since it combines interactions between quantum fields which are treated in probabilistic terms, with a classical gravitational field relying on well determined values [121]. Regarding field equations, the backreaction problem deals with the search for self-consistent solutions of both matter field equations and the Einstein field equations [42]. In the presence of gravitation the notions of "vacuum" and "particles" are also inherently ambiguous (in fact one observer's particle may be another observer's vacuum). Conversely, quantities defined directly through the field expectation values in some appropriate state are unambiguous. In this sense field observables are more fundamental than particle occupation numbers [119]. Hence in this framework the stress-energy tensor, with the expectation value of the matter field as source, is the central object to assess the importance of quantum fields on the dynamics of the gravitational field itself [46].

4.2 Semiclassical Field Equations

This section discusses how the ideas mentioned so far can be translated into the field equations. In what follows only situations where the fluctuations of the gravitational field are negligible will be considered. In this case a classical metric $g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle$ is assumed just as in general relativity [122]. Now suppose that this classical spacetime is populated by a collection of quantum fields $\Phi(x)$ (called matter fields) assumed to be in a given quantum state $|\psi\rangle$. In this context a semiclassical version of the Einstein field equations is proposed replacing the classical energy-momentum tensor by the expectation value of the energy-momentum tensor of the relevant quantized fields in the chosen state [123], that is

$$G_{\mu\nu}\left(g_{\alpha\beta}\right) = 8\pi \langle \psi | \hat{T}_{\mu\nu}\left(\Phi, g_{\alpha\beta}\right) | \psi \rangle \tag{4.1}$$

The field equations are assumed to hold as long as the fluctuations on the stress-energy tensor are sufficiently small and curvatures are small compared with the Planck scale[46], where quantum gravity effects are expected to dominate and the theory of quantum fields in classical spacetime breaks down [44, 119].

As frequently done in ordinary quantum mechanics[47], classical solutions are usually obtained in the limit $\hbar \to 0$, namely

$$T_{\mu\nu} \equiv \lim_{\hbar \to 0} \langle \psi | \hat{T}_{\mu\nu} \left(\Phi, g_{\alpha\beta} \right) | \psi \rangle. \tag{4.2}$$

Therefore it makes sense to define

$$\mathcal{Q}_{\mu\nu} \equiv \langle \psi | \hat{T}_{\mu\nu} \left(\Phi, g_{\alpha\beta} \right) | \psi \rangle - T_{\mu\nu}, \tag{4.3}$$

which implies that $\mathcal{Q}_{\mu\nu}$ has order $\mathcal{O}(\hbar)$. Solving the semiclassical equations is a challenging task because even for simple spacetimes, the evaluation of the stress-energy tensor is not straightforward [119]. Besides that some solutions, although consistent, are not physically relevant. For instance, there are known cases of "runaway" solutions when gravity generates a large expectation value for the stress-energy tensor, giving rise to more curvature and to an even stronger vacuum polarization, ad infinitum [119]. In fact, all exact solutions available are trivial [123]:

- Minkowski space with all the quantum fields in the vacuum state (imposing a cosmological constant renormalized to zero).
- de Sitter space with all the quantum fields in the vacuum state (imposing a fine tuned renormalized cosmological constant).
- Starobinsky inflation, which is a special case of de Sitter space with all the quantum fields in the vacuum state. Limiting the calculations to massless conformally-coupled quantum fields, one can then obtain the renormalized cosmological constant in a self-consistent manner via the trace (conformal) anomaly.
- Classical solutions obtained by taking the limit $\hbar \to 0$.

One might be skeptical about the semiclassical approximation[48], arguing that the fluctuations of the gravitational sector should be considered. In this case the metric should be treated as an operator $\hat{g}_{\mu\nu}$

[46] Nevertheless one should be careful with this association between quantum gravity effects and the Planck scale, since it is only reasonable in the absence of matter [124]. Even though it seems to leave much scope for a semiclassical theory since the Planck length is about twenty powers of ten below the size of an atomic nucleus [46].

[47] In the early days of quantum mechanics it was perceived that the limit $\hbar \to 0$ should play some role in connecting the classical and the quantum realm (although strictly speaking it is a dimensional constant). As observed by Landau and Lifshitz, this brings a curious aspect of quantum mechanics: *it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation* [125].

[48] It is worth mentioning that this has nothing to do with the WKB approximation, which is also sometimes referred to as the semiclassical approximation [126].

and the wave-functional as a tensor product $|\psi, g\rangle \equiv |\psi\rangle \otimes |g\rangle$. In this case the field equations assume the form

$$\langle\psi, g|G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right)|\psi, g\rangle = 8\pi\langle\psi, g|\hat{T}_{\mu\nu}\left(\Phi, g_{\alpha\beta}\right)|\psi, g\rangle. \tag{4.4}$$

In general $\langle\psi, g|G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right)|\psi, g\rangle \neq G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right)$ since $G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right)$ is a non-linear function of the metric operator [127]. Even so it is legitimate to consider an expansion around a classical solution such as

$$\hat{g}_{\mu\nu} = g_{\mu\nu}\mathbb{1} + \hat{\gamma}_{\mu\nu}. \tag{4.5}$$

Then it is possible to express the difference between $\langle\psi, g|G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right)|\psi, g\rangle$ and $G_{\mu\nu}[\hat{g}]$ (keeping only terms quadratic in $\hat{\gamma}_{\mu\nu}$) as

$$G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right) - \langle\psi, g|G_{\mu\nu}\left(\hat{g}_{\alpha\beta}\right)|\psi, g\rangle = 8\pi\langle\hat{t}_{\mu\nu}\left(\hat{\gamma}_{\alpha\beta}\right)\rangle, \tag{4.6}$$

where $\hat{t}_{\mu\nu}$ is responsible for taking into account the contribution to the stress-energy tensor due to quantum fluctuations of the metric. The operational version of the field equations can then be written as[49]

$$G_{\mu\nu}\left(\langle\hat{g}_{\alpha\beta}\rangle\right) = 8\pi\left(\langle\hat{T}_{\mu\nu}\rangle + \langle\hat{t}_{\mu\nu}\left(\hat{\gamma}_{\alpha\beta}\right)\rangle\right). \tag{4.7}$$

Therefore in situations in which the matter-gravitational state is highly populated (in the sense that macroscopic values of the fields can be well approximated by coherent states), $\langle\hat{\gamma}\rangle$ can be neglected and the assumption that $g_{\mu\nu} = \langle\hat{g}_{\mu\nu}\rangle$ is justified [127]. Since developing exact solutions for the semiclassical field equations is an arduous task, most investigations rely on approximation techniques. The most common are [123, 127]:

- Linearization: The metric is expanded around a flat Minkowski space as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ represents a small perturbation. The quantum field Φ is considered to be in some vacuum state $|0\rangle$, so that $T_{\mu\nu} = 0$. Keeping only linear terms, the perturbation obeys the equation

$$\Box\left(h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\right) = -16\pi\mathcal{Q}_{\mu\nu} + \mathcal{O}\left(h^2\right) \tag{4.8}$$

  So the perturbation $h_{\mu\nu}$ behaves as a usual quantum field in flat space. This method can be used, for instance, to calculate graviton-matter scattering. A disadvantage of this approach is the non-renormalizability of general relativity, so not every Green function is well defined.

- Test field limit: This technique is similar to what is done in quantum field theory in curved spacetime. The background metric assumes the form of any solution of the classical vacuum Einstein equations. As in the previous case, the quantum field is also in a vacuum state with $T_{\mu\nu} = 0$. Then $\mathcal{Q}_{\mu\nu}$ is calculated after performing a quantization of this field of the background, ignoring its backreaction on the background metric.

[49] The validity of this operatorial expression can not be properly addressed without a full theory of quantum gravity [127].

- Backreaction: One starts with a background field which is any solution of the Einstein equations in vacuum. From the classical metric $\mathcal{Q}_{\mu\nu}$ is computed. Then the metric is allowed to fluctuate, and the fluctuations are computed from the linearized field equations with $\mathcal{Q}_{\mu\nu}$ as source. After this, one can re-compute $\mathcal{Q}_{\mu\nu}$ using the classical metric plus the perturbations and re-iterate the previous steps until convergence is obtained.

The discussion made so far does not mention a prescription on how to calculate $\mathcal{Q}_{\mu\nu}$. The most undesirable property regarding its calculation is that it often diverges [127]. This is not a novelty of curved spacetime, in fact this is also the case in Minkowski spacetime. In order to illustrate the origin of these divergences, consider the stress-energy tensor of a scalar field, namely [127]

$$T_{\mu\nu}(x) = \frac{1}{2}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - \frac{1}{2}g_{\mu\nu}V\left(\phi(x)\right). \tag{4.9}$$

Quantum fields are, mathematically speaking, operator-valued distributions and a product between two distributions is not a well-defined mathematical concept [127]. The usual approaches for dealing with these issues in ordinary quantum field theory (namely normal ordering or Casimir subtraction) do not work in curved spacetime. Hence other techniques should be used to compute $\langle\hat{T}_{\mu\nu}\rangle$. The three most common are dimensional regularization, Riemann-zeta function renormalization and Point-splitting renormalization[50] [127]. For instance, in the Point-splitting renormalization one deals with terms like $\phi(x)\phi(x)$ by introducing an auxiliary point and a geodesics γ_0 connecting the points x and x'. Hence the stress-energy tensor in the vacuum state assumes the form [127]:

$$\langle 0|\hat{T}_{\mu\nu}(x)|0\rangle := \lim_{x'\to x}\langle 0|\hat{T}_{\mu\nu}\left(x,x';\gamma_0\right)|0\rangle \tag{4.10}$$

Since the energy-momentum tensor is quadratic in the fields, it is assumed that one of them is evaluated at x and the other at x' [127]. It can be demonstrated that locally (that is, considering small distances between x and x') this approach does not depend on the particular choice for γ_0.

The outcome of this procedure is the following structure[51] [127]

$$\langle 0|\hat{T}_{\mu\nu}(x)|0\rangle = \mathcal{D}_{\mu\nu}\left(x,x';\gamma_0\right)G^{(1)}\left(x,x';\gamma_0\right); \tag{4.11}$$

$$G^{(1)}\left(x,x';\gamma_0\right) := \langle 0|\{\phi(x),\phi(x')\}|0\rangle. \tag{4.12}$$

In the above expression $G^{(1)}\left(x,x';\gamma_0\right)$ is the Hadamard Green function for the field $\phi(x)$. This function is is to have the Hadamard form whenever it can be expressed as [128]

$$G^{(1)}\left(x,x'\right) = \frac{U(x,x')}{\sigma} + V(x,x')\ln\sigma + W(x,x'), \tag{4.13}$$

[50] The point-splitting technique has the advantage of being valid beyond Riemannian manifolds [127].

[51] Under the assumption that the energy-momentum tensor is constructed with products of two fields and contains second-order derivatives [127].

where U, V and W are regular functions for all choices of x and x'. The function W is responsible for carrying the information about the quantum state, while U and V are geometrical quantities, therefore independent of the particular quantum state [128]. A state is called a Hadamard state if the singular behavior of the associated Green function is the natural generalization for a curved spacetime of its singular structure in Minkowski spacetime. Such states are the ones usually considered as physical [129]. The question for which spacetimes and for what states the Hadamard Green function has the Hadamard form is still open, nevertheless it is usually believed that it acquires this form for a wide class of systems [46]. The expectation value of the energy-momentum tensor can be constructed as a limit of derivatives of the Hadamard Green function. For instance consider a massless, minimally coupled scalar field written as

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi. \tag{4.14}$$

So the formal expectation value of the above expression is [128]

$$\langle T_{\mu\nu} \rangle = \frac{1}{2} \lim_{x' \to x} \left\{ \left[\partial_\mu \partial_{\nu'} - \frac{1}{2} g_{\mu\nu} \partial_\alpha \partial^{\alpha'} \right] G^{(1)} \left(x, x' \right) \right\}, \tag{4.15}$$

where ∂_μ represents a derivative with respect to x^μ and $\partial_{\nu'}$ represents a derivative with respect to x'^ν. Equation (4.15) is only formal since it diverges when $x = x'$. When these two quantities are different however, the above equation can be taken as a regularized form of the expectation value of the energy-momentum tensor [128].

In equation (4.11), $\mathcal{D}_{\mu\nu} \left(x, x'; \gamma_0 \right)$ is some second-order derivative operator constructed upon covariant derivatives at x and x'. At x', the covariant derivatives must be parallel transported back to x in order for the differential operator to qualify as a proper geometrical object [123]. In other words, $\mathcal{D}_{\mu\nu} \left(x, x'; \gamma_0 \right)$ depends on the particular geodesic employed. Therefore it is exclusively determined by the background geometry (as a consequence it is regular) and in general is described by very complicated expressions. As a matter of curiosity, in the particular case of a conformally-coupled massless scalar field, $\mathcal{D}_{\mu\nu} \left(x, x'; \gamma_0 \right)$ assumes the form [123]:

$$\begin{aligned}
\mathcal{D}_{\mu\nu} \left(x, x'; \gamma_0 \right) \equiv & \frac{1}{6} \left(\nabla^x_\mu g^\alpha_\nu \left(x, x'; \gamma_0 \right) \nabla^{x'}_\alpha + g^\alpha_\mu \left(x, x'; \gamma_0 \right) \nabla^{x'}_\alpha \nabla^x_\nu \right) \\
& - \frac{1}{12} g_{\mu\nu}(x) \left(g^{\alpha\beta} \left(x, x'; \gamma_0 \right) \nabla^x_\alpha \nabla^{x'}_\beta \right) \\
& - \frac{1}{12} \left(\nabla^x_\mu \nabla^x_\nu + g^\alpha_\mu \left(x, x'; \gamma_0 \right) \nabla^{x'}_\alpha g^\beta_\nu \left(x, x'; \gamma_0 \right) \nabla^{x'}_\beta \right) \\
& + \frac{1}{48} g_{\mu\nu}(x) \left(g^{\alpha\beta}(x) \nabla^x_\alpha \nabla^x_\beta + g^{\alpha\beta}(x') \nabla^{x'}_\alpha \nabla^{x'}_\beta \right) \\
& - \mathcal{R}_{\mu\nu}(x) + \frac{1}{4} g_{\mu\nu}(x) \mathcal{R}(x).
\end{aligned} \tag{4.16}$$

It is evident by looking to equation (4.11) that the renormalization of the energy-momentum tensor will be dependent on the success in renormalizing the Green function, yielding [127]

$$\langle 0|\hat{T}_{\mu\nu}(x)|0\rangle_{ren} = \mathcal{D}_{\mu\nu}\left(x, x'; \gamma_0\right) G^{(1)}_{ren}\left(x, x'; \gamma_0\right). \tag{4.17}$$

It is also desirable to restrict the expectation value of the stress-energy tensor to obey *Wald's physical axioms* [42, 44, 122, 127]:

1. Covariant conservation: $\nabla^{\mu}\langle\psi|\hat{T}_{\mu\nu}\left(\Phi, g_{\alpha\beta}\right)|\psi\rangle_{ren} = 0$ (All states must be compatible with the contracted Bianchi identity in general relativity).
2. Causality: For a fixed "in" ("out") state, $\langle\psi|\hat{T}_{\mu\nu}\left(\Phi, g_{\alpha\beta}\right)|\psi\rangle_{ren}$ at a point p must depend only on the spacetime geometry to the past (future) of p.
3. Standard results for off-diagonal matrix elements: The value of $\langle\psi|\hat{T}_{\mu\nu}\left(\Phi, g_{\alpha\beta}\right)|\xi\rangle$ must be preserved for two orthonormal states (that is, $\langle\psi|\xi\rangle = 0$) $|\psi\rangle$ and $|\xi\rangle$ for which this quantity is finite.
4. Standard results in Minkowski spacetime: In the absence of curvature, the expectation value of the energy-momentum tensor must converge consistently to its normal ordered version in Minkowski spacetime.

Wald demonstrated that any $\langle\hat{T}_{\alpha\beta}\rangle$ that satisfies the first three axioms is unique up to the addition of a local, geometrical conserved tensor. Such a tensor can always be decomposed and the coefficient associated with each term can be interpreted as providing the renormalization for the bare quantities in the left side of the field equations [127]. Consequently, the renormalization of the energy-momentum tensor provides a renormalization for the matter and gravitation couplings, inducing higher order curvature terms even if they are absent in the bare action [127]. From the action point of view one is left with

$$S = S_{grav} + W = \left(S_{grav}\right)_{ren} + W_{ren}, \tag{4.18}$$

where W represents the one loop effective action for the matter fields and the divergent parts of the two actions cancels.

4.2.1 Semiclassical Action

Similarly to what is done in general relativity, the semiclassical field equations can also be motivated through an action principle. Fortunately, the divergent parts of $\langle T_{\alpha\beta}\rangle$ discussed earlier can be absorbed by the renormalization of counterterms in the gravitational action [128]. The first step is to show how the idea of a classical energy-momentum tensor can be replaced by its quantum expectation value.

For this purpose consider, as frequently done in quantum mechanics, the propagator for a "particle" to go from the position x' at time t' to position x'' at time t'', which can be formally expressed as a sum over all possible paths connecting these position [126],

$$\langle x'', t''|x', t'\rangle = \int \mathcal{D}x(t)e^{iS\left[x(t)\right]/\hbar}. \tag{4.19}$$

This approach can be formally generalized for quantum field theory. Consider now a scalar field $\phi(x)$. In this case, instead of the above expression, one has (setting $\hbar = 1$) [126]

$$Z\left[\phi(x)\right] = \int \mathcal{D}\phi(x)e^{iS\left[\phi(x)\right]}, \tag{4.20}$$

where $Z\left[\phi(x)\right]$ is the usual notation to abbreviate the path integral in the field-theoretical approach, which may refer, for instance, to in-out transition amplitudes or to partition sums [126].

In many calculations it is appropriate to work in the four-dimensional Euclidean space. Equation (4.20) can be translated into the Euclidean formulation through a Wick rotation $t \to -i\tau$, leading to $iS\left[\phi(x)\right] = -S_E\left[\phi(x)\right]$. For a quantum system with a coordinate $\hat{q}$ interacting with a classical background field J, the Euclidean effective action $\Gamma_E\left[J(\tau)\right]$ can be determined through the expression [119]

$$e^{-\Gamma_E\left[J(\tau)\right]} = \int \mathcal{D}q e^{-S_E\left[q(\tau),J(\tau)\right]}, \tag{4.21}$$

where $S_E\left[q, J\right]$ is the Euclidean classical action for the variable q including its interaction with the background, and τ is called the Euclidean time. Obviously, obtaining a Lorentzian analogue is straightforward. One needs to perform an analytic continuation that involves replacing $\tau = it$, where t is the Lorentzian time. The Lorentzian effective action is simply defined as the analytic continuation of the Euclidean effective action with an extra imaginary factor, that is

$$\Gamma_L\left[J(t)\right] \equiv i\Gamma_E\left[J(\tau)\right]_{\tau=it}. \tag{4.22}$$

Formally this allows one to replace previous Euclidean path integral by its Lorentzian analogue and write

$$e^{i\Gamma_L\left[J(t)\right]} = \int e^{iS\left[q(t),J\right]}\mathcal{D}q. \tag{4.23}$$

The above relation consists in a merely symbolic representation of the analytic continuation of the Euclidean path integral, since the Lorentzian path integral is ill-defined. On the other hand it is intuitively easier to manipulate the Lorentzian path integral directly, as if it were well-defined, computing, for example, functional derivatives of Γ_L or changing variables in the path integral. These operations should be understood as the respective manipulations on the Euclidean path integral, followed by the analytic continuation to the Lorentzian time [119].

Now consider a quantum field $\hat{\phi}$ and the role of the background is played by the metric $g_{\mu\nu}$. The backreaction on the metric can be found through the effective action $\Gamma_L\left[g_{\mu\nu}\right]$ via [119],

$$e^{i\Gamma_L\left[g_{\mu\nu}\right]} = \int e^{iS^{(m)}\left[g_{\mu\nu},\phi\right]}\mathcal{D}\phi, \tag{4.24}$$

where the action on the right hand side is associated with the matter field in the presence of gravitation. The backreaction modifies the vacuum Einstein field equations in the following way:

$$\frac{\delta S^{EH}}{\delta g^{\alpha\beta}} + \frac{\delta\Gamma_L\left[g_{\mu\nu}\right]}{\delta g^{\alpha\beta}} = \frac{\sqrt{-g}}{16\pi}G_{\alpha\beta} + \frac{\delta\Gamma_L\left[g_{\mu\nu}\right]}{\delta g^{\alpha\beta}} = 0. \tag{4.25}$$

and the term carrying the functional derivative of $\Gamma_L\left[g_{\mu\nu}\right]$ describes the quantum corrections in the theory [129]. Now, using the expression for the stress-energy tensor in general relativity, namely[52]

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S^m}{\delta g^{\alpha\beta}}, \tag{4.26}$$

it is convenient to express the expectation value of the quantum stress-energy tensor as [119]

$$\langle T_{\alpha\beta}\rangle = -\frac{\int T_{\alpha\beta}e^{iS}\mathcal{D}\phi}{\int e^{iS}\mathcal{D}\phi} = -e^{-i\Gamma_L}\frac{2}{i\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}e^{i\Gamma_L} = -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_L}{\delta g^{\alpha\beta}}. \tag{4.27}$$

The next step in the study of the semiclassical field equations is to analyze the gravitational part of the semiclassical action, which has the generic form [42]

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\mathcal{R} - 2\Lambda + \left(a\mathcal{R}^2 + b\mathcal{R}^{\alpha\beta}\mathcal{R}_{\alpha\beta} + c\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\alpha\beta\mu\nu}\right)\right], \tag{4.28}$$

where a, b and c are constants. These quantities before renormalization are bare[53] and their observable physical counterparts are renormalized and fixed by experimental data [42, 119]. The semiclassical action differs from the standard Einstein-Hilbert action by extra higher order curvature terms that are needed in order to cancel ultraviolet differences which arise in the matter quantum fields. These extra terms lead to a much larger number of solutions than those of classical general relativity, and often they also lead to solutions which may follow the classical solution for a while but after some time they become unstable and substantially deviate from it [42].

In four dimensional spacetime, using the generalized Gauss-Bonnet theorem, it is possible to find a relation between the constants a, b and c. It can be demonstrated that general relativity is reobtained by setting $a = b = 0$ [42].

[52] See Appendix E.

[53] The bare coupling constants are never observable since the quantum field is always present and the backreaction can not be "switched off" [119] .

The action can be alternatively expressed in terms of the Weyl tensor as [42]

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\mathcal{R} - 2\Lambda + \left(a\mathcal{R}^2 + \alpha_2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\alpha\beta} + \gamma C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu}\right)\right], \tag{4.29}$$

where $\alpha_2 = b + c$ and $\gamma = 3c/2$. The field equations are obtained by taking the functional derivative of the action with respect to $g_{\alpha\beta}$. The quantities $\mathcal{R}^2$, $\mathcal{R}^{\alpha\beta}\mathcal{R}_{\alpha\beta}$ and $C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu}$ give rise to the following terms respectively [42]

$$^{(1)}H_{\alpha\beta} = -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\int d^4x\sqrt{-g}\mathcal{R}^2 \tag{4.30}$$

$$= -2g_{\alpha\beta}\Box\mathcal{R} + 2\nabla_\alpha\nabla_\beta\mathcal{R} - 2\mathcal{R}\mathcal{R}_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}\mathcal{R}^2; \tag{4.31}$$

$$^{(2)}H_{\alpha\beta} = -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\int d^4x\sqrt{-g}\mathcal{R}^{\alpha\beta}\mathcal{R}_{\alpha\beta} \tag{4.32}$$

$$= \frac{1}{2}g_{\alpha\beta}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \Box\mathcal{R}_{\alpha\beta} - \frac{1}{2}\Box\mathcal{R}g_{\alpha\beta} + \nabla_\alpha\nabla_\beta\mathcal{R} - \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\alpha\nu\beta}; \tag{4.33}$$

$$^{(C)}H_{\alpha\beta} = -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\int d^4x\sqrt{-g}C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu} \tag{4.34}$$

$$= -\nabla^\mu\nabla^\nu C_{\alpha\beta\mu\nu} + 2R^{\mu\nu}C_{\alpha\beta\mu\nu}. \tag{4.35}$$

It is useful to group the sum of these terms

$$\mathcal{H}_{\alpha\beta} = a\,{}^{(1)}H_{\alpha\beta} + \alpha_2\,{}^{(2)}H_{\alpha\beta} + \gamma\,{}^{(C)}H_{\alpha\beta}. \tag{4.36}$$

In conformally flat spacetimes in four dimensions, where the Weyl tensor vanishes, ${}^{(1)}H_{\alpha\beta}$ and ${}^{(2)}H_{\alpha\beta}$ no longer remain linearly independent. It follows that ${}^{(1)}H_{\alpha\beta} = 3\,{}^{(2)}H_{\alpha\beta}$ and a new quantity appears [42]

$$^{(3)}H_{\alpha\beta} = -\frac{1}{12}R^2g_{\alpha\beta} + R^{\mu\nu}R_{\mu\alpha\nu\beta} \tag{4.37}$$

$$= -R_\alpha{}^\mu R_{\mu\beta} + \frac{2}{3}RR_{\alpha\beta} + \frac{1}{2}R_{\mu\nu}R^{\mu\nu}g_{\alpha\beta} - \frac{1}{4}R^2g_{\alpha\beta}, \tag{4.38}$$

which is conserved only in conformally flat spacetimes (not as a consequence of a variational derivation). In this case $\mathcal{H}_{\alpha\beta}$ can be rewritten as

$$\mathcal{H}_{\alpha\beta} = a\hbar\,{}^{(1)}H_{\alpha\beta} + \alpha_2\hbar\,{}^{(2)}H_{\alpha\beta} + \beta\hbar\,{}^{(3)}H_{\alpha\beta} + \mathcal{O}\left(\hbar^2\right), \tag{4.39}$$

where $\alpha = a + \alpha_2/3$. Since in the presence of ${}^{(3)}H_{\alpha\beta}$, ${}^{(1)}H_{\alpha\beta}$ and ${}^{(2)}H_{\alpha\beta}$ are not linearly independent one can put the above equation in the form [42]

$$\mathcal{H}_{\alpha\beta} = \alpha\hbar\,{}^{(1)}H_{\alpha\beta} + \beta\hbar\,{}^{(3)}H_{\alpha\beta} + \mathcal{O}\left(\hbar^2\right). \tag{4.40}$$

All things considered, the semiclassical field equations have the generic form

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi\left(\langle T_{\alpha\beta}\rangle + \mathcal{H}_{\alpha\beta}\right). \tag{4.41}$$

Although $\mathcal{H}_{\mu\nu}$ does not vanish in general, in static situations, like the stellar applications to be discussed next, its contribution is negligible [39]. Since $\mathcal{H}_{\alpha\beta}$ arises from new curvature terms, it can be reasonably considered as a geometrical part of the field equations, and so be written in its left-hand side. So, if one wishes, the field equations can be conveniently rewritten as:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} + \Xi_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} + \mathcal{Q}_{\alpha\beta}\right), \tag{4.42}$$

where $\Xi_{\alpha\beta} := -8\pi\mathcal{H}_{\alpha\beta}$. In the right hand side the quantum expectation value is separated in its classical and quantum contributions. In this way the field equations can be interpreted as providing quantum corrections to the classical theory [99]. To summarize, a self-consistent semiclassical theory can be formulated in the following way. A quantum field has a nonzero vacuum expectation of the stress-energy tensor induced by the metric. Then one computes the value of the stress-energy tensor for this metric, requiring that this metric must satisfy the semiclassical field equations sourced by expectation value of the same stress-energy tensor [119].

There are different ways to justify the semiclassical Einstein equations, the two most common are [42, 43, 44, 130, 131, 132, 133]:

- Axiomatic: Having in mind Wald's axioms mentioned earlier, it is the only consistent way to couple quantum matter to a classical metric.
- Large N expansion[54]: Consider gravity coupled with N identical and independent scalar matter fields (each coupled to gravity with a coupling proportional to $1/N$), all of which are in the same quantum state. In the limit in which N goes to infinity and the gravitational constant is appropriately rescaled, the leading order[55] theory reproduces semiclassical gravity.

Regarding the large N expansion, the value of N is essentially arbitrary, as pointed out by Tomboulis, it is there just to classify graphs in a gauge invariant way [133]. Nonetheless, a few comments are pertinent. As previously mentioned, the limit $\hbar \to 0$ is traditionally used as a guide to study the emergence of classical physics from quantum theory. Although formalizing the transition from quantum to classical physics is a highly non-trivial issue, an analogous discussion can be made considering a quantum system when it becomes large, symbolically represented by limit $N \to \infty$. In this sense, strictly classical

[54] This expansion has also been successfully used in other contexts, for instance, in quantum chromodynamics, to compute some non-perturbative results [130].

[55] As a matter of curiosity, the next order provides another formalism, known as stochastic gravity [42]. In this formalism, the symmetrized stress-energy bitensor and its expectation value known as the noise kernel, plays a similar role to the one played by the quantum expectation value in semiclassical gravity. The theory can be used to investigate how the noise associated with the fluctuations of quantum matter fields seed the structures of the universe, how they affect fluctuations of the black hole horizon and the backreaction of Hawking radiation on the black hole dynamics, as well as the implications on trans-Planckian physics [42].

behavior would be an idealization obtained in the limit where the size is infinite. However, this notion is not problematic and shares some similarities with discussions about the derivation of thermodynamics from statistical mechanics. For instance, in theory phase transitions only happen in infinite systems, but in reality they can be observed in ordinary phenomena, making it reasonable to approximate 10^{23} boiling water molecules by an infinite number of such molecules [134]. The underlying idea is that without infinite systems as an idealization, the differentiation between microscopic and macroscopic systems is unclear. Likewise, classical behavior does not emerge from quantum theory associated with finite systems (with $\hbar > 0$ fixed), but only in infinite ones [134].

This discussion about the transition between quantum and classical physics is fundamental, and may involve both limits, for instance, when dealing with black holes, which can be described as quantum critical systems but at the same time they are also solutions in general relativity and therefore must have a classical interpretation [117].

4.2.2 The Semiclassical Source

As pointed out in the previous section, the expectation value of stress-energy tensor is the quantity responsible to incorporate the backreaction of the quantum fields on the metric [119]. Now it is time to introduce the expectation value of the energy-momentum tensor that is applied to deduce the hydrostatic equilibrium equations in semiclassical gravity. Aiming to simplify the calculations in what follows it is assumed that the field does not depend on angular variables, reducing the system to a $(1+1)$-dimensional spacetime [119, 135].

In order to set up the problem, consider a massless scalar field ϕ defined over a globally hyperbolic[56] $(1+1)$-dimensional spacetime manifold $\mathcal{M}$, obeying the Klein-Gordon equation [122, 129]

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right] = 0. \tag{4.43}$$

Equation (4.43) is invariant under conformal transformations of the metric which leaves the field unchanged. This equation can be deduced from the action

$$S\left[\phi, g\right] = -\frac{1}{2}\int d^2x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{4.44}$$

In order to see that the above action is conformally invariant, observe that under the substitution [119]

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}, \tag{4.45}$$

[56] A globally hyperbolic spacetime is one which admits a Cauchy surface Σ and consequently it is diffeomorphic to the product $\mathbb{R}\times\Sigma$. A Cauchy surface is a spacelike hypersurface with the property that any causal curve (non-spacelike) intersects Σ exactly once [57].

then it follows that:

$$\sqrt{-g} \to \Omega^2\sqrt{-g}, \quad g^{\alpha\beta} \to \Omega^{-2}g^{\alpha\beta}. \tag{4.46}$$

Therefore dependence in Ω is canceled. This invariance implies a major simplification in the subsequent theory which does not happen in $(3+1)$ dimensions, allowing exact calculations to be performed [136].

Taking advantage that in $(1+1)$ dimensions all metrics are conformal to the flat metric [46], that is

$$g_{\mu\nu} = C(x)\eta_{\mu\nu}, \tag{4.47}$$

the line element can be written in null coordinates (u, v) as [129]

$$ds^2 = -C\,(u,v)\,dudv = -C\left(dt^2 - dx^2\right). \tag{4.48}$$

In null coordinates, the equations of motion for a massless scalar field are simply [136]

$$\frac{\partial^2\phi}{\partial u\partial v} = 0. \tag{4.49}$$

While searching for solutions and listing a complete set of normal modes, it is important to distinguish the various types of boundary conditions [136]. When there are no boundaries (that is, x, u and v vary from $-\infty$ to ∞), a set of positive-norm mode solutions is given by (running waves) [136, 129]

$$\phi^u_\omega := \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega u}, \;\; \phi^v_\omega := \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v}, \;\; \omega \in \mathbb{R}^+. \tag{4.50}$$

These are positive-norm solutions with respect to a complexified space where it is possible to define a pseudo-inner Klein-Gordon product[57], namely [129]

$$(\phi_1, \phi_2)_{KG} := i\int_\Sigma d\Sigma^\mu \left(\phi_1^*\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1^*\right), \tag{4.51}$$

with Σ denoting a Cauchy surface. By applying a null Cauchy surface in the integral (4.51), it is straightforward to observe that the mode solutions (4.50) satisfy

$$\left(\phi^a_\omega, \phi^b_{\omega'}\right) = \delta\left(\omega - \omega'\right)\delta_{ab}; \tag{4.52}$$

$$\left(\phi^{a*}_\omega, \phi^{b*}_{\omega'}\right) = -\delta\left(\omega - \omega'\right)\delta_{ab}; \tag{4.53}$$

$$\left(\phi^a_\omega, \phi^{b*}_{\omega'}\right) = 0. \tag{4.54}$$

So the general solution of the Klein-Gordon equation can be written as

$$\phi = \int_0^\infty d\omega\left(a^u_\omega\phi^u_\omega + a^v_\omega\phi^v_\omega + H.c.\right). \tag{4.55}$$

[57] This product is not positive defined. If ϕ has a positive norm, then ϕ^* has a negative norm [129].

These orthonormal modes have positive frequency with respect to the timelike vector field $\partial_t := \partial_u + \partial_v$ [129]

$$i\partial_t \phi^a_\omega = \omega \phi^a_\omega. \tag{4.56}$$

Now the two main approaches available in the literature to obtain an expression for the renormalized energy-momentum tensor of a massless scalar field in $(1+1)$-dimensional will be briefly outlined [129]. The details are unnecessary because all that is needed to study hydrostatic equilibrium in semiclassical gravity is a covariant tensorial expression closely connected to one of these methods.

The first method consists in a direct construction of an exact solution of the scalar wave equation in normal modes [136]. In this case one uses (4.10) with the expectation value as written in equation (4.11). To eliminate the divergent behavior at $x \to x'$, one simply subtracts a function with the same divergent behavior in this limit [129]. The Hadamard Green function can be constructed using equation (4.50) and the renormalized energy-momentum tensor takes the form [46, 129]

$$\begin{aligned}
\langle \hat{T}_{uu} \rangle^{(2)} &= -\frac{1}{12\pi} C^{\frac{1}{2}} \partial_u^2 C^{-\frac{1}{2}} = \frac{1}{24\pi}\left(-\frac{3}{2C^2}\left(\partial_u C\right)^2 + \frac{1}{C}\partial_u^2 C\right); \\
\langle \hat{T}_{vv} \rangle^{(2)} &= -\frac{1}{12\pi} C^{\frac{1}{2}} \partial_v^2 C^{-\frac{1}{2}} = \frac{1}{24\pi}\left(-\frac{3}{2C^2}\left(\partial_v C\right)^2 + \frac{1}{C}\partial_v^2 C\right); \\
\langle \hat{T}_{uv} \rangle^{(2)} &= \langle \hat{T}_{vu} \rangle^{(2)} = \frac{1}{96\pi}\mathcal{R}^{(2)} g_{uv} = -\frac{\mathcal{R}C}{96\pi}.
\end{aligned} \tag{4.57}$$

where $C = C(u, v)$ is the conformal factor in null coordinates as expressed in equation (4.48).

Another path is to obtain the quantum expectation value through an effective action. As a starting point, consider the Polyakov action [119]

$$\Gamma_E\left[\gamma_{\mu\nu}\right] = \frac{1}{96\pi}\int d^2x \sqrt{\gamma}\mathcal{R}\Box_\gamma^{-1}\mathcal{R} \tag{4.58}$$

$$= \frac{1}{96\pi}\int d^2x \sqrt{\gamma(x)} d^2y \sqrt{\gamma(y)} \mathcal{R}(x)\mathcal{R}(y) G_E(x, y), \tag{4.59}$$

where $\gamma_{\mu\nu}$ is the positive defined metric. The Lorentzian metric and the positive defined metric are related by an analytic continuation:

$$\gamma_{\mu\nu} = -g^{(E)}_{\mu\nu} = -g_{\mu\nu}|_{t\to -i\tau}. \tag{4.60}$$

So the idea is to perform an analytic continuation of $\Gamma_E\left[g_{\mu\nu}\right]$ back to the Lorentzian time and use the Feynman Green function[58] instead of the Euclidean one. The result is the Lorentzian effective action, as expressed in (4.27) [119]. The first step is to return to the original Euclidean metric $g^{(E)}_{\mu\nu}$ before applying the analytic continuation. This requires:

$$\sqrt{\gamma} = \sqrt{g^{(E)}}, \quad \mathcal{R}(\gamma) = -\mathcal{R}\left(g^{(E)}\right), \quad \Box_\gamma = -\Box_{g^{(E)}}. \tag{4.61}$$

[58] The Feynman Green function is given by $iG_F(x, y) \equiv \langle 0|T\phi(x), \phi(y)|0\rangle = \theta\left(y_0 - x_0\right) G^-(x, y) + \theta\left(x_0 - y_0\right) G^+(x, y)$ [127].

The overall effect is a sign change in the action [119]

$$\Gamma_E\left[\gamma_{\mu\nu}\right] = \frac{1}{96\pi}\int d^2x\sqrt{-\gamma}\mathcal{R}(\gamma)\Box_\gamma^{-1}\mathcal{R}(\gamma) \tag{4.62}$$

$$= -\frac{1}{96\pi}\int d^2x^{(E)}\sqrt{g^{(E)}}\mathcal{R}\left(g^{(E)}\right)\Box_{g^{(E)}}^{-1}\mathcal{R}\left(g^{(E)}\right). \tag{4.63}$$

Now, the standard procedure is to replace the Euclidean Green's function by the Feynman Green's function, resulting in the Lorentzian effective action [119]

$$\Gamma_L\left[g_{\mu\nu}\right] = i\Gamma_E\left[g_{\mu\nu}^{(E)}\right]_{\tau=it} \tag{4.64}$$

$$= -i\frac{1}{96\pi}\int id^2x_1\int id^2x_2\sqrt{-g(x_1)}\mathcal{R}(x_1)\frac{1}{i}G_F\left(x_1,x_2\right)\sqrt{-g(x_2)}\mathcal{R}(x_2) \tag{4.65}$$

$$= \frac{1}{96\pi}\int d^2x\sqrt{-g}\mathcal{R}\Box_g^{-1}\mathcal{R}. \tag{4.66}$$

Similarly one can simply notice that the analytic continuation involves the replacement of $\Box_{g^{(E)}}^{-1}$ by $\Box_{g^{-1}}$, therefore

$$\Gamma_L\left[g_{\mu\nu}\right] = i\Gamma_E\left[g_{\mu\nu}^{(E)}\right]_{\tau=it} \tag{4.67}$$

$$= -\frac{i}{96\pi}\int id^2x\sqrt{-g}\mathcal{R}\Box_g^{-1}\mathcal{R} = \frac{1}{96\pi}\int d^2x\sqrt{-g}\mathcal{R}\Box_g^{-1}\mathcal{R}. \tag{4.68}$$

As expressed in equation (4.27), the quantum energy-momentum tensor is obtained by functionally differentiating Γ_L, which yields [129]

$$\langle\hat{T}_{\mu\nu}\rangle = \frac{1}{24\pi}\left(\mathcal{R}g_{\mu\nu} - \nabla_\mu\nabla_\nu\left(\Box^{-1}\mathcal{R}\right) + \frac{1}{2}\nabla_\mu\left(\Box^{-1}\mathcal{R}\right)\nabla_\nu\left(\Box^{-1}\mathcal{R}\right) - \frac{1}{4}g_{\mu\nu}\nabla^\alpha\left(\Box^{-1}\mathcal{R}\right)\nabla_\alpha\left(\Box^{-1}\mathcal{R}\right)\right), \tag{4.69}$$

which is a nonlocal expression. However, it can be converted into a local form by introducing a scalar field [129]

$$\varphi(x) := -\left(\Box^{-1}\mathcal{R}\right)(x) = -\int d^2x' G_F\left(x,x'\right)\mathcal{R}(x'). \tag{4.70}$$

Therefore

$$\langle\hat{T}_{\mu\nu}\rangle_\varphi = \frac{1}{24\pi}\left(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi + \frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{4}g_{\mu\nu}\nabla^\alpha\varphi\nabla_\alpha\varphi\right), \tag{4.71}$$

where the subindex φ indicates that choosing a specific solution of the inhomogenous equation (obeying some physical requirements)

$$\Box\varphi = -\mathcal{R} \tag{4.72}$$

is equivalent to selecting a specific Feynman Green function, as well as its respective vacuum state whose quantum energy-momentum tensor is given by equation (4.71).

Now the two results (equations (4.57) and (4.71)) from both approaches will be confronted. The first step is to build a covariant expression that reduces to equation (4.57) when evaluated in normal coordinates [129]. A possible path is to search for a geometric quantity that could be associated with the vacuum selection. Now, consider the timelike Killing vector field[59] $\xi \coloneqq \partial_t = \partial_u + \partial_v$. Its norm (which enters the positive-frequency condition 4.56) is given by

$$g\left(\xi,\xi\right) = -|\xi|^2 = -C\left(u,v\right). \tag{4.73}$$

This identity allows one to rewrite equation (4.57) as

$$\langle \hat{T}_{ab}\rangle_\xi^{(2)} = \frac{1}{48\pi}\left(\frac{1}{2}\mathcal{R}^{(2)}g_{ab} + A_{ab} - \frac{1}{2}g_{ab}A\right), \tag{4.74}$$

where

$$A_{ab} \coloneqq -4|\xi|\nabla_a\nabla_b|\xi|^{-1}. \tag{4.75}$$

Observing that it is convenient for calculations, equation (4.74) will be used for treating hydrostatic equilibrium in the next section. For completeness, aiming to show that equation (4.74) is equivalent to (4.57) it is important to express to write the Christoffel symbols in normal coordinates, that is

$$\Gamma^u_{uu} = \frac{1}{C}\partial_u C, \quad \Gamma^v_{vv} = \frac{1}{C}\partial_v C. \tag{4.76}$$

It follows that

$$\mathcal{R} = -2\Box \ln C(u,v). \tag{4.77}$$

Therefore the relation

$$\varphi = \ln|\xi|^2, \tag{4.78}$$

guarantees that equation (4.72) is satisfied [129]. The above relation implies that:

$$\frac{1}{2}\nabla_\mu\nabla_\nu\varphi = -\frac{1}{|\xi|^2}\nabla_\mu\nabla_\nu|\xi| + \frac{1}{|\xi|}\nabla_\mu\nabla_\nu|\xi|, \tag{4.79}$$

and

$$\frac{1}{4}\nabla_\mu\varphi\nabla_\nu\varphi = \frac{1}{|\xi|^2}\nabla_\mu|\xi|\nabla_\nu|\xi|. \tag{4.80}$$

Therefore

$$\frac{1}{|\xi|}\nabla_\mu\nabla_\nu|\xi| = \frac{1}{2}\nabla_\mu\nabla_\nu\varphi + \frac{1}{4}\nabla_\mu\varphi\nabla_\nu\varphi. \tag{4.81}$$

By combining this expression with equation (4.72) it follows that

$$24\pi\langle \hat{T}_{ab}\rangle_\xi^{(2)} = \frac{1}{2}\mathcal{R}g_{\mu\nu} + \nabla_\mu\nabla_\nu\varphi + \frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\left(\Box\varphi + \frac{1}{2}\nabla_\alpha\varphi\nabla^\alpha\varphi\right) = 24\pi\langle \hat{T}_{ab}\rangle_\varphi^{(2)}. \tag{4.82}$$

That is, the two expressions for the quantum energy-momentum tensor are actually equal [129].

[59] Killing vector fields are the ones which satisfy the equation $\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0$, called the Killing equation.

4.3 Semiclassical Hydrostatic Equilibrium

It is a natural guess to imagine that semiclassical gravity, being an extension of general relativity, must lead to a generalization of the usual TOV equations. One should expect that at low densities the classical equations are recovered, but in principle at high densities semiclassical effects (which although negligible in most situations are widely expected to have a physical reality) could not be disregarded. Aiming to fill a gap since, except for cosmological problems, studies of the backreaction in spacetime originated by the effects of quantum vacuum polarization in the presence of a gravitational field were practically non-existent, Raúl Carballo-Rubio has proposed to analyze hydrostatic equilibrium in semiclassical gravity [39]. The main result is that the polarization of the quantum vacuum is able to produce new static configurations, acting as a new kind of pressure of quantum mechanical origin (analogous to the degeneracy pressure).

First, consider the semiclassical field equations

$$G_{\mu\nu} = 8\pi \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle. \tag{4.83}$$

If the characteristic radius of curvature L in a given spacetime region is much greater than the Planck length $L_P = \sqrt{\hbar}$, the quantum expectation value of the stress-energy tensor can be expanded using a small parameter $\kappa = \left(L_P/L\right)^2 \ll 1$, keeping only terms up to first order in κ [122]. In this case, the first term is simply the energy-momentum tensor for a classical field, while the next order term (containing a factor $\hbar$) carries the main contribution due to quantum effects [122]. In this linear approximation the contributions of all fields are additive and thus can be studied independently.

All things considered, in this context the semiclassical field equations can be expressed as

$$G_{\mu\nu} = 8\pi \left(T_{\mu\nu} + \hbar N Q_{\mu\nu}\right). \tag{4.84}$$

The classical source is treated at the same manner as in general relativity and the phenomenon of quantum vacuum polarizations are encoded via the expectation value of the renormalized energy-momentum tensor of N non-interacting scalar fields.

The solution is obtained under two assumptions:

1. Spherical Symmetry: This is the same assumption that was made for deriving the TOV equations (see Appendix A).

The second approximation is related to the fact that in order to include backreaction effects in the Boulware vacuum, an expression for the renormalized stress-energy tensor is needed, and there are no such expressions available in four dimensions [137, 138]. In this context the following approximation, which preserves all qualitative features of all relevant vacuum states, is very useful [139, 140]

2. s-wave Polyakov approximation[60]: A well known approximation in black-hole physics that basically neglects quantum fluctuations that are not spherically symmetric, as well as effects of backscattering by means of a projection to a two-dimensional manifold.

It is useful to write the spherically symmetric line element as

$$ds^2_{(4)} = ds^2_{(2)} + r^2 d\Omega^2 = g_{ab}(y)dy^a dy^b + r^2(y)d\Omega^2(\theta,\phi), \tag{4.85}$$

where the $ds^2_{(2)}$ is the two-dimensional projection that is used to evaluate the effects from quantum vacuum polarization. For spherically symmetric spacetimes the projected line element can always be put in the form [39]

$$ds^2_{(2)} = -C(r)dt^2 + \frac{dr^2}{1-2m(r)/r}. \tag{4.86}$$

Obviously the whole theory could not be developed in two dimensions, since in this case the Einstein tensor vanishes identically [136]. A two-dimensional stress-energy tensor can be defined analogously to the four-dimensional case, namely [142]

$$T^{(2)}_{\alpha\beta} = -\frac{2}{\sqrt{-g^{(2)}}}\frac{\delta S^m}{\delta g^{\alpha\beta}_{(2)}}. \tag{4.87}$$

It follows that the renormalized stress-energy tensor and its two-dimensional projection are related by [39, 137]

$$Q_{\mu\nu} = \frac{\delta^a_\mu \delta^b_\nu}{4\pi r^2} Q^{(2)}_{ab} \tag{4.88}$$

The two-dimensional projection $Q^{(2)}_{ab}$, under the aforementioned assumptions, is computed via the closed tensorial expression (4.74), alongside with equation (4.75) [39, 129]. The Killing vector ξ can be associated with the Boulware state, which for (4.86) is given by $\xi = \partial_t$, therefore $\xi = \sqrt{C}$. Equation (4.74) provides the following components

$$Q^{(2)}_{rr} = -\frac{1}{96\pi}\left(\frac{C'}{C}\right)^2, \tag{4.89}$$

$$Q^{(2)}_{tt} = -\frac{1}{24\pi}\left[\left(1-\frac{2m}{r}\right)C'' - C'\left(\frac{m}{r}\right)' - \frac{3}{4}\left(1-\frac{2m}{r}\right)\frac{C'^2}{C}\right], \tag{4.90}$$

$$Q^{(2)}_{tr} = Q^{(2)}_{rt} = 0, \tag{4.91}$$

[60] For the spherical symmetric case a decomposition in terms of spherical harmonics effectively allows the reduction from a four-dimensional theory to a set of two-dimensional theories characterized by different values of the angular momentum. These two-dimensional theories are an interesting scheme to infer general features of sophisticated systems, hard to analyze in the four-dimensional case. In some spherically symmetric systems the main effects come from the “s-wave sector”, that is, the $l = 0$ mode [141].

where the prime denotes $f' = \frac{df}{dr}$ for any function f. It can be checked that the semiclassical source is identically conserved [39]. So the Bianchi identities imply the conservation of the classical source, which furnishes the usual continuity equation (equivalent to the respective equation obtained in Appendix A)

$$p' = -\frac{1}{2}(\varepsilon + p)\frac{C'}{C}, \tag{4.92}$$

The (t,t) component of the semiclassical field equation yields

$$\frac{2m'}{r^2} = 8\pi\varepsilon + \frac{\ell_P^2}{r^2}\left[\left(1 - \frac{2m}{r}\right)\frac{C''}{C} - \frac{C'}{C}\left(\frac{m}{r}\right)' - \frac{3}{4}\left(1 - \frac{2m}{r}\right)\left(\frac{C'}{C}\right)^2\right], \tag{4.93}$$

where ℓ_P is a "renormalized " Planck length defined by $\ell_P^2 = \hbar N/12\pi$. The (r,r) component, at the other side gives

$$\frac{C'}{rC} - \frac{2m}{r^2(r-2m)} = \frac{8\pi p}{1 - \frac{2m}{r}} - \frac{\ell_P^2}{4}\left(\frac{C'}{rC}\right)^2. \tag{4.94}$$

The above equation when combined with the continuity equation provides [61]:

$$p'\left(1 - \frac{\ell_P^2}{2r}\frac{p'}{\varepsilon + p}\right) = -\frac{(\varepsilon + p)\left(m + 4\pi r^3 p\right)}{r(r-2m)}, \tag{4.95}$$

which is the hydrostatic equilibrium equation for relativistic stars in semiclassical gravity, with modifications induced by vacuum polarization. If one wishes, this equation can be written in a reduced form in terms of enthalpy, through the relation $h'/h = p'/(p+\varepsilon)$ [39].

4.4 The Role of Quantum Vacuum Polarization Effects

As stated previously, although general relativity is a completely classical theory, beyond some energy scale quantum-gravitational effects are expected. It is often assumed that such effects should only take place near the Planck scale. This idea comes from the fact that near the Planck length, the Schwarzschild radius is of the order of the Compton wavelength of a black hole and classicality is lost [32]. Nevertheless, it has consistently been argued that between the Planck scale and the ones accessible by current experiments, new physics can hide. For instance, if gravity is fundamentally a higher-dimensional interaction, the Planck length can be significantly larger. Besides that, compact objects may present some effects with a logarithmic dependence on the Planck length [32]. Even when restricted to moderate curvature where the semiclassical theory is justified, effects such as particle production and vacuum polarization may contribute substantially to the curvature of spacetime.

[61] The derivation of the semiclassical TOV equation is outlined in Appendix C.

The vacuum does not present any inherent structure in classical physics. Nevertheless the vacuum has been revealed to be incredibly crowded by all sorts of processes in relativistic quantum physics [143]. From the early Dirac sea of negative-energy states to the picture of virtual particles constantly being created and annihilated, the vacuum has acquired fundamental conceptual importance for a consistent description of nature [143]. Unfortunately its direct effects are usually so subtle that its structure remains almost as evasive as in classical physics, demanding careful experiments. Nevertheless, for instance the Casimir effect, according to which objects experience a force between them solely due to the fact that their presence changes the vacuum energy in a position-dependent way, has been successfully observed in laboratory [143].

Semiclassical gravity incorporates the effects of quantum matter field process such as the vacuum fluctuation (e.g., particle creation) exerted on the classical background spacetime [119]. However, more important to our purposes, is the effect known as gravitational vacuum polarization [140, 144, 145, 146, 147]. Gravity changes the properties of the vacuum state of the quantum fields. Whether or not particles are being produced, the local field observables such as the vacuum expectation value of the stress-energy tensor at some point x differ from their Minkowski values. This modification of the vacuum state induced by the influence of the classical background is called the polarization of vacuum [119]. It should be emphasized that this effect can not be removed by a coordinate transformation [67]. As mentioned earlier, the expectation value of the stress-energy tensor is able to describe the backreaction of the quantum fields on the metric through the semiclassical field equations. This backreaction affects the metric and it follows that the vacuum polarization also changes. Vacuum polarization effects in strong gravity can lead to important phenomena even in the absence of actual particle creation, a scenario similar to what is found in quantum electrodynamics [46].

Although in flat spacetime normal ordering gives an entirely satisfactory prescription for defining an stress-energy tensor on a suitable class of states using the standard Fock representation, no satisfactory generalization is available for curved spacetimes [44]. As emphasized earlier, in curved spacetime there is no preferred "vacuum state". Vacuum polarization complicates this situation a bit further. Even for cases where a natural vacuum state can be picked out, one would not expect $\langle \hat{T}_{\alpha\beta} \rangle = 0$ for this state since vacuum polarization effects would be expected to make $\langle \hat{T}_{\alpha\beta} \rangle \neq 0$. Now consider that the gravitational field (distorting the quantum vacuum and shifting the expectation value of the stress-energy tensor) contains an event horizon. In that case there are at least four different natural definitions of the quantum mechanical vacuum state [144]:

- Hartle–Hawking vacuum [thermal equilibrium at infinity].
- Boulware vacuum [empty at infinity].
- Unruh vacuum [evaporating black hole].

- Vacuum cleaner vacuum [accreting black hole].

These different states correspond to different definitions of normal ordering on the spacetime. If the spacetime does not possess an event horizon (like in a star or planet) then a more clear scenario rises where you only have one vacuum state to deal with, the Boulware vacuum (which corresponds to normal ordering with respect to the usual static time coordinate)[148].

Semiclassical gravity has been invoked in the astrophysical context to write another chapter in the history of the confrontation between general relativity and quantum physics [29]. This history has already shown that quantum mechanical effects in matter can prevent the formation of black holes in situations in which classically such formation would seem unavoidable. As observed earlier, without quantum mechanics, objects such as white dwarfs and neutron stars would have never been predicted in the first place. This reinforces the idea that any fundamental description of Nature has to include quantum effects. Guided by the notion that the vacuum, being a dynamical entity, gravitates, some fascinating studies have been developed to include vacuum effects to investigate the ultimate fate of relativistic stars, accumulating interesting results [77, 116, 143, 149]. For example, that such effects can play an unexpected central role in the formation of these stars, possibly leading the vacuum energy density of a quantum field to an exponential growth [143]. In particular the gravitational collapse in semiclassical gravity is extremely rich, possibly allowing alternative end points in which new stable configurations of compact stars could emerge [27, 29].

The picture of a semiclassical collapse governed by its field equations, being relevant whenever the expectation value of the stress-energy tensor becomes comparable with its classical counterpart, goes as follows [29]. Imagine that at some point a star begins to collapse. Initially the evolution proceeds as in general relativity, but with some extra contributions since, as pointed out previously, spacetime dynamics may disturb the behavior of the vacuum associated with the quantum fields present, giving place to both particle production and additional vacuum polarization effects. It can be shown that the quantum state corresponding to the physical collapse is indistinguishable from the Boulware vacuum [29]. However in order to this quantum effects to prevent further collapse, a state where the horizon formation is approached sufficiently slowly is necessary [29]. This slow down of the collapse is attributed to matter-related high energy physics, providing an interval long enough for the vacuum polarization to grow and finally modify the evolution of the collapse [29]. Having this in mind a new compact object was proposed, which can be considered the most compact and quantum mechanical kind of star, called black star. These horinzoless objects are filled with matter, instead of being simply voids in space, and therefore completely accessible to astrophysical measurements [124]. The Newtonian counterpart of these black stars, usually called dark stars, have a very long history in astrophysics, dating back to Michell and Laplace [29]. It should also be emphasized that the black star is not the same as the object called in the literature as gravastar, since the former consists in a compact aglomerate of matter and the latter has a de Sitter-like interior [29].

Thus after the degeneracy pressure of electrons (giving birth to white dwarfs), and the degeneracy pressure of neutrons (giving birth to neutron stars), quantum vacuum effects would be the last process able to produce stable horizonless compact objects. Since the understanding of the physics of matter for objects of such densities is not available, it is not possible to reliably assert anything about the details with respect to the stability of black stars. Gravitational wave observations can rule out the existence of such hypothetical compact stars as data from scenarios where these effects should have been triggered become available. In particular the aforementioned gravitational wave echoes in the post-merger signal can offer evidence of quantum corrections at the relevant scale [109].

Chapter 5

Ultracompact Stars in Semiclassical Gravity

As discussed in the last chapter, semiclassical gravity is an approximate theory. Nevertheless the gravitational collapse of a star seems to be an appropriate environment to apply semiclassical gravity, where quantum fields are found in a classical curved spacetime and quantum gravity effects are expected to be relatively small[62] [127]. The purpose of this chapter is to elucidate if the semiclassical solutions are able to provide an alternative scenario, with significant deviations in comparison to general relativity, producing ultracompact stellar configurations. It is worth mentioning that for 1.4 $M_\odot$ neutron stars (the type that seems to be favored by observations) different equations of state predict values for R/R_S between 2 and 3.8. Thus already in this regime a neutron star although larger than its photon sphere, is not necessarily by very much [100].

It is valid to review the range of systems where the semiclassical treatment is pertinent. In order to clarify this consider a star of mass M and density ρ in hydrostatic equilibrium. As mentioned in the previous chapter, in this case the appropriate quantum state is called the Boulware vacuum (a state with zero particle content for static observers and regular everywhere in the absence of an event horizon, also known as static or Schwarzschild vacuum) [29]. For sufficiently dilute stars (with $R \gg 2M$) this state is virtually indistinguishable from the Minkowski vacuum and the renormalized stress energy tensor will be negligible throughout the entire spacetime [29]. On the other hand, David Boulware has shown, by studying a stationary compact star, that the closer the star gets to its gravitational radius, the larger the vacuum renormalized stress energy tensor near its surface [29, 151]. In other words, quantum corrections become significant if the star is not much larger than its gravitational radius. The basic idea is that matter alters the zero-point energy density of the quantum fields present, which is no longer exactly canceled and the excess amount is said to be caused by vacuum polarization [29]. In more extreme scenarios, most

[62] Even so, some authors have explored the possibility that quantum gravity effects might take place at densities exceeding the nuclear one, as found in neutron stars and other exotic compact stars [150].

believe that if ρ reaches the Planck density ρ_P (which is defined by $\frac{c^5}{\hbar G^2} = 5 \times 10^{93} \mathrm{g/cm}^3$), spacetime must be treated from the point of view of a quantum theory and the semiclassical approach (where spacetime remains classical) is no longer valid [68]. So, the gravitationally-induced quantum effects considered in what follows take place far away from the Planck regime.

One might ask how these semiclassical effects might evade the traditional analysis employed to model compact stars. It turns out that the semiclassical configurations are highly dependent on the details of the gravitational collapse, which is very difficult to explore and the models available strongly rely on approximations. So, for instance, although some compact star can be well described by a simple self-gravitating perfect fluid, its formation process is way more complex than the gravitational collapse of a perfect fluid. Such collapses inevitably involve intricate microphysics and quantum effects. The point is that although an equilibrium solution may be well described by simple matter fields, it does not imply that its formation process is equally simple, where new physics can hide [32].

It may also be argued that the collapse associated with most compact star models is too quick to produce strong semiclassical corrections. However it has been shown that the timescale for a particular semiclassical effect, known as gravity-induced vacuum dominance, is of tiny fractions of a second in certain astrophysical scenarios (differently from cosmological problems where it could require a few billion years) [116]. Moreover, in realistic collapses, it is reasonable to believe that widely neglected effects like dissipation might affect the search for new equilibrium configurations [124]. Since each new recollapsing phase starts from a position closer to its Schwarzschild radius than the previous one, at some point semiclassical effects should star being relevant, inducing the formation of new (stable or metastable) equilibrium objects [124].

All things considered, our current understanding concerning compact objects leaves space to propose a pleasant theoretical scheme, where the appropriate formalism is delimited by transitions which are attached to physical properties, that is:

- General Relativity $\Longrightarrow$ objects without photon spheres or event horizons (for instance, regular neutron stars);
- Semiclassical Gravity $\Longrightarrow$ objects with photon spheres but no event horizons (ultracompact stars);
- Quantum Gravity $\Longrightarrow$ objects with both photon spheres and event horizons (black holes).

The rest of the chapter is dedicated to the novelty in this work, namely, to analyze the semiclassical stellar output for different equations of state available in the literature, verifying the possibility of achieving ultracompact solutions.

5.1 LinEos in Semiclassical Gravity

The goal of this section, as a first attempt to produce ultracompact configurations, is to study hydrostatic equilibrium in semiclassical gravity using the LinEos as presented in Chapter 3, namely

$$p = \omega\left(\varepsilon - \varepsilon_0\right), \tag{5.1}$$

and compare the results with the ones produced in general relativity. As mentioned in Chapter 3, the LinEos has been used to model very exotic phases of baryonic matter like abnormal matter and Q-matter [81].

The analytical expression for the LinEos permits one to rewrite equation (4.95) in the form:

$$p' = -\frac{\left(\varepsilon + p\right)\left(m + 4\pi r^3 p\right)}{r\left(r - 2m\right)} + \frac{\ell_P^2}{2r}\frac{\omega^2 \varepsilon'^2}{\varepsilon + p}. \tag{5.2}$$

and use the last term as a semiclassical correction to the usual TOV equation.

The results will be displayed using units frequently adopted in the astrophysical literature, where energy density is expressed in $\mathrm{g/cm^3}$ and pressure in $\mathrm{dyne/cm^2}$ [1]. Radius and mass are expressed in km and $\mathrm{M}_\odot$, respectively. Compactness will be expressed as a dimensionless quantity, so it's worth mentioning, as pointed out earlier, that $1\,\mathrm{M}_\odot = 1.4766$ km. We have also used the energy density of normal symmetric nuclear matter $\varepsilon_S = 2.51 \times 10^{14}\ \mathrm{g/cm^3}$ (saturation density[63] of nuclear matter) as a reference value [1].

Figures 5.1-5.4 illustrate the pressure and energy density inside the star, for both general relativity and semiclassical gravity, as a function of radius for different values of the ω parameter.

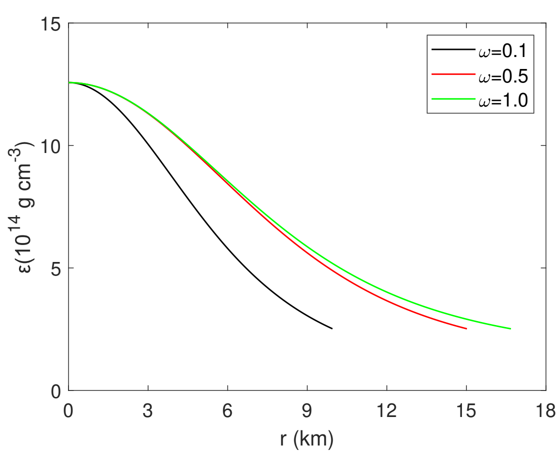

Figure 5.1: *Energy density profiles in general relativity for different values of ω and $\varepsilon_0 = \varepsilon_S$. A central energy density $\varepsilon_c = 5\,\varepsilon_S$ was adopted.*

[63] The saturation density is the one at which the binding energy is minimized [16].

What should be noticed here, is that the semiclassical solutions preserve the monotonically decreasing behavior for such functions. In fact, maintaining the general behavior found in general relativity is desirable for the semiclassical solutions regarding all physical quantities, avoiding the unphysical "runaway" solutions mentioned in the last chapter [119]. Differently from regular neutron stars, which satisfy $\varepsilon(r = R) = 0$, the energy density does not vanish at the surface. This is a well known behavior of self-bound[64] configurations [22].

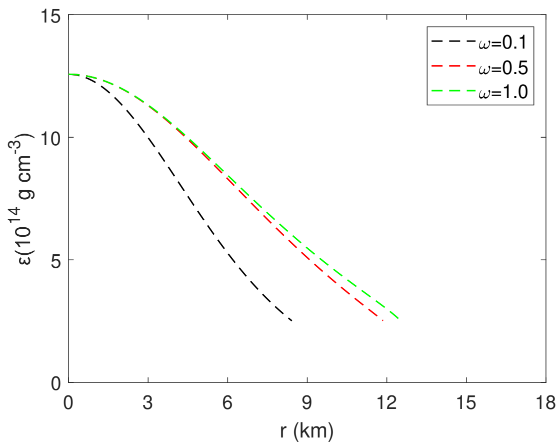

Figure 5.2: *Energy density profiles in semiclassical gravity for different values of ω and $\varepsilon_0 = \varepsilon_S$. A central energy density $\varepsilon_c = 5\,\varepsilon_S$ was adopted.*

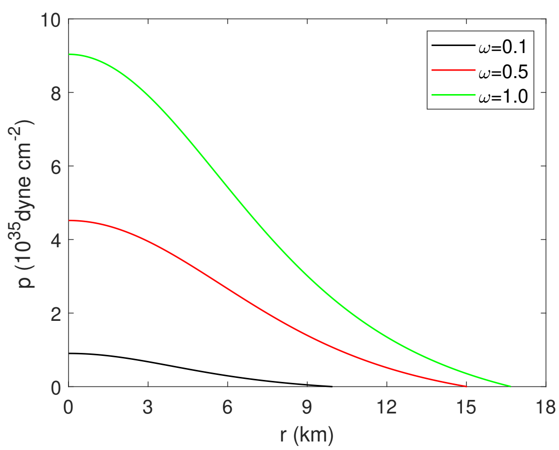

Figure 5.3: *Pressure profiles in general relativity for different values of ω and $\varepsilon_0 = \varepsilon_S$. A central energy density $\varepsilon_c = 5\,\varepsilon_S$ was adopted.*

[64] A self-bound star is a star that would be bound even in the absence of gravity [1].

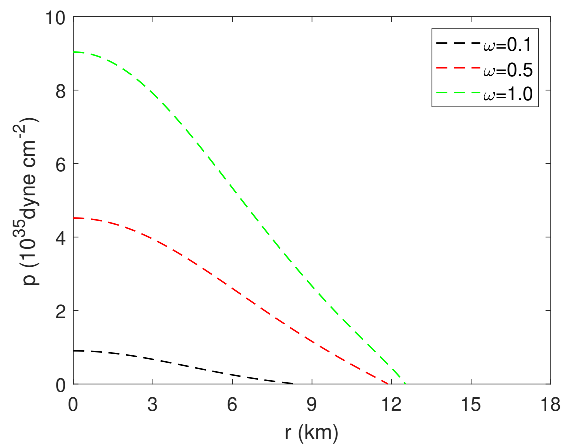

Figure 5.4: *Pressure profiles in semiclassical gravity for different values of ω and $\varepsilon_0 = \varepsilon_S$. A central energy density $\varepsilon_c = 5\,\varepsilon_S$ was adopted.*

In Figure 5.5 we compare the gravitational total mass versus the total radius for different values of the ω parameter. Each point represents a different star corresponding to some initial central energy density, unlike Figures 5.1-5.4 where each line depicts the interior of a single star.

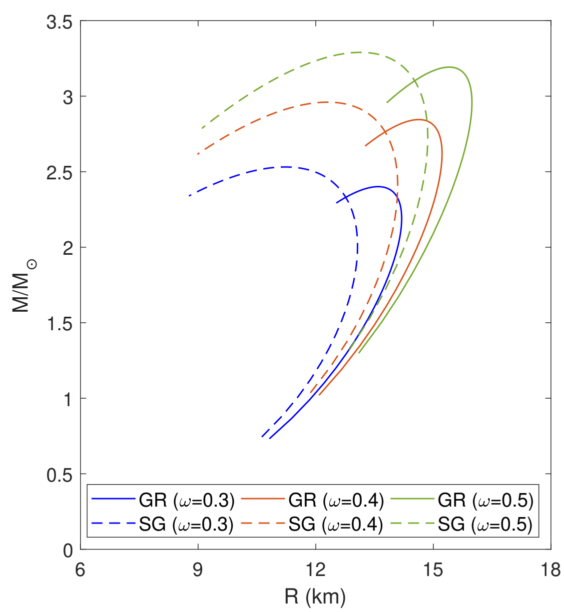

Figure 5.5: *Total gravitational mass and total radius in general relativity and semiclassical gravity for different values of ω and $\varepsilon_0 = \varepsilon_S$. The interval $[1.25 - 10.0]\,\varepsilon_S$ for the central energy density was adopted.*

The semiclassical solutions provide stars with a greater mass and smaller radius, in other words, the stars are more compact. The impact on the radius is more prominent, therefore it is the main responsible for the increase in compactness. The semiclassical solutions preserve the well known behavior for self-bound equations of state, where the radius increases as mass increases for stable solutions. Regular neutron stars present the opposite property in general, the radius decreases as mass increases [152]. Figure 5.6 compares the mass functions inside the star between the two frameworks for $\varepsilon_c = 3.0\,\epsilon_S$ and $\omega = 1.0$. Although the two mass functions converge as one gets close to the center, at some point the semiclassical solution overcomes, providing more mass than the classical solution.

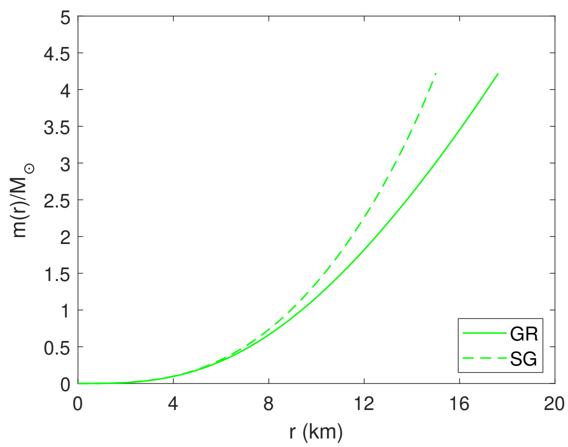

Figure 5.6: *Mass function comparison between general relativity and semiclassical gravity for* $\omega = 1.0$ *and* $\varepsilon_0 = \varepsilon_S$*. A central energy density* $\varepsilon_c = 3.0\,\varepsilon_S$ *was adopted.*

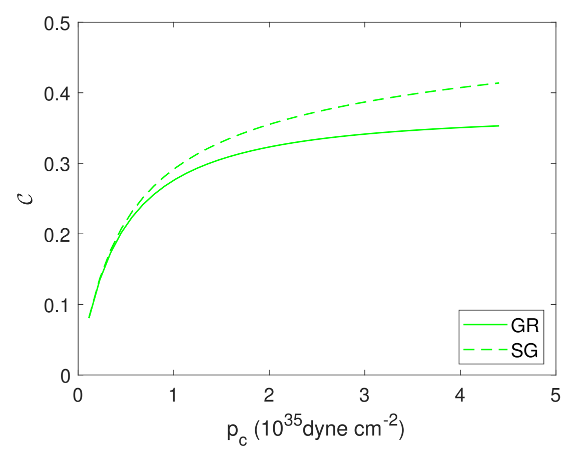

Figure 5.7: *Compactness versus central pressure in general relativity and semiclassical gravity for* $\omega = 1.0$ *and* $\varepsilon_0 = \varepsilon_S$*. A central energy density* $\varepsilon_c = [1.05 - 3.0]\,\varepsilon_S$ *was adopted.*

Figure 5.7 depicts compactness against the central pressure, indicating that the semiclassical solutions allow higher central pressures. Although the TOV equations (either in general relativity or semiclassical gravity) assure hydrostatic equilibrium, they do not assure dynamical stability[65] [153]. Dynamical instabilities forbid stable stars with central densities in the range about $10^9 - 10^{14}\mathrm{g/cm^3}$, thus separating the white dwarf from the neutron star regime [153]. It can be proved that along the sequence of equilibrium configurations of the Tolman-Oppenheimer-Volkoff equations, perfect fluid stars can pass from stability to instability with respect to any radial mode of oscillation only at a value of the central density at which the equilibrium mass is stationary, that is [1]

$$\frac{\partial M\left(\varepsilon_c\right)}{\partial \varepsilon_c} = 0, \tag{5.3}$$

which is a fundamental result in the discussion of stability.

In addition it can also be shown that a necessary condition for stability is given by

$$\frac{\partial M\left(\varepsilon_c\right)}{\partial \varepsilon_c} > 0. \tag{5.4}$$

Figure 5.8 depicts the total mass as a function of the central energy density for some values of ω. Regions that satisfy the condition (5.4) can be found for all allowed values of ω. Therefore despite the increase in compactness, the semiclassical solutions present roughly the same behavior found in general relativity, at least for the LinEos.

Tables 5.1 and 5.2 compare the solutions given by general relativity and semiclassical gravity for the configurations where the maximum gravitational mass is obtained for different values of ω and $\varepsilon_0 = \varepsilon_S$. This reinforces the systematic increase in compactness provided by the semiclassical solutions since they have higher masses and smaller radius. This increase in compactness is sufficient to expand the range of ultracompact solutions, which can be obtained starting from about $\omega \approx 0.3$.

It is important as a consistency check to confront those results with the literature. Fortunately the LinEos is a well documented equation of state, especially on the case saturating the causality bound, the so called *maximally compact equation of state* obtained when the ω parameter in the LinEos is set to unity [154]. The maximum compactness found, $\mathcal{C} = 0.354$, is a well known upper bound for fluid stars constrained by causality [38]. In addition, for the maximally compact equation of state, Urbano pointed out the following approximate relations for the maximum mass and its respective radius [38]

$$M \simeq 3 \times \left(\frac{\varepsilon_0}{5\times 10^{14}\mathrm{g/cm^3}}\right)^{-\frac{1}{2}} \mathrm{M}_{\odot}, \tag{5.5}$$

$$R \simeq 12.5 \times \left(\frac{\varepsilon_0}{5\times 10^{14}\mathrm{g/cm^3}}\right)^{-\frac{1}{2}} \mathrm{km}, \tag{5.6}$$

[65] To prove dynamical stability an additional evaluation of the radial vibrational modes (acoustical modes) of the star is required. This type of analysis is not going to be performed here.

which for the case treated in Table 5.1 gives respectively $4.23\,\mathrm{M}_\odot$ and 16.94 km. Therefore for both cases it was found a reasonable agreement keeping in mind that the above expressions are approximate relations. Also, another article shows that this maximum mass should occur at an energy density of $3.034\,\varepsilon_s$, which is precisely the value that provided the maximum mass in Table 5.1 [154].

Although it is most likely a coincidence, it is an interesting property of the semiclassical solutions that the transition to ultracompact stars occurs close to $\omega = 1/3$, which is potentially applicable to quark stars or stars with quark cores [154]. The semiclassical configurations are also promising for gravitational echoes, although the direct evaluation is certainly needed for further investigations. The reason for this is that semiclassical solution for $\omega = 0.4$ already presents higher compactness than the maximally compact case in general relativity. The corresponding maximum compactness found for the maximally compact case in semiclassical gravity, $\mathcal{C} = 0.407$, is only obtained in general relativity using unphysical values of ω [38].

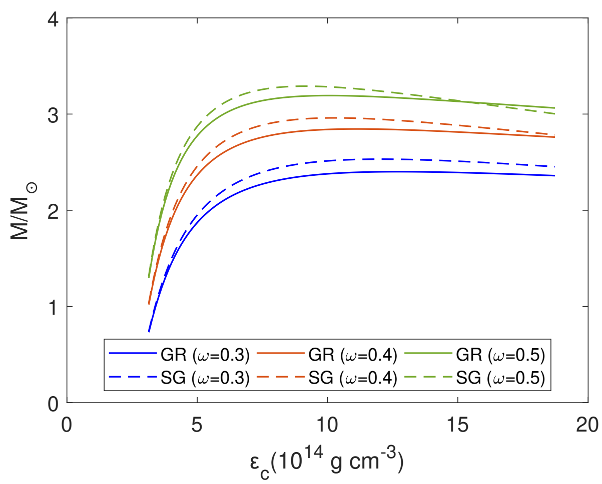

Figure 5.8: *Total gravitational mass versus central energy density in general relativity and semiclassical gravity for different values of ω and $\varepsilon_0 = \varepsilon_S$. The interval $[1.25 - 7.5]\,\varepsilon_S$ for the central energy density was adopted.*

One could argue that the analysis made with respect to the compact parameter is misleading because the exterior state in semiclassical gravity is associated with the Boulware state outside the star, so the description of the photon orbits made using general relativity in Chapter 3 is no longer valid. Fortunately the exterior semiclassical solution is well understood and the geometry can be approximated by the Schwarszchild geometry until very close to the Schwarszchild radius [137, 142]. Therefore it makes sense to apply the same compactness condition to define ultracompact stars in both formalisms. All things considered, semiclassical gravity is capable of producing a class of stable solutions which are compact enough to accommodate the presence of a photon sphere, opening the possibility of trapping gravitational radiation and affecting the ringdown phase of a merger event [38].

Table 5.1: *Results for the maximum gravitational mass configurations for different values of ω with $\varepsilon_0 = \varepsilon_S$ using general relativity.*

ω	$\varepsilon_c(\varepsilon_0)$	$M_{Max}(\mathrm{M}_\odot)$	R(km)	$\mathcal{C}$
0.1	8.15	0.982	9.556	0.152
0.2	6.1	1.808	12.090	0.221
0.3	5.1	2.401	13.578	0.261
0.4	4.5	2.845	14.603	0.288
0.5	4.0	3.192	15.408	0.306
0.6	3.7	3.473	16.019	0.320
0.7	3.5	3.706	16.505	0.331
0.8	3.3	3.903	16.933	0.340
0.9	3.2	4.072	17.265	0.348
1.0	3.034	4.220	17.590	0.354

Table 5.2: *Results for the maximum gravitational mass configuration for different values of ω with $\varepsilon_0 = \varepsilon_S$ using semiclassical gravity.*

ω	$\varepsilon_c(\varepsilon_0)$	$M_{Max}(\mathrm{M}_\odot)$	R(km)	$\mathcal{C}$
0.1	8.6	1.086	7.398	0.217
0.2	6.0	1.943	9.731	0.295
0.3	4.8	2.532	11.213	0.333
0.4	4.1	2.961	12.292	0.356
0.5	3.75	3.290	13.017	0.373
0.6	3.3	3.553	13.827	0.379
0.7	3.1	3.770	14.338	0.388
0.8	2.9	3.952	14.831	0.393
0.9	2.75	4.108	15.244	0.398
1.0	2.75	4.242	15.403	0.407

In the next section, a more specific type of self-bound configuration, namely, strange stars in the color flavor locked phase, will be analyzed from the semiclassical point of view.

5.2 Semiclassical CFL Strange Stars

The LinEos studied in the last section has among its subcases the well-known MIT bag model equation of state. In this section, a model will be constructed relying on strange-quark matter explicitly. Strange stars, being completely made of quark matter, are the most extreme scenario for quark matter in compact stars [16]. They are held together by both the strong interaction and gravity, although the first has the major role in bounding such systems [7]. They are motivated by the so-called strange matter hypothesis, developed independently by Bodmer and Witten, which asserts that the true ground state of the strong interaction is strange quark matter, composed of an approximately equal proportion of up, down, and strange quarks [88, 89]. Since Witten's work the MIT bag model with unpaired strange quark matter has been widely used to study strange stars. The model considers a gas of free relativistic quarks where confinement is achieved through a vacuum pressure, called the bag constant B.

The MIT bag model can be generalized, introducing a more sophisticated quark dynamics, by applying the BCS mechanism to quark matter. The strong interaction among quarks is very attractive in some channels and quarks are expected to form Cooper pairs[66] easily (which in quark matter always implies color superconductivity) [155, 156, 157]. Although many pairing schemes have been proposed, if central densities in compact stars are sufficient to support quark matter, it will probably manifest itself through the most symmetrical state, namely the color flavor locked (CFL) phase. In this phase all three quarks composing strange quark matter are paired on an approximately equal footing and form a color condensate.

Quark matter in the CFL phase has been applied in different astrophysical contexts, including the possibility of supporting exotic structures like wormholes [158]. When applied to strange stars, it has been shown that the CFL state affects the mass-radius relationship considerably, allowing configurations with large maximum masses [159]. Although CFL strange stars may not be as extreme as the hypothetical models for black-hole mimickers available in the literature (like black stars), they still offer a high density environment combined with a solid theoretical background, avoiding the problems usually found in more extreme proposals [39]. Also, it has been argued that the low mass companion (about 2.6 $M_\odot$) of the black hole in the source of GW190814 could be a strange star [51, 160].

In what follows, the model will be constructed differently from what was made using the LinEos in the last section. The semiclassical TOV equation, being quadratic with respect to the pressure gradient, has two distinct ways of expressing hydrostatic equilibrium. This certainly opens new possibilities regarding compact stellar models. The idea is to develop a simple model which associates to each hydrostatic

[66] Cooper pairs are a microscopic explanation for superfluidity and superconductivity originated in BCS theory (abbreviation for Bardeen–Cooper–Schrieffer). Cooper pairs emerge when an arbitrarily small interaction produces an instability of the Fermi surface. Compact stars could, at least in principle, produce Cooper pairs of neutrons, protons, hyperons and quarks [16].

equilibrium solution an appropriate source. In addition, the semiclassical equations for stellar equilibrium have two quantities which are not present in the respective equations in general relativity, namely, $\hbar$ and N. It was observed in Chapter 4 that the limits $\hbar \to 0$ and $N \to \infty$ can be viewed as alternative measures of classicality. It is interesting to investigate if it is possible to construct a model taking advantage of these limits, with non-trivial contributions surviving this "classicalization" process. Nevertheless, before going into details, in the next section the equation of state for CFL quark matter will be discussed.

5.2.1 The Color Flavor Locked Equation of State

The CFL equation of state is a nonlinear generalization of the unpaired version of the MIT bag model, proposed in the context of color superconductivity. Different parametrizations are possible depending on the values of B (the bag constant), Δ (the gap of the QCD Cooper pairs) and m_S (the strange quark mass), which are not accurately known and are taken as free parameters. All results that will be presented in this section use the set of parametrizations presented in Ref. [159], which are also displayed in Table 5.3.

It is customary to use a semi-empirical model in which the thermodynamic potential to order Δ^2 can be expressed as [161, 162]

$$\begin{aligned}\Omega_{\text{CFL}} &= \Omega_{free} - \frac{3}{\pi^2}\Delta^2\mu^2 + B \\ &= \frac{6}{\pi^2}\int_0^\nu [p-\mu]\, p^2 dp + \frac{3}{\pi^2}\int_0^\nu \left[\left(p^2+m_s^2\right)^{\frac{1}{2}} - \mu\right] p^2 dp - \frac{3}{\pi^2}\Delta^2\mu^2 + B,\end{aligned} \tag{5.7}$$

where Ω_{free} represents the non-paired state, μ is the chemical potential and ν is the common Fermi momentum, given by

$$\nu = 2\mu - \left(\mu^2 + \frac{m_s^2}{3}\right)^{\frac{1}{2}}. \tag{5.8}$$

Pressure and energy density can be obtained using the relations [161]

$$p_{\text{CFL}} = -\Omega_{\text{CFL}}, \tag{5.9}$$

$$\varepsilon_{\text{CFL}} = \Omega_{\text{CFL}} - \mu\frac{\partial\Omega_{\text{CFL}}}{\partial\mu}. \tag{5.10}$$

From the above equation, pressure and energy density can be analytically expressed to order m_s^2 as

$$p_{\text{CFL}} = \frac{3\mu^4}{4\pi^2} + \frac{9\alpha\mu^2}{2\pi^2} - B, \tag{5.11}$$

$$\varepsilon_{\text{CFL}} = \frac{9\mu^4}{4\pi^2} + \frac{9\alpha\mu^2}{2\pi^2} + B, \tag{5.12}$$

with

$$\alpha = -\frac{m_S^2}{6} + \frac{2\Delta^2}{3}. \tag{5.13}$$

Although this semi-empirical model is enough for the present purposes, it is important to mention that other effects may be included, for instance, a pQCD contribution from one-gluon exchange for gluon interaction [163].

Parametrization	$B\left(\text{MeV}/fm^3\right)$	$\Delta(\text{MeV})$	$m_S(\text{MeV})$
CFL1	60	50	0
CFL2	60	50	150
CFL3	60	100	0
CFL4	60	100	150
CFL5	60	150	0
CFL6	60	150	150
CFL7	80	100	0
CFL8	80	100	150
CFL9	80	150	0
CFL10	80	150	150
CFL11	100	50	0
CFL12	100	100	0
CFL13	100	100	150
CFL14	100	150	0
CFL15	100	150	150
CFL16	120	100	0
CFL17	120	150	0
CFL18	120	150	150
CFL19	140	150	0

Table 5.3: *Set of parametrizations for CFL matter.*

It is straightforward to express pressure and energy density as a one parameter equation of state of the form $\varepsilon(p)$, namely

$$\varepsilon_{\text{CFL}} = 3p_{\text{CFL}} + 4B - \frac{9\alpha\mu^2}{\pi^2}, \tag{5.14}$$

with the chemical potential expressed by

$$\mu^2 = -3\alpha + \left[\frac{4\pi^2}{3}\left(B + p_{\text{CFL}}\right) + 9\alpha^2\right]^{\frac{1}{2}}. \tag{5.15}$$

Alternatively, in a similar fashion $p(\varepsilon)$ is given by

$$p_{\text{CFL}} = \frac{\varepsilon_{\text{CFL}}}{3} - \frac{4B}{3} + \frac{3\alpha\mu^2}{\pi^2}, \tag{5.16}$$

where the chemical potential is now expressed in terms of the energy density, namely

$$\mu^2 = -\alpha + \left[\frac{4\pi^2}{9}\left(\varepsilon_{\text{CFL}} - B\right) + \alpha^2\right]^{\frac{1}{2}}. \tag{5.17}$$

Also, the speed of sound can be obtained straightforwardly,

$$c^2_{s,\text{CFL}} = \frac{dp_{\text{CFL}}}{d\varepsilon_{\text{CFL}}} = \frac{1}{3} + \frac{2\alpha}{3}\left(\frac{1}{\mu^2 + \alpha}\right). \tag{5.18}$$

The gap parameter plays a central role in the CFL phase because as the parameter increases the equation of state gets stiffer, allowing configurations with higher maximum masses when compared with regular strange stars [161]. The CFL matter is also significantly more bound than ordinary quark matter, being a candidate for the true ground state of hadronic matter for a much wider range of the parameters of the model than the state without any pairing [159, 162].

5.2.2 Introducing Semiclassical Effects in CFL Strange Stars

The structure equation for semiclassical stars, under the conditions discussed in Chapter 4, are given by

$$p'\left(1 - \frac{\ell_P^2}{2r}\frac{p'}{\varepsilon + p}\right) = -\frac{(\varepsilon + p)\left(m + 4\pi r^3 p\right)}{r\,(r - 2m)}, \tag{5.19}$$

$$\frac{2m'}{r^2} = 8\pi\varepsilon + \frac{\ell_P^2}{r^2}\left[\left(1 - \frac{2m}{r}\right)\frac{C''}{C} - \frac{C'}{C}\left(\frac{m}{r}\right)' - \frac{3}{4}\left(1 - \frac{2m}{r}\right)\left(\frac{C'}{C}\right)^2\right]. \tag{5.20}$$

It follows from equation (5.19) that the pressure gradient has two solutions in semiclassical gravity, namely

$$p'_{\pm} = \frac{r\,(\varepsilon + p)}{\ell_P^2}\left(1 \pm \sqrt{1 + \frac{2\ell_P^2}{r^3}\frac{m + 4\pi r^3 p}{1 - 2m/r}}\right). \tag{5.21}$$

After a "classicalization process" taking advantage of the limits $\hbar \to 0$ and $N \to \infty$, these equations assume the following form[67]

$$m' = 4\pi r^2 \varepsilon, \tag{5.22}$$

$$p'_{\pm} = \pm\,(\varepsilon + p)\,\frac{m + 4\pi r^3 p}{r\,(r - 2m)}. \tag{5.23}$$

So, the p'_{-} solution recovers the TOV equation at the limit $\hbar \to 0$ [39]. This equation will be associated with CFL quark matter and treated just as in general relativity. Even though a classicalization process was performed, the resulting structure equations differ from the system obtained in general relativity. Since the only modification in the semiclassical field equations is a source associated with vacuum polarization effects, one is lead to concluded that the additional equation in the system, namely, the p'_{+} solution, are ultimately connected to this physical effect. In the same way that in general relativity the equation of state is a phenomenological representation of the quantum physics of matter, the equation of state associated with this new solution can be seen as a phenomenological representation of the semiclassical effects in the macroscopic system. Therefore, additionally, it will be hypothesized that the environment produced by CFL matter is able to ignite a semiclassical correction, absent in general relativity, with a pressure gradient obeying the p'_{+} solution. This solution differs only by a sign change with respect to the usual TOV equation. Even so, the forms of matter that can be applied in this case must be analyzed.

Exactly like the usual TOV equation, the equation describing p'_{+} has also to fulfill the condition $r > 2m$ at any point inside the star, thus forbidding the presence of a Schwarzschild black hole at any radius r. Now observe that, for ordinary forms of matter with $\varepsilon > 0$ and $p > 0$, the TOV equation describes a negative pressure gradient and the pressure is a monotonically decreasing function of the radius which eventually vanishes. It is not difficult to realize that in order to have a similar picture regarding p'_{+}, an unconventional equation of state should be employed. Consider, for instance, a linear equation of state with negative pressure (also called tension),

$$\tilde{p} = -\omega\tilde{\varepsilon}, \tag{5.24}$$

where ω is a positive parameter smaller than unity. For configurations where $m > |4\pi r^3 p|$ the pressure gradient is positive and assures that the absolute value of the pressure (and also the energy density) decreases as the radius increases.

The presence of a negative pressure component may sound unphysical, but this is certainly not the case. For example, in a rubber band, the component of the stress tensor along the band is negative [69]. The mechanical action of a negative pressure is that the internal volume forces in the matter are attractive, instead of repulsive (which is the usual situation for an observational media consisting of particles), thus allowing more compact configurations [64][68]. It has been long ago proposed that such an effect may have a role in ultradense matter, possibly reached through a gravitational collapse, providing additional attraction between material elements [64]. This idea couples very well with the possible unconventional sources that one may introduce via the new root in which the semiclassical effects are dominant. Also, as exposed

[67] See Appendix C.

[68] Eventually, this might bring confusion when one thinks of gravitationally repulsive fluids with negative pressure, like dark energy. It is important to observe that there is no contradiction between a particular type of fluid being gravitationally repulsive while presenting an attractive mechanical action [164]. These aspects are two separate matters.

in Chapter 2, one of the main differences between Newtonian gravity and general relativity, is that in the latter is that pressure not only opposes gravity but also enhances its effects. This phenomenon is called *regeneration of the pressure* [104]. Here, in the semiclassical scenario proposed, the negative pressure may be a continuation of this phenomenon, but now it is physically attributed to vacuum effects. So, conceptually the present model explores the hypothesis that the usual repulsive fields between material elements are affected by an additional component with a semiclassical origin, phenomenologically described via a negative pressure fluid, which is reached during the gravitational collapse of the ultradense matter present in CFL strange stars, allowing ultracompact configurations to emerge.

It is worth mentioning that the negative pressure fluid introduced by equation (5.24) is a subclass of what is sometimes called a γ-fluid, which satisfies the γ-law equation of state [165]

$$p = (\gamma - 1)\,\varepsilon, \quad 0 \leqslant \gamma \leqslant 2. \tag{5.25}$$

In this sense, the semiclassical solutions treated hereafter can be seen as describing γ-CFL strange stars. Other important subclasses of the γ-law equation of state include stiff matter ($\gamma = 2$), blackbody radiation ($\gamma = 4/3$) and vacuum energy ($\gamma = 0$). The thermodynamical properties can easily be derived by considering γ-fluids as some kind of generalized radiation [165]. It can be shown, for instance, that such fluids obey generalized versions of the usual Wien and Stefan-Boltzmann laws. Moreover, if fluids with $\gamma < 1$ could be somehow confined in a vessel, they would try to pull the walls inwards, instead of outwards as it happens with an ordinary gas.

5.2.3 Minimal Geometric Deformation

The proposed scenario to describe γ-CFL strange stars ultimately demands a decoupling of the gravitational sources. Such a simplification may seem absurd due to the highly nonlinear structure of the field equations. Fortunately this can be achieved, at least for the spherically symmetric and static case, through a technique called minimal geometric deformation. So, before numerically implementing the model outlined in the last subsection, it is worth to show how the CFL sector and the semiclassical one can be decoupled, as long as the sources interact only gravitationally [166, 167]. Therefore, aiming to incorporate more intricate gravitational sources to CFL strange stars, consider the field equations in the following form

$$G_{\mu\nu} = 8\pi \left(T^{\text{CFL}}_{\mu\nu} + \beta\theta_{\mu\nu} \right), \tag{5.26}$$

where $\theta_{\mu\nu}$ describes geometrical or physical contributions of any additional source that may arise due to the presence of extra interactions whose coupling to gravity is proportional to the constant β [168]. Here it will describe a sector where the semiclassical effects are dominant, namely

$$\theta_{\mu}^{\ \nu} = \tilde{T}_{\mu}^{\ \nu} + \hbar N Q_{\mu}^{\ \nu} + \frac{\delta_{\mu}^{\ a}\,\delta_{a}^{\ \nu}}{8\pi r^2}. \tag{5.27}$$

This new source includes a perfect fluid obeying equation (5.24), represented by $\tilde{T}_{\mu\nu}$. The last term in the right hand side is a geometrical factor (evaluated with respect to $ds^2_{(2)}$ in a similar fashion to $Q_{\mu\nu}$) introduced in order to preserve the spherically symmetric form of the Einstein tensor (thus respecting the Bianchi identities) in both sectors after the geometrical deformation is performed [166, 167].

It can be identified by inspection that the combination of the two fluids provide an effective energy density and a effective pressure specified by the relations

$$\varepsilon = \varepsilon_{\text{CFL}} + \beta\tilde{\varepsilon}, \tag{5.28}$$

$$p = p_{\text{CFL}} + \beta\tilde{p}. \tag{5.29}$$

The minimal geometric deformation can be introduced via a linear decomposition of the form [166, 167]:

$$e^{-2\lambda(r)} = C(r) + \beta D(r) \tag{5.30}$$

The linear decomposition splits the system into two sets. The first, with $\beta = 0$, is obviously the CFL strange star solution as obtained using general relativity. The other set corresponds to the additional source, with (t,t) and (r,r) components obeying

$$\begin{aligned} \frac{D-1}{r^2} + \frac{D'}{r} &= 8\pi\tilde{T}_t^{\ t} + 8\pi\hbar N Q_t^{\ t}, \\ \frac{D-1}{r^2} + 2D\frac{\Phi'}{r} &= 8\pi\tilde{T}_r^{\ r} + 8\pi\hbar N Q_r^{\ r}. \end{aligned} \tag{5.31}$$

These are simply the (t,t) and (r,r) components obtained through equation (4.84) under spherical symmetry, from which the semiclassical hydrostatic equilibrium is directly derived. Since this is the sector where semiclassical effects are assumed to be dominant, it is logical to choose the p'_+ solution in equation (5.23) to represent its hydrostatic equilibrium.

5.2.4 γ-CFL Strange Stars

Hydrostatic equilibrium, as expressed in equations (5.22) and (5.23), will be applied to CFL quark matter and to an additional γ-fluid with negative pressure, producing a γ-CFL strange star where both contributions interact only gravitationally [166, 167]. Numerical solutions are obtained similarly to the standard procedure in general relativity, keeping in mind that here each fluid satisfies its own hydrostatic equilibrium equation. The system is solved from the center, with $m(r = 0) = 0$ for both contributions, and the energy densities are needed as input. The integration stops when the effective pressure vanishes, defining the final mass (which is the total gravitational mass from its two contributions) and radius. It is worth mentioning that although the CFL strange star will be embedded in a negative pressure environment, all solutions considered have an non-negative effective pressure and energy density.

Figure 5.9 illustrates the impact of the β parameter on the effective equation of state using the CFL5 and CFL15 parametrizations. The cases with $\beta \neq 0$ adopt $\omega = 0.05$, while the $\beta = 0$ case is simply the CFL equation of state as expressed in equation (5.14). A dashed green line corresponding to the ultrarelativistic limit $p = \varepsilon/3$ was added to the figure as a reference. The initial nonlinear behavior (associated with the μ^2 term in equation (5.14)) is preserved in the effective equation of state, as well as the asymptotic linear behavior [159]. Increasing β translates into a lower pressure for a given energy density. In other words, the γ-fluid reduces the effective pressure of the system, softening the equation of state. From this, however, one must not infer that configurations with $\beta \neq 0$ will necessarily produce solutions with lower maximum masses, when compared with general relativity, since in this case hydrostatic equilibrium is expressed differently. Hereafter, in order to restrict the role of semiclassical effects in the model, the value of $\beta\tilde{\varepsilon}$ at the center is taken to be 10% of the main contribution coming from the CFL phase.

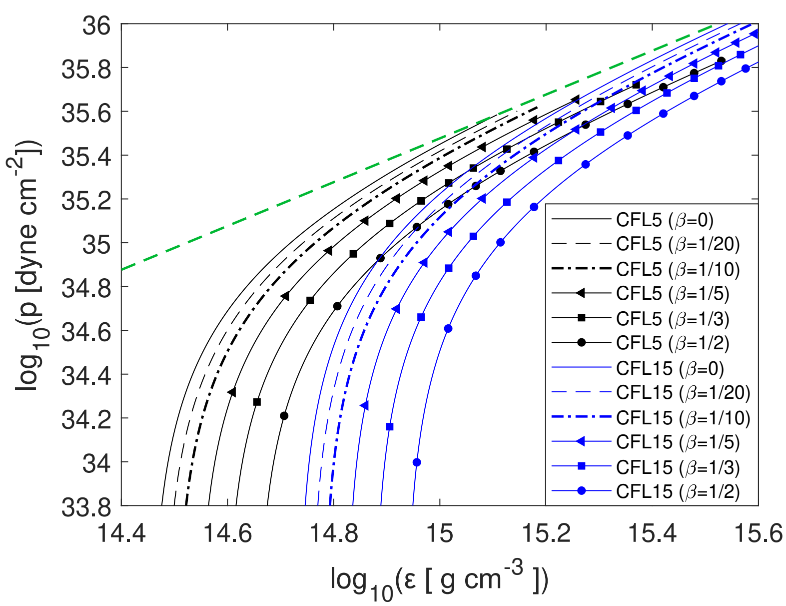

Figure 5.9: *Impact of the β parameter on the effective equation of state for the CFL5 and CFL15 parametrizations. The slashed green line denotes the ultrarelativistic limit $p = \varepsilon/3$.*

Since semiclassical stellar models are still in their early days, a detailed comparison with recent astronomical observations may not be a primary concern. Even though, in order to provide some perspective, a few comments are pertinent. In a combined effort by the North American Nanohertz Observatory for Gravitational Waves (NANOGrav) and the Canadian Hydrogen Intensity Mapping Experiment (CHIME)/Pulsar, the mass of the PSR J0740+6620 pulsar has been updated to the range $M = 2.08 \pm 0.07 M_{\odot}$, which is the highest reliably determined mass for a pulsar so far [169]. Also, by combining datasets from the X-ray telescopes NICER and XMM-Newton, in Ref. [84] the radius of PSR

J0740+6620 has been constrained to be $R = 12.39^{+1.30}_{-0.98}$ km.

Another important value for constraining equations of state is the radius of a $1.4\,M_\odot$ neutron star. The analysis made in Ref. [85] found $R_{1.4} = 12.18^{+0.56}_{-0.79}$ km, adopting a model based on the speed of sound in a neutron star. These values were included in Figures 5.10 and 5.11 as reference. However, the confrontation between those numbers and the strange star solutions presented here may not be immediate. For example, some works have discussed the idea that neutron and strange stars could coexist as two separate families, obviously resulting in distinct mass-radius relations [160]. Besides that, NICER data has also been used to study configurations at the threshold which marks the ultracompact regime [87]. Although such stars are not currently an observational reality, they may become important in the near future.

As illustrated in Figures 5.10 and 5.11, the main feature of introducing the semiclassical correction is the possibility of finding ultracompact configurations (defined by $\mathcal{C} > 1/3$) throughout all parametrizations without imposing significant deviations in the low mass-radius region. Besides that, all semiclassical solutions present an upper bound to the mass of stable configurations. In Figures 5.10 and 5.11, a grey line corresponding to $\mathcal{C} = 1/3$ was added to demarcate the ultracompact region. Specifically, Figure 5.10 confronts the mass-radius curve obtained using general relativity and semiclassical gravity for some parametrizations of the CFL equation of state, while the ω parameter is fixed in 0.05 (in $G = c = 1$ units where the parameter is dimensionless). The semiclassical configurations are more massive and smaller, crossing to the ultracompact region.

On the other hand, in Figure 5.11 the ω parameter is varied while the parametrization is fixed. The goal is to illustrate that the maximum mass can not be increased indefinitely within the model. Due to the negative pressure associated with the component carrying the semiclassical corrections, after some value of ω that depends on the inputs for the central energy densities, the solutions become less massive. Also, the physically acceptable solutions found are restricted to the subset of equation (5.24) where $\omega \ll 1$. In another context, this equation of state has also been used to describe the total effect of a mixture of cold dark matter and dark energy [170].

Figures 5.12-5.17 compare some internal functions for the maximum mass configurations when using general relativity and semiclassical gravity with $\omega = 0.05$. A subset of the parametrizations presented in Table 5.3 was chosen aiming for a better visualization. The semiclassical solutions for the energy density and pressure (Figures 5.13 and 5.15) preserve the characteristic monotonically decreasing behavior found in general relativity (Figures 5.12 and 5.14), although they possess higher central energy densities and pressures. Besides that, just as in the previous model, the energy density does not vanish at the surface, since strange stars are also self-bound configurations[69].

[69] The surface density in strange stars is equal to the value for strange quark matter at zero pressure. It is about fourteen orders of magnitude larger than the surface density of regular neutron stars [81].

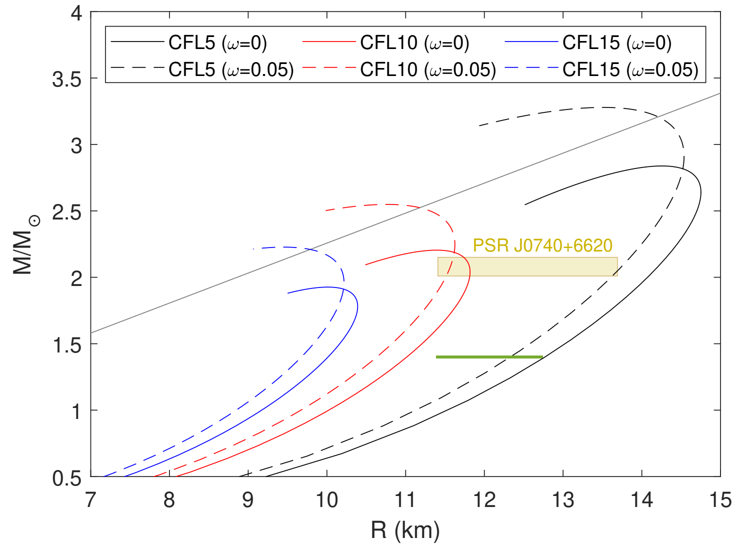

Figure 5.10: *Comparison of the mass-radius relationship obtained in general relativity and semiclassical gravity for some parametrizations of the CFL equation of state. The rectangular region corresponds to mass and radius constraints for the pulsar PSR J0740+6620. The green horizontal line denotes radius constraints for a $1.4\,M_{\odot}$ neutron star. The gray line demarcates the threshold for the ultracompact region ($\mathcal{C} = 1/3$).*

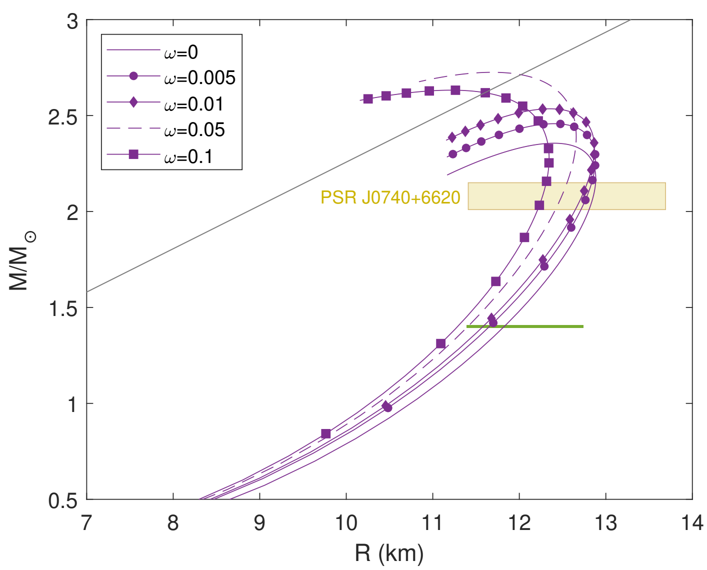

Figure 5.11: *Impact of the ω parameter on the mass-radius curve for the CFL3 parametrization. Other elements displayed are the same as those exhibited in Figure 5.10.*

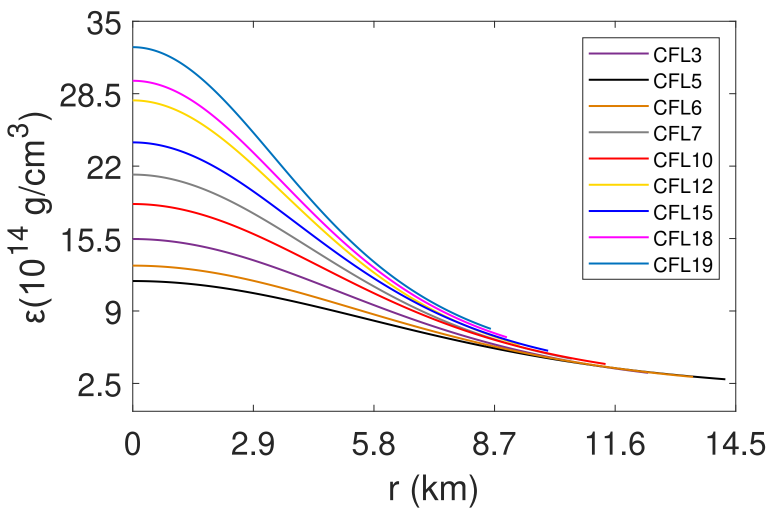

Figure 5.12: *Energy density profiles for different CFL parametrizations using general relativity.*

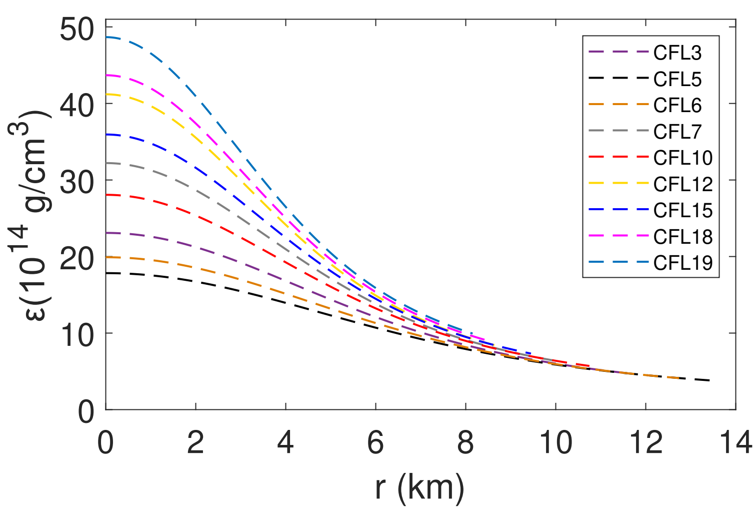

Figure 5.13: *Energy density profiles for different CFL parametrizations using semiclassical gravity* $(\omega = 0.05)$.

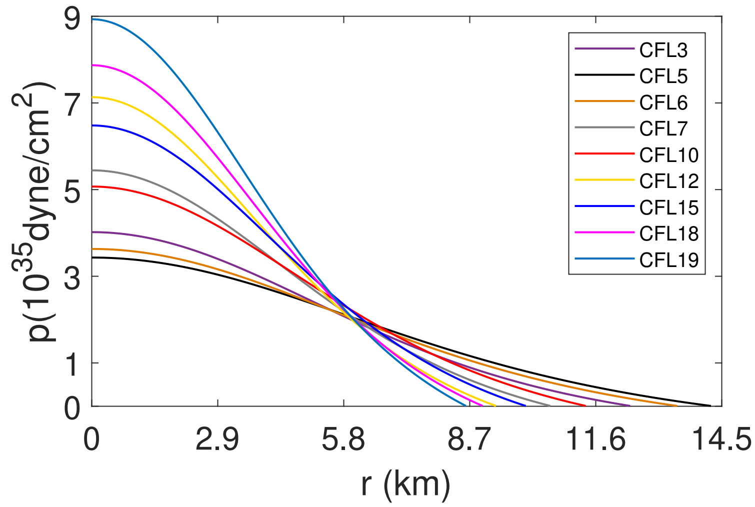

Figure 5.14: *Pressure profiles for different CFL parametrizations using general relativity.*

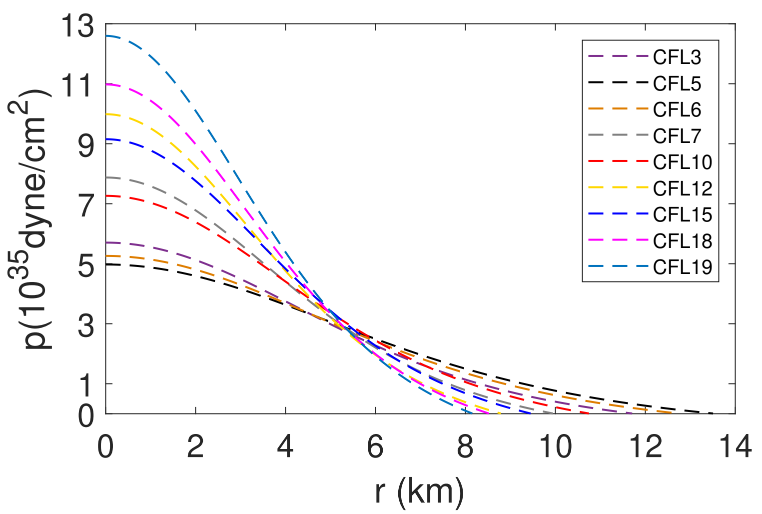

Figure 5.15: *Pressure profiles for different CFL parametrizations using semiclassical gravity* $(\omega = 0.05)$.

The pattern for the mass function inside the star, as illustrated in Figures 5.16 and 5.17, is also preserved within the model. In both situations the mass continuously increases with increasing radius, but the γ-CFL stars are about $1\,$km smaller and $0.3\,\mathrm{M}_\odot$ heavier.

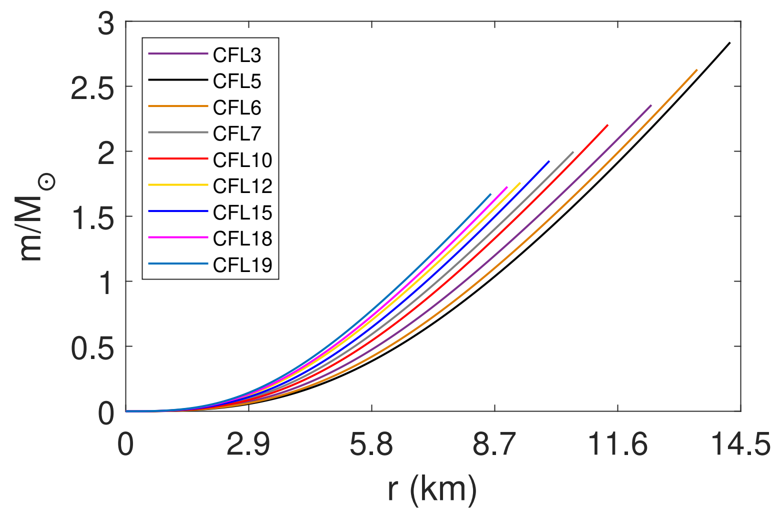

Figure 5.16: *Mass function profiles for different CFL parametrizations using general relativity.*

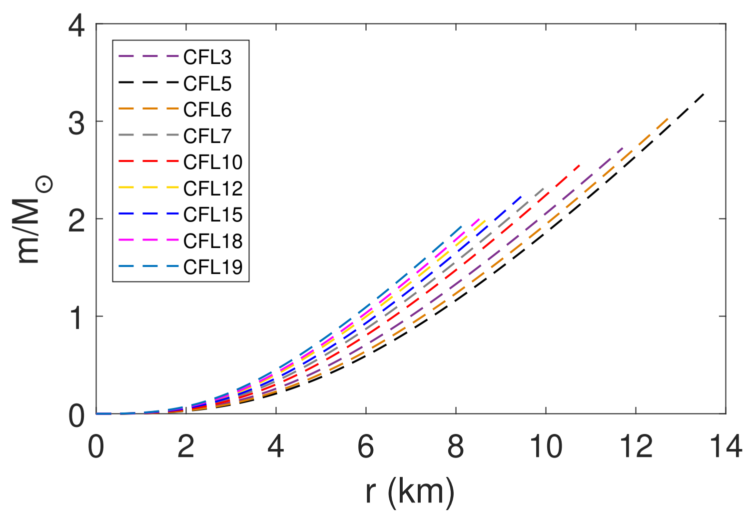

Figure 5.17: *Mass function profiles for different CFL parametrizations using semiclassical gravity* ($\omega = 0.05$).

In order to check that the causality constraint is not violated, in Figure 5.18 the sound speed squared is plotted as a function of the pressure, for different values of the ω parameter, using the CFL5 parametrization (which is the one with the highest sound speed [159]).

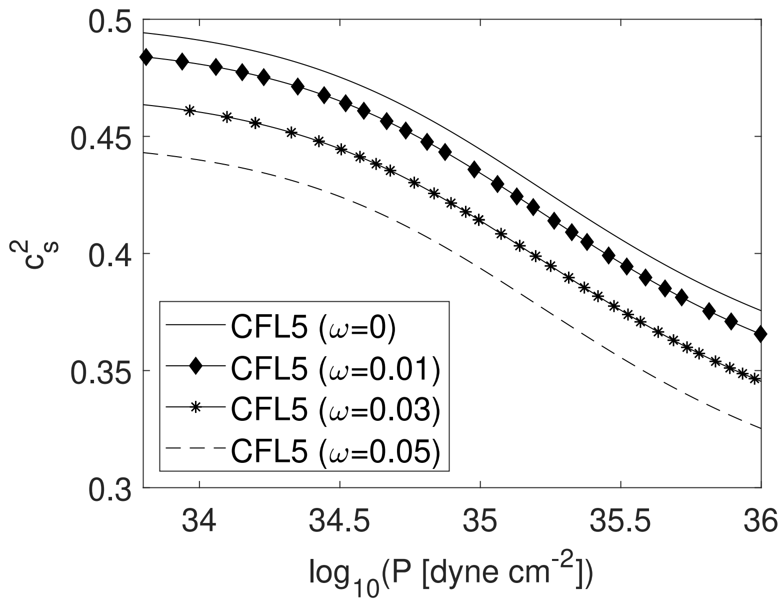

Figure 5.18: *Sound speed squared as a function of pressure for different values of the ω parameter using the CFL5 parametrization.*

The sound speed squared associated with equation (5.24) is simply $-\omega$ (strictly valid only for constant ω [171]), therefore the sound speed of CFL matter inevitably constraints the ω parameter, providing an upper limit, namely

$$\omega < \frac{1}{3} + \frac{2\alpha}{3}\left(\frac{1}{\mu^2+\alpha}\right). \tag{5.32}$$

The reason behind this condition is that the effective sound speed squared must be non-negative, otherwise the composition of the compact star would be microscopically unstable [38].

Table 5.4 summarizes the results for the nineteen parametrizations of the CFL equation. The data corresponds to the maximum mass configurations. In order to interpret the implications of such solutions, it is interesting to observe that semiclassical gravity and general relativity can produce, for different parametrizations, similar results for mass and radius (compare for example the CFL9 parametrization in general relativity and the CFL1 parametrization in semiclassical gravity). Probably a mass measurement would not be able to distinguish among such objects, even though they would be physically different, since the semiclassical solutions would present the physical signatures associated with the ultracompact regime.

Parametrization	General Relativity $(M(\mathrm{M}_\odot), R(\mathrm{km}), \mathcal{C})$	Semiclassical Gravity $(M(\mathrm{M}_\odot), R(\mathrm{km}), \mathcal{C})$
CFL1	(2.051, 11.08, 0.27)	(2.375, 10.43, 0.34)
CFL2	(1.830, 10.09, 0.27)	(2.120, 9.47, 0.33)
CFL3	(2.357, 12.38, 0.28)	(2.725, 11.68, 0.36)
CFL4	(2.127, 11.41, 0.27)	(2.462, 10.75, 0.34)
CFL5	(2.842, 14.24, 0.29)	(3.278, 13.51, 0.36)
CFL6	(2.631, 13.46, 0.29)	(3.039, 12.74, 0.35)
CFL7	(1.994, 10.52, 0.28)	(2.309, 9.95, 0.34)
CFL8	(1.821, 9.79, 0.27)	(2.111, 9.22, 0.34)
CFL9	(2.365, 11.98, 0.29)	(2.735, 11.36, 0.35)
CFL10	(2.202, 11.36, 0.29)	(2.548, 10.76, 0.35)
CFL11	(1.571, 8.51, 0.27)	(1.823, 8.01, 0.34)
CFL12	(1.754, 9.29, 0.28)	(2.034, 8.79, 0.34)
CFL13	(1.616, 8.70, 0.27)	(1.875, 8.20, 0.34)
CFL14	(2.055, 10.49, 0.29)	(2.379, 9.95, 0.35)
CFL15	(1.922, 9.98, 0.28)	(2.227, 9.44, 0.35)
CFL16	(1.582, 8.40, 0.28)	(1.835, 7.95, 0.34)
CFL17	(1.834, 9.42, 0.29)	(2.125, 8.94, 0.35)
CFL18	(1.722, 8.98, 0.28)	(1.997, 8.51, 0.35)
CFL19	(1.667, 8.60, 0.29)	(1.934, 8.16, 0.35)

Table 5.4: *Comparison between the maximum mass configurations using general relativity and semiclassical gravity. For the semiclassical solutions it was adopted* $\omega = 0.05$.

In Ref. [160] it was discussed the possible tension between the evidence of the existence compact stars satisfying $R \lesssim 11.6$ km at $1.4\mathrm{M}_\odot$ (suggested by some analyses on thermonuclear bursts and X-ray binaries), and the possibility of very massive stars with $M \sim 2.6\mathrm{M}_\odot$. None of the parametrizations can accommodate, using general relativity, a family satisfying $R_{1.4} \lesssim 11.6$km and also stars with masses as high as $2.6\mathrm{M}_\odot$. For the semiclassical solutions this is not necessarily the case. The CFL3 parametrization, for instance, predicts a radius of 11.40km for a star with $M = 1.4\mathrm{M}_\odot$ and a maximum mass of $2.725\mathrm{M}_\odot$, being able to deal with both conditions at once.

In order to further explore the model in a complementary direction it is important to consider the dimensionless tidal deformability Λ as presented in Chapter 2, namely

$$\Lambda = \frac{2}{3} k_2 \mathcal{C}^{-5}, \tag{5.33}$$

which is strongly dependent on the star composition. In the above equation, k_2 is a dimensionless quantity called the Love number. The configurations with $\omega \neq 0$ are computed using the standard procedure for self-bound stars (see for example Ref. [97]), just replacing the physical quantities by their effective counterparts[70].

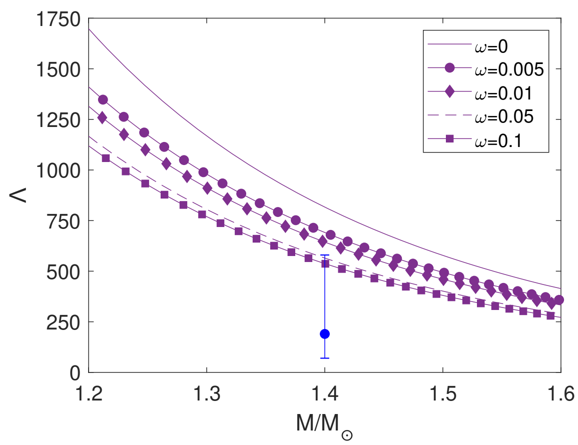

Figure 5.19: *Dimensionless tidal deformability as a function of the total mass using different values of the ω parameter for the CFL3 parametrization. Full circle: $\Lambda_{1.4} = 190^{+390}_{-120}$.*

Analysis based on the gravitational-wave event GW170817 implies that $70 < \Lambda_{1.4} < 580$, providing an important upper limit on stiffness [172, 173]. All parametrizations, except the four with the largest maximum masses (i.e, CFL3, CFL5, CFL6 and CFL9), satisfy $70 < \Lambda_{1.4} < 580$ already in general relativity. As a consequence, it is not possible to establish a minimum value for ω in such cases, since the referred range is also satisfied for arbitrarily small ω. However, differently from general relativity, it is also possible to find compatible semiclassical solutions for CFL3 and CFL9, but ω must be at least 0.034 and 0.003, respectively. No physically acceptable solutions were found for the CFL5 and CFL6 because they would violate the maximum value for ω as expressed by the inequality (5.32).

Figure 5.19 shows the dimensionless tidal deformability as a function of the total mass M for the CFL3 parametrization. So, for example, the previously discussed result for $\omega = 0.05$ has also the virtue of being within the limits imposed by the event GW170817, since in this case $\Lambda_{1.4} = 564$. It is not difficult to visualize why the semiclassical configurations present smaller tidal deformabilities for the proposed model. The Love number is, roughly speaking, inversely proportional to the compactness for the range $\mathcal{C} = 0.1 - 0.3$, hence the tidal deformability dependence on the compactness is about $\Lambda \propto \mathcal{C}^{-6}$

[70] This procedure is only valid when $\delta\theta_{\mu\nu} \approx \delta\tilde{T}_{\mu\nu}$ is assumed. In these situations the effective source can be directly substituted in equation 2.46.

[23]. Considering that the semiclassical solutions presented here have higher compactness than its general relativistic counterparts, the tidal deformability decrease is expected.

Therefore, results have shown the possibility of horizonless ultracompact configurations, a class that has received much attention in the literature [38, 39, 40, 109]. These results were achieved without imposing drastic modifications on the low mass-radius regime. Within the semiclassical framework it is also possible to find families of stars satisfying $R_{1.4} \lesssim 11.6$ km, $70 < \Lambda_{1.4} < 580$ and masses as high as $2.6 M_{\odot}$, which is very difficult within regular neutron star or even strange star models [160].

5.3 Relaxing Isotropy in Semiclassical Gravity

In order to motivate a final application, observe that spherical symmetry does not require an isotropic source [26]. In fact, the most general source compatible with spherical symmetry has the form

$$T^{\mu}{}_{\nu} = diag\{-\varepsilon, p_r, p_{\perp}, p_{\perp}\}, \tag{5.34}$$

which describes an anisotropic fluid, as discussed in Chapter 2. The number of physical processes from which anisotropy might emerge is quite large. Such deviations from local isotropy could occur, for instance, in exotic phase transitions that may take place in the gravitational collapse of highly dense systems [174]. Further motivation comes from the observation that nuclear matter has a tendency to become anisotropic at densities of order 10^{15} $\mathrm{g/cm^3}$ [175]. Anisotropy could also take place in low density systems, like in the processes of stellar formation.

The rest of this section is dedicated to construct a simple semiclassical anisotropic model, relying on the nonlocal equation of state (NLES), in which the components of the energy-momentum tensor at a given point have a functional dependence throughout the enclosed configuration, instead of being simply a function at that point [176].

This differs from the traditional analysis in classical continuum theories, where one relies on the premise that the state of an object is completely determined by the behavior of an arbitrary infinitesimal neighborhood centered at any of its material points [176]. There is also the assumption that any section of the material can be extrapolated as a representation of its totality and the underlying physics is valid for every part of body, independently of how small.

This non-local character can dominate the macroscopic behavior of matter under many circumstances found in modern classical continuum mechanics and fluid dynamics, being vital to tackle many problems in science and engineering. Just to cite a few examples in which a non-local approach is useful: to study damage and cracking in materials, surface phenomena between two liquids or two phases, blood flow and colloidal suspensions [176]. It has also been shown that under particular circumstances, a general relativistic spherically symmetric bounded distribution of matter could obey a non-local equation of state [177].

5.3.1 Non-local Equation of State

In the static limit, the non-local equation of state can be written as [176, 177]

$$p_r(r) = \varepsilon(r) - \frac{2}{r^3}\int_0^r u^2\varepsilon(u)du. \tag{5.35}$$

It is not difficult to see that the above equation describes a collective relation between the radial pressure and energy density. The radial pressure is not simply a function of the energy density at a given point, it is a functional that depends on the entire configuration, indicating the non-local behavior of these variables. Any change in the radial pressure considers the effects associated with the variations of the energy density within the entire volume [176, 177].

Equation (5.35) can alternatively expressed as a differential equation, namely

$$\varepsilon(r) - 3p_r(r) + r\left(\varepsilon'(r) - p_r'(r)\right) = 0. \tag{5.36}$$

By looking at the above equation is easy to see that causality can be stated as:

$$c_s^2 < 1 \implies \frac{\varepsilon - 3p_r}{r\varepsilon'} < 0. \tag{5.37}$$

For further purposes, it is interesting to state the following energy conditions:

$$\text{Strong energy condition}: \ \varepsilon + p_r + 2p_\perp \geqslant 0; \tag{5.38}$$

$$\text{Trace energy condition}: \ \varepsilon + p_r + 2p_\perp \geqslant 0. \tag{5.39}$$

In particular, the trace energy condition is more restrictive than the strong energy condition for imperfect fluids [178].

5.3.2 The Tangential Pressure

As mentioned in Appendix A, employing an anisotropic source demands an additional relation. The usual procedure is to consider an ansatz to express the tangential pressure. Similarly to what was assumed in

Ref. [39], let is consider that the quantity inside the square root in the differential equation for the radial pressure is equal to a constant. Therefore one can writes

$$p_r' = \frac{2\Delta}{r} + \frac{r\left(\varepsilon + p_r\right)}{\ell_P^2}\left(1 - \sqrt{1 + \frac{2\ell_P^2}{r^3}\frac{m + 4\pi r^3 p_r}{1 - 2m/r}}\right) = \frac{2\Delta}{r} + \frac{r\left(\varepsilon + p_r\right)}{\ell_P^2}\tau. \tag{5.40}$$

where τ is some constant. By combining the above result with the non-local equation of state (5.36) one finds

$$p_\perp = \frac{1}{2}\left[\varepsilon - p_r + r\varepsilon' - \frac{r^2\tau}{\ell_P^2}\left(\varepsilon(r) + p_r\right)\right] \tag{5.41}$$

This implies that at the center $p_{\perp c} = \left(\varepsilon_c - p_{rc}\right)/2$. Regarding the ansatz, throughout this section it will be assumed that $\tau = -1/N$.

5.3.3 Semiclassical Solutions Using the NLES

Now it is time to verify if the proposed model is able to produce credible solutions. The possibility of ultracompact solutions is already verified in Figure 5.20. The compactness function is displayed as a function of the radius, presenting two solutions entering the ultracompact region. The final radius is practically constant for configurations with central energy densities in the range $1\varepsilon_S - 6.5\varepsilon_S$, being about 12.40 km. It is worth noticing that despite the simplicity of the model, the typical size of the solutions presented is compatible with regular compact stars. The total mass, on the other hand, varies from $0.64\,M_\odot$ to $4.17\,M_\odot$. So the maximum mass overcomes the usual limits considered for compact stars.

Figure 5.21 exemplify the behavior of the function that describes the causality constraint for the NLES, using $\varepsilon_c = 6\varepsilon_S$, starting from a point very close to the center. The relevant function remains negative throughout the entire configuration, as demanded by (5.37). Figure 5.22 shows that the same configuration also satisfies both the strong and the trace energy conditions at all radii. All things considered, bounded ultracompact gravitational sources can also materialize in semiclassical gravity when anisotropic sources are considered, expanding the range of hydrodynamic scenarios available.

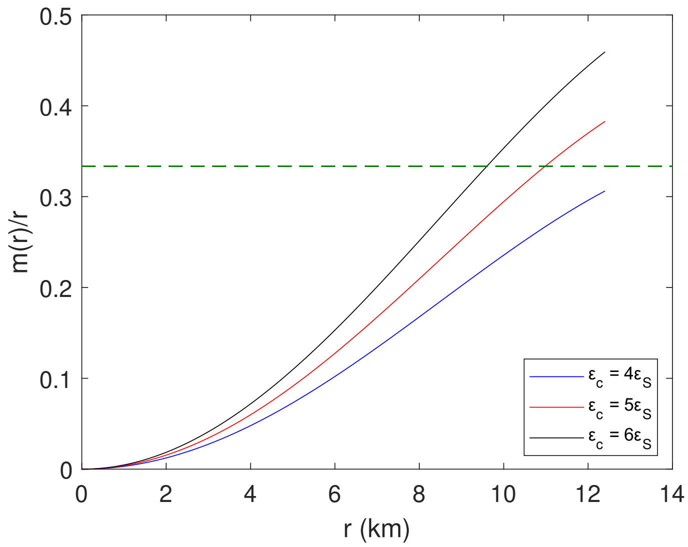

Figure 5.20: *Compactness function versus radius using different values for the central energy density. The green dashed line represents the ultracompact limit $\mathcal{C} = 1/3$.*

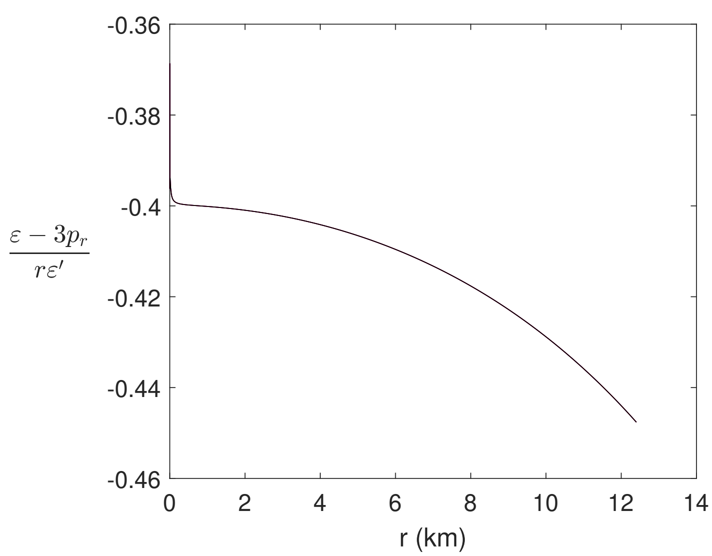

Figure 5.21: *Relevant quantity to verify the causality condition for the NLES using $\varepsilon_c = 6\varepsilon_S$.*

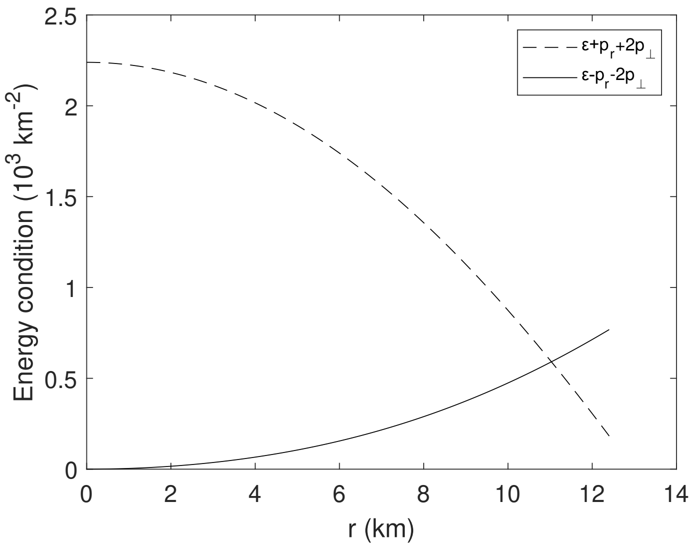

Figure 5.22: *Relevant quantities to verify the Strong (dashed) and Trace (solid) energy conditions using* $\varepsilon_c = 6\varepsilon_S$.

5.4 Forthcoming Research

Many different paths might be taken to explore more deeply the impacts of the semiclassical formalism on the ultracompact regime. First, the anisotropic results presented here are only preliminary and a more detailed study is needed. Also, a semiclassical study of gravitational wave echoes requires a careful examination that has not yet been done. Such study is fundamental to investigate if these solutions are able to generate gravitational echoes like those that characterize the relaxation phase of a putative black hole mimicker [38].

However, to evaluate this properly it is important to generalize our results beyond the spherically symmetric static configurations, which is an idealization. The presence of rotation breaks spherical symmetry and the configurations become axisymmetric. Through rotation it is possible to verify, among other issues, if the solutions are not plagued with ergoregion instability [38]. In that sense, to develop a Hartle-Thorne semiclassical solution would be a natural and interesting sequence for this work. The solution is basically a perturbed solution of the Schwarzschild metric. In that case the spherical symmetric line element is replaced by [72]

$$\begin{aligned} ds^2 = & - e^{2\Phi}\Big\{1 + 2\Big[h_0(r) + h_2(r)P_2\left(\cos(\theta)\right)\Big]\Big\}dt^2 \\ & + \left[1 - \frac{2m(r)}{r}\right]^{-1}\left\{1 + \frac{2\Big[m_0(r) + m_2(r)P_2\left(\cos(\theta)\right)\Big]}{r - 2m(r)}\right\}dr^2 \\ & + r^2\Big\{1 + 2\left[v_2(r) - h_2(r)\right]P_2\left(\cos(\theta)\right)\Big\}\Big\{d\theta^2 + \sin^2(\theta)\left[d\phi^2 - \omega(r)dt\right]^2\Big\} + \mathcal{O}\left(\Omega^3\right). \end{aligned} \quad (5.42)$$

In the above equation the functions $m(r)$ and $\Phi(r)$ are the same of the non-rotating case, Ω is an uniform angular velocity, ω is the dragging potential, $h_0(r)$ and $m_0(r)$ are associated with monopole deformations and $h_2(r)$, $m_2(r)$, $v_2(r)$ describe the radial dependence of the quadrupole deformations. The quantity $P_2(\cos(\theta))$ is the second order Legendre polynomial, defined by [21]

$$P_2(x) = \frac{3x^2 - 1}{2}. \quad (5.43)$$

Regarding the classical source, the energy density and the pressure can be written as [72]

$$p(r,\theta) = p(r) + (\varepsilon + p)\left[p_0(r) + p_2(2)P_2(\cos(\theta))\right] + \mathcal{O}\left(\Omega^4\right); \quad (5.44)$$

$$\varepsilon(r,\theta) = \varepsilon(r) + (\varepsilon + p)\frac{d\varepsilon}{dp}\left[p_0(r) + p_2(r)P_2(\cos(\theta))\right] + \mathcal{O}\left(\Omega^4\right). \quad (5.45)$$

Unfortunately, the closed tensorial expression used so far for the expectation value of the stress-energy tensor to include backreaction is deeply attached to the spherical symmetry and an equivalent expression suitable for the Hartle-Thorne approximation can be incredibly hard to find. Anticipating future problems it may be useful to develop the semiclassical effects for this purpose under an alternative formalism. An interesting choice is to consider an algebraic extension of general relativity using pseudo-complex numbers, defined by [77]

$$\mathbb{P} = \{X = x_1 + I\,x_2;\ x_1, x_2 \in \mathbb{R},\ I \notin \mathbb{R};\ I^2 = 1\}, \quad (5.46)$$

and the subsequent theory is called pseudo-complex general relativity. It can be shown that this algebraic extension demands a modified variational principle that translates into an extra source term in the field equations, which may be physically interpreted as the average contribution of the quantum vacuum. According to the authors: *the main point is that pseudo-complex general relativity predicts that mass not only curves the space but also changes the vacuum structure of the space itself.* The similarity with semiclassical gravity is evident. The advantage of this framework is that the Hartle-Thorne solution can be obtained straightforwardly [179]. So a major simplification of the rotational problem can be achieved via a direct mathematical connection between semiclassical gravity and the pseudo-complex general relativity.

Another necessary improvement is to expand our analysis to other types of horizonless ultracompact objects. Another two candidates can be found in Rubio's original work about stellar equilibrium in semiclassical gravity. The exact solutions developed apparently combine aspects of the black star (stars with an interior made of extremely dense matter supported by quantum vacuum polarization) and gravastar (empty interior with a large vacuum polarization) proposals [39, 27, 180, 181, 67], being therefore natural candidates for future studies.

Chapter 6

Concluding Remarks

In this thesis, the possibility of achieving ultracompact configurations was investigated by analyzing hydrostatic equilibrium as portrayed in semiclassical gravity. This program is ultimately related with one of the greatest open problems in gravitational physics, namely, the outcome of extreme gravitational collapse, as well as possible deviations from general relativity, a theory that has remained unmodified in more than a century. Central in this discussion is the scope of the gravitational influence on quantum phenomena. The generalization studied here is in some sense in tune with Einstein's doubts about the reality of his field equations in the face of quantum physics, particularly the right hand side (a phenomenological representation of matter), while he believed that the left hand side (obtained from first principles using geometrical quantities) contained a deeper truth. To express this contrast he even used to say that the first was made of low grade wood and the second of fine marble.

Quantum effects are often quite subtle and it is commonly assumed that in gravity they only play relevant roles at the Planck scale, but current challenges presented to the theory of quantum effects in black holes, as well as cosmological spacetimes, suggest otherwise [67]. In particular, there are at least two macroscopic systems where quantum effects may be crucial towards its understanding, cosmological dark energy and the gravitational collapse, which are the main obstacles to reconcile general relativity with quantum mechanics on macroscopic scales [67]. This work adds to many efforts in *awakening the vacuum* to investigate the fate of some relativistic stars [29, 39, 77, 116, 143, 149]. This fate crucially depends on how the vacuum backreacts on spacetime. With this in mind, the study was developed using the phenomena of quantum vacuum polarization as a new type of stabilizing effect, analogous to the degeneracy pressure, thus imposing quantum corrections to general relativity. This framework was applied in equilibrium solutions with interesting properties situated between regular neutron stars and black holes.

Three distinct models were proposed in Chapter 5. The results presented support that, in general, semiclassical solutions have higher compactness, which can be seen as a measure of the strength of its gravity. All discussed models have horizonless ultracompact objects as a viable byproduct. The first model was a simple evaluation of the LinEos under semiclassical gravity, in which the range of ultracompact solutions was substantially expanded when compared with general relativity. The second model took advantage that in semiclassical gravity there are two solutions for the pressure gradient and these equations were combined to study a CFL strange star with an extra component, described by a negative pressure fluid, carrying the semiclassical effects. Some solutions were not only ultracompact, but also able to fit some up-to-date observational data. The last model has shown that it is also possible to explore more general sources, being a preliminary attempt to produce anisotropic solutions. Such solutions may sound too exotic, with no place in real astrophysical phenomena. It is interesting to remember that until the 1960s (with the discovery of quasars and pulsars) this was the general view regarding black holes and neutron stars [182]. In face of the results obtained it is legitimate to pursue a more detailed study of ultracompact stars in semiclassical gravity.

Gravitational waves are providing a new window to observe the Universe, particularly in the strong gravity regime. As pointed out in the Introduction, discoveries like the one recently announced by the LIGO collaboration of an object inside the mass gap (with $\approx 2.6 M_{\odot}$) are very stimulating [51]. Although it is usually assumed that the object must be the lightest known black hole or the heaviest known neutron star, it could as well be a new kind of compact object since there are no shortage of theoretical proposals in the field. Whether or not ultracompact stars or any other theoretical proposal actually exist in nature is an issue to be settled ultimately by observation. Even so, the mere mathematical possibility of such interesting objects is in itself fascinating. Besides that, models are getting refined as our knowledge about the details regarding the equation of state for matter at the relevant pressures and densities, as well as the relevance of other quantum effects, improves. All things considered, the future seems promising towards new observational evidence able to elucidate the influence of gravity on quantum phenomena, revolutionizing the general view of such effects.

Appendix A

Schwarzschild Stars

A.1 Derivation of the Tolman-Oppenheimer-Volkoff Equations

This appendix is devoted to the derivation of the general relativistic stellar structure equations. The analysis is valid for static and spherically symmetric stars. Under these assumptions the spacetime can be expressed as [10]

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{A.1}$$

a line element[71] with just two unknown radial functions, $\Phi(r)$ and $\lambda(r)$.

Due to the highly symmetric character of the Schwarzschild spacetime, the calculations will be relative to the local orthonormal one-form basis, which is naturally attached to the metric element

$$\Theta^0 = e^{\Phi(r)}dt, \quad \Theta^1 = e^{\lambda(r)}dr, \tag{A.2}$$

$$\Theta^2 = rd\theta, \qquad \Theta^3 = r\sin\theta d\phi. \tag{A.3}$$

Since the basis is orthonormal, the associated connection forms satisfy [10]

$$\omega_{ab} + \omega_{ba} = 0, \quad \omega_{ab} = \eta_{ac}\omega^c_b, \quad a, b = 0, 1, 2, 3. \tag{A.4}$$

The external derivatives are given by

$$d\Theta^0 = \Phi' e^{\Phi} dr \wedge dt, \tag{A.5}$$

$$d\Theta^1 = 0, \tag{A.6}$$

[71] These coordinates are called Schwarzschild coordinates (or curvature coordinates), where the basic idea in a nutshell is: (Schwarzschild r-coordinate)=(proper circumference)/2π [54].

$$d\Theta^2 = dr \wedge d\theta, \tag{A.7}$$

$$d\Theta^3 = \sin\theta dr \wedge d\phi + r\cos\theta d\theta \wedge d\phi. \tag{A.8}$$

By expressing the above equations in terms of the orthonormal frames we obtain

$$d\Theta^0 = \Phi' e^{-\lambda}\Theta^1 \wedge \Theta^0, \tag{A.9}$$

$$d\Theta^1 = 0, \tag{A.10}$$

$$d\Theta^2 = r^{-1}e^{-\lambda}\Theta^1 \wedge \Theta^2, \tag{A.11}$$

$$d\Theta^3 = r^{-1}\left[e^{-\lambda}\Theta^1 \wedge \Theta^3 + \cot\theta\Theta^2 \wedge \Theta^3\right]. \tag{A.12}$$

Comparing this with the first Cartan structure equation

$$d\Theta^a = -\omega^a_b \wedge \Theta^b, \tag{A.13}$$

it is immediately to see that

$$\omega^0_2 = \omega^2_0 = \omega^0_3 = \omega^3_0 = 0. \tag{A.14}$$

The other connection coefficients are

$$d\Theta^0 = -\omega^0_1 \wedge \Theta^1 \implies \tag{A.15}$$

$$-\Phi' e^{-\lambda}\Theta^0 \wedge \Theta^1 = -\omega^0_1 \wedge \Theta^1 \implies \tag{A.16}$$

$$\omega^0_1 = \Phi' e^{-\lambda}\Theta^0; \tag{A.17}$$

$$d\Theta^2 = -\omega^2_1 \wedge \Theta^1 \implies \tag{A.18}$$

$$-r^{-1}e^{-\lambda}\Theta^2 \wedge \Theta^1 = -\omega^2_1 \wedge \Theta^1 \implies \tag{A.19}$$

$$\omega^2_1 = r^{-1}e^{-\lambda}\Theta^2; \tag{A.20}$$

$$d\Theta^3 = -\omega^3_1 \wedge \Theta^1 - \omega^3_2 \wedge \Theta^2 \implies \tag{A.21}$$

$$-r^{-1}\left[e^{-\lambda}\Theta^3 \wedge \Theta^1 + \cot\theta\Theta^3 \wedge \Theta^2\right] = -\omega^3_1 \wedge \Theta^1 - \omega^3_2 \wedge \Theta^2 \implies \tag{A.22}$$

$$\omega^3_1 = r^{-1}e^{-\lambda}\Theta^3 \implies \tag{A.23}$$

$$\omega^3_2 = -r^{-1}\cot\theta\Theta^3. \tag{A.24}$$

and (A.4) yields

$$\omega^0_1 = \omega^1_0;\;\; \omega^1_2 = -\omega^2_1;\;\; \omega^1_3 = -\omega^3_1;\;\; \omega^2_3 = -\omega^3_2. \tag{A.25}$$

From the second Cartan structure equation we obtain the curvature two-form ${\Omega^a}_b$, which is also an element of the Lie algebra of the Lorentz group [10]. It follows that:

$$\Omega^0_1 = d\omega^0_1 \tag{A.26}$$

$$= \left(\Phi'' e^{-\lambda} - \lambda'\Phi' e^{-\lambda}\right) dr \wedge \Theta^0 + \Phi' e^{-\lambda} d\Theta^0 \tag{A.27}$$

$$= \left(\Phi'' e^{-\lambda} - \lambda'\Phi' e^{-\lambda}\right) e^{-\lambda}\Theta^1 \wedge \Theta^0 + \left(\Phi' e^{-\lambda}\right)^2 \Theta^1 \wedge \Theta^0 \tag{A.28}$$

$$= -e^{-2\lambda}\left(\Phi'' - \lambda'\Phi' + \Phi'^2\right)\Theta^0 \wedge \Theta^1; \tag{A.29}$$

$$\Omega^0_2 = \omega^0_1 \wedge \omega^1_2 \tag{A.30}$$

$$= -e^{-2\lambda}\frac{\Phi'}{r}\Theta^0 \wedge \Theta^2; \tag{A.31}$$

$$\Omega^0_3 = \omega^0_1 \wedge \omega^1_3 \tag{A.32}$$

$$= -e^{-2\lambda}\frac{\Phi'}{r}\Theta^0 \wedge \Theta^3; \tag{A.33}$$

$$\Omega^1_2 = d\omega^1_2 \tag{A.34}$$

$$= \left(r^{-2}e^{-\lambda} + r^{-1}\lambda' e^{-\lambda}\right) dr \wedge \Theta^2 - r^{-1}e^{-\lambda}d\Theta^2 \tag{A.35}$$

$$= e^{-2\lambda}\left(r^{-2} + r^{-1}\lambda'\right)\Theta^1 \wedge \Theta^2 - r^{-2}e^{-2\lambda}\Theta^1 \wedge \Theta^2 \tag{A.36}$$

$$= e^{-2\lambda}\left(r^{-1}\lambda'\right)\Theta^1 \wedge \Theta^2; \tag{A.37}$$

$$\Omega^1_3 = d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 \tag{A.38}$$

$$= e^{-2\lambda}\left(r^{-1}\lambda'\right)\Theta^1 \wedge \Theta^3 - r^{-2}e^{-\lambda}\cot\theta\Theta^2 \wedge \Theta^3 + r^{-2}e^{-\lambda}\cot\theta\Theta^2 \wedge \Theta^3 \tag{A.39}$$

$$= e^{-2\lambda}\left(r^{-1}\lambda'\right)\Theta^1 \wedge \Theta^3; \tag{A.40}$$

$$\Omega^2_3 = d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 \tag{A.41}$$

$$= \left(e^{-\lambda}r^{-2}\cot\theta - e^{-\lambda}r^{-2}\cot\theta\right)\Theta^1 \wedge \Theta^3 + r^{-2}\left(\frac{1}{\sin^2\theta} - \cot^2\theta + e^{-2\lambda}\right)\Theta^2 \wedge \Theta^3 \tag{A.42}$$

$$= r^{-2}\left(1 - e^{-2\lambda}\right)\Theta^2 \wedge \Theta^3. \tag{A.43}$$

Having the above results it is straightforward to obtain the components of the Riemann tensor using

$$\Omega^a{}_b = \frac{1}{2}\mathcal{R}^a{}_{bcd}\Theta^c \wedge \Theta^d. \tag{A.44}$$

It follows that

$$\mathcal{R}^0{}_{101} = -\mathcal{R}^1{}_{010} = -e^{-2\lambda}\left(\Phi'' - \lambda'\Phi' + \Phi'^2\right); \tag{A.45}$$

$$\mathcal{R}^0{}_{202} = -\mathcal{R}^2{}_{020} = -e^{-2\lambda}\frac{\Phi'}{r}; \tag{A.46}$$

$$\mathcal{R}^0{}_{303} = -\mathcal{R}^3{}_{030} = -e^{-2\lambda}\frac{\Phi'}{r}; \tag{A.47}$$

$$\mathcal{R}^1{}_{212} = \mathcal{R}^2{}_{121} = e^{-2\lambda}\left(r^{-1}\lambda'\right); \tag{A.48}$$

$$\mathcal{R}^1{}_{313} = \mathcal{R}^3{}_{131} = e^{-2\lambda}\left(r^{-1}\lambda'\right); \tag{A.49}$$

$$\mathcal{R}^2{}_{323} = \mathcal{R}^3{}_{232} = r^{-2}\left(1 - e^{-2\lambda}\right). \tag{A.50}$$

The components of the Ricci tensor can be obtained via $\mathcal{R}_{bd} = \mathcal{R}^a{}_{bad}$

$$\mathcal{R}_{00} = \mathcal{R}^1{}_{010} + \mathcal{R}^2{}_{020} + \mathcal{R}^3{}_{030} = e^{-2\lambda}\left(\Phi'' + \Phi'\left(\frac{2}{r} - \lambda'\right) + \Phi'^2\right); \tag{A.51}$$

$$\mathcal{R}_{11} = \mathcal{R}^0{}_{101} + \mathcal{R}^2{}_{121} + \mathcal{R}^3{}_{131} = -e^{-2\lambda}\left(\Phi'' - \lambda'\left(\frac{2}{r} + \Phi'\right) + \Phi'^2\right); \tag{A.52}$$

$$\mathcal{R}_{22} = \mathcal{R}^0{}_{202} + \mathcal{R}^1{}_{212} + \mathcal{R}^3{}_{232} = -\frac{e^{-2\lambda}}{r}\left(\Phi' - \lambda' + \frac{1}{r}\right) + \frac{1}{r^2}; \tag{A.53}$$

$$\mathcal{R}_{33} = \mathcal{R}^0{}_{303} + \mathcal{R}^1{}_{313} + \mathcal{R}^2{}_{323} = -\frac{e^{-2\lambda}}{r}\left(\Phi' - \lambda' + \frac{1}{r}\right) + \frac{1}{r^2}. \tag{A.54}$$

One further contraction gives the scalar of curvature

$$\mathcal{R} = -2e^{-2\lambda}\left(\Phi'' - \lambda'\Phi' + \Phi'^2 + \frac{2}{r}\left(\Phi' - \lambda'\right) + \frac{1}{r^2}\right) + \frac{2}{r^2}. \tag{A.55}$$

Now the components of the Einstein tensor can now be easily calculated using

$$G_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\eta_{ab}\mathcal{R}. \tag{A.56}$$

This yields the following non-zero components

$$G^0{}_0 = e^{-2\lambda}\left(\frac{1}{r^2} - 2\frac{\lambda'}{r}\right) - \frac{1}{r^2}; \tag{A.57}$$

$$G^1{}_1 = e^{-2\lambda}\left(\frac{1}{r^2} + 2\frac{\Phi'}{r}\right) - \frac{1}{r^2}; \tag{A.58}$$

$$G^2{}_2 = G^3{}_3 = e^{-2\lambda}\left(\Phi'' + \Phi'^2 - \lambda'\Phi' + \frac{1}{r}\left(\Phi' - \lambda'\right)\right). \tag{A.59}$$

As pointed out in Chapter 2, to high precision the matter inside the star can be approximated by a perfect fluid. The static condition implies that [54]:

$$u^r = u^\theta = u^\phi = 0. \tag{A.60}$$

The energy-momentum components in the fluid's orthonormal frame are given by

$$T^a{}_b = diag\{-\varepsilon, p, p, p\}. \tag{A.61}$$

Therefore the Einstein field equations

$$G^a{}_b = 8\pi T^a{}_b, \tag{A.62}$$

assume the explicit form

$$e^{-2\lambda}\left(\frac{1}{r^2} - 2\frac{\lambda'}{r}\right) - \frac{1}{r^2} = -8\pi\varepsilon; \tag{A.63}$$

$$e^{-2\lambda}\left(\frac{1}{r^2} + 2\frac{\Phi'}{r}\right) - \frac{1}{r^2} = 8\pi p; \tag{A.64}$$

$$e^{-2\lambda}\left(\Phi'' + \Phi'^2 - \lambda'\Phi' + \frac{1}{r}\left(\Phi' - \lambda'\right)\right) = 8\pi p. \tag{A.65}$$

Observe that equation (A.63) can be put in the form

$$r^{-2}\frac{d}{dr}\left[r\left(1 - e^{-2\lambda}\right)\right] = 8\pi\varepsilon. \tag{A.66}$$

It is useful to define the quantity

$$2m(r) := r\left(1 - e^{-2\lambda}\right), \quad e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}, \tag{A.67}$$

so (A.66) can be rewritten as

$$\frac{2}{r^2}m' = 8\pi\varepsilon. \tag{A.68}$$

Integrating the above equation gives

$$m(r) = \int_0^r 4\pi\varepsilon r^2 dr + m(0). \tag{A.69}$$

Since it is desirable to have a smooth space geometry at the origin we assign a zero value to the constant of integration[72]. The quantity $m(r)$ represents the "total mass-energy inside the radius r"[73].

Replacing $e^{-2\lambda}$ by $\left(1 - \frac{2m}{r}\right)$ and solving for Φ' one obtains

$$\Phi' = \frac{m + 4\pi p r^3}{r\,(r - 2m)}. \tag{A.70}$$

It is useful to define some auxiliary quantities based on the Ricci tensor in order to obtain the hydrostatic equilibrium equation:

$$\alpha_0 = \Phi'' + 2\frac{\Phi'}{r} - \lambda'\Phi' + \Phi'^2; \tag{A.71}$$

$$\alpha_1 = \Phi'' - 2\frac{\lambda'}{r} - \lambda'\Phi' + \Phi'^2; \tag{A.72}$$

$$\alpha_2 = \frac{e^{-2\lambda}}{r}\left(\Phi' - \lambda' + \frac{1}{r}\right) - \frac{1}{r^2}. \tag{A.73}$$

The components of the Einstein tensor can be rewritten as:

$$G^0{}_0 = -\frac{1}{2}\left(e^{-2\lambda}\alpha_0 - e^{-2\lambda}\alpha_1 - 2\alpha_2\right); \tag{A.74}$$

$$G^1{}_1 = \frac{1}{2}\left(e^{-2\lambda}\alpha_0 - e^{-2\lambda}\alpha_1 + 2\alpha_2\right); \tag{A.75}$$

$$G^2{}_2 = \frac{1}{2}\left(e^{-2\lambda}\alpha_0 + e^{-2\lambda}\alpha_1\right). \tag{A.76}$$

A few manipulations are necessary. First, observe that

$$\alpha_0 - \alpha_1 = \frac{2}{r}\left(\Phi' + \lambda'\right) \implies \Phi' + \lambda' = \frac{r}{2}\left(\alpha_0 - \alpha_1\right). \tag{A.77}$$

[72] A non-zero value produces a geometry with a singularity at the origin and there is no local Lorentz frame at $r = 0$ [54].

[73] For a Newtonian star $m(r)$ is simply the "the mass inside radius r", but for a relativistic star $m(r)$ splits into a combination of three factors: rest-mass energy, internal energy and gravitational potential energy [54].

From the above equation it is immediately to see that

$$\Phi'' + \lambda'' = \frac{1}{2}\left[(\alpha_0 - \alpha_1) + r\left(\alpha_0' - \alpha_1'\right)\right]. \tag{A.78}$$

Therefore α_1 can be rewritten as

$$\alpha_1 = \frac{1}{2}\left[(\alpha_0 - \alpha_1) + r\left(\alpha_0' - \alpha_1'\right)\right] - \lambda'' - 2\frac{\lambda'}{r} - \lambda'\left[\frac{r}{2}(\alpha_0 - \alpha_1) - \lambda'\right] + \left[\frac{r}{2}(\alpha_0 - \alpha_1) - \lambda'\right]^2 \tag{A.79}$$

$$= \left[\frac{\alpha_0 - \alpha_1}{2} + \frac{r}{2}\left(\alpha_0' - \alpha_1'\right) - \frac{3}{2}\lambda' r(\alpha_0 - \alpha_1) + \frac{r^2}{4}(\alpha_0 - \alpha_1)^2\right] - \left(\lambda'' - 2\lambda'^2 + \frac{2}{r}\lambda'\right). \tag{A.80}$$

Now observe that

$$\frac{e^{2\lambda}}{r}\left(re^{2\lambda}\right)'' = -2\left(\lambda'' + \frac{2}{r}\lambda' - 2\lambda'^2\right), \tag{A.81}$$

but since

$$\left(re^{2\lambda}\right)' = e^{-2\lambda}\left(1 - 2\lambda' r\right) \tag{A.82}$$

$$= r^2\alpha_2 + 1 - \frac{r^2}{2}e^{-2\lambda}(\alpha_0 - \alpha_1), \tag{A.83}$$

it is also true that

$$\frac{e^{2\lambda}}{r}\left(re^{2\lambda}\right)'' = 2e^{2\lambda}\alpha_2 + re^{2\lambda}\alpha_2' - (\alpha_0 - \alpha_1) + \lambda' r(\alpha_0 - \alpha_1) - \frac{r}{2}\left(\alpha_0' - \alpha_1'\right). \tag{A.84}$$

Therefore

$$-\left(\lambda'' + \frac{2}{r}\lambda' - 2\lambda'^2\right) = e^{2\lambda}\alpha_2 + \frac{r}{2}e^{2\lambda}\alpha_2' - \frac{(\alpha_0 - \alpha_1)}{2} + \frac{r}{2}\lambda'(\alpha_0 - \alpha_1) - \frac{r}{4}\left(\alpha_0' - \alpha_1'\right). \tag{A.85}$$

So equation (A.80) assumes the form:

$$\alpha_1 = \frac{r^2}{4}(\alpha_0 - \alpha_1)^2 + e^{2\lambda}\alpha_2 - r\lambda'(\alpha_0 - \alpha_1) + \frac{r}{4}\left(\alpha_0' - \alpha_1'\right) + \frac{r}{2}e^{2\lambda}\alpha_2'. \tag{A.86}$$

The Einstein field equations implies the following relations

$$(\alpha_0 - \alpha_1) = 8\pi e^{2\lambda}(p + \varepsilon); \tag{A.87}$$

$$\left(\alpha_0' - \alpha_1'\right) = 16\pi\lambda' e^{2\lambda}(p + \varepsilon) + 8\pi e^{2\lambda}\left(p' + \varepsilon'\right); \tag{A.88}$$

$$\alpha_2 = 4\pi(p - \varepsilon); \tag{A.89}$$

$$\alpha_2' = 4\pi\left(p' - \varepsilon'\right); \tag{A.90}$$

$$\alpha_1 = 4\pi e^{2\lambda}(p - \varepsilon). \tag{A.91}$$

Substituting the above results in (A.86) we obtain

$$\begin{aligned} 4\pi e^{2\lambda}(p - \varepsilon) =& \frac{r^2}{4}64\pi^2 e^{4\lambda}(p + \varepsilon)^2 + 4\pi e^{2\lambda}(p - \varepsilon) - 8\pi\lambda' re^{2\lambda}(p + \varepsilon) \\ &+ \frac{r}{4}\left(16\pi\lambda' e^{2\lambda}(p + \varepsilon) + 8\pi e^{2\lambda}\left(p' + \varepsilon'\right)\right) + \frac{r}{2}e^{2\lambda}4\pi\left(p' - \varepsilon'\right). \end{aligned}$$

The above equation can be reduced to

$$-p' = 4\pi r e^{2\lambda} \left(p+\varepsilon\right)^2 - \lambda' \left(p+\varepsilon\right). \tag{A.92}$$

Considering that

$$e^{2\lambda} = \left(1-\frac{2m}{r}\right)^{-1} \implies \lambda' = \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{m'r-m}{r^2}\right), \tag{A.93}$$

and substituting in (A.92) the above relations results

$$\frac{dp}{dr} = -\frac{\left(\varepsilon+p\right)\left(m+4\pi r^3 p\right)}{r\left(r-2m\right)} = -\left(\varepsilon+p\right)\frac{d\Phi}{dr} \tag{A.94}$$

which is the general relativistic hydrostatic equilibrium equation. The above equation can be easily rearranged to

$$\frac{dp(r)}{dr} = -\frac{m(r)\varepsilon}{r^2}\left[1+\frac{p(r)}{\varepsilon(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{m(r)}\right]\left[1-\frac{2m(r)}{r}\right]^{-1} \tag{A.95}$$

as presented in Chapter 2.

Regarding the behavior of the mass function and pressure gradient when $r \to 0$, observe that their expressions can be expanded as [183]

$$m \to \frac{4\pi}{r}\varepsilon_0 r^3 \tag{A.96}$$

$$\frac{dp}{dr} \to -\frac{4\pi}{3}\varepsilon_0^2 r\left(1+\frac{p_0}{\varepsilon_0}\right)\left(1+\frac{3p_0}{\varepsilon_0}\right) \tag{A.97}$$

Since both expressions are proportional to r they vanish at the origin.

For a given equation of state, responsible for describing the quantum physics of matter (i.e. the non-gravitational part), the hydrostatic equilibrium equation can be integrated (usually numerically) from the origin with initial conditions and an arbitrary value for the central density (or pressure), until the pressure vanishes at some radius R [23]. Regarding the matter properties, the TOV equations depend only on bulk quantities, namely the energy density and pressure. Other properties like the number density or the composition of matter are not relevant for modelling Schwarzschild stars in general relativity [23]. It is also important to emphasize that to each possible equation of state there is a unique family of stars parametrized by the central density, in other words, a sequence of stellar models is obtained [10].

A.2 Exterior Solution

For radii exceeding the mass distribution, $r \geqslant R$, the pressure vanishes and $m(r) = M$. It follows from applying these relations on the results obtained on the previous section that

$$e^{-2\lambda} = 1-\frac{2M}{r}; \tag{A.98}$$

$$\Phi' = \frac{M}{r^2\left(1-\frac{2M}{r}\right)}. \tag{A.99}$$

It is reasonable to impose, since the solutions represent bounded systems, that the spacetime must be asymptotically flat. This fact can be mathematically expressed by the asymptotic regularity conditions [69]

$$\lim_{r\to\infty}\lambda(r) = \lim_{r\to\infty}\Phi(r) = 0. \tag{A.100}$$

The above equation implies that $e^{2\Phi} \to 1$ when $r \to \infty$. With this in mind, equation (A.99) can be integrated, which yields

$$e^{2\Phi} = 1 - \frac{2M}{r}. \tag{A.101}$$

This is the famous Schwarzschild solution

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\phi^2\right), \tag{A.102}$$

uniquely determined by the central mass of the object. The quantity $R_S = 2M$ is called the Schwarzschild radius. In terms of the Schwarzschild radius the above line element can be rewritten as

$$ds^2 = -\left(1-\frac{R_S}{r}\right)dt^2 + \left(1-\frac{R_S}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\phi^2\right), \tag{A.103}$$

as presented in Chapter 3.

A.3 Anisotropic Case

As a matter of completeness, the perfect fluid solution for hydrostatic equilibrium can be easily generalized for the anisotropic case. As presented in Chapter 2, in this case the source acquires the form

$$T_{\mu\nu} = (\varepsilon + p_\perp)\, u_\mu u_\nu + p_\perp g_{\mu\nu} + (p_r - p_\perp)\, k_\mu k_\nu, \tag{A.104}$$

The modifications are restricted to the source, so the components of the field equations are

$$e^{-2\lambda}\left(\frac{1}{r^2} - 2\frac{\lambda'}{r}\right) - \frac{1}{r^2} = -8\pi\varepsilon; \tag{A.105}$$

$$e^{-2\lambda}\left(\frac{1}{r^2} + 2\frac{\Phi'}{r}\right) - \frac{1}{r^2} = 8\pi p_r; \tag{A.106}$$

$$e^{-2\lambda}\left(\Phi'' + \Phi'^2 - \lambda'\Phi' + \frac{1}{r}\left(\Phi' - \lambda'\right)\right) = 8\pi p_\perp. \tag{A.107}$$

For an isotropic fluid it was obtained that $p' + \Phi' (\varepsilon + p) = 0$. For an anisotropic source this equation is no longer valid, but its counterpart can be easily computed. First observe that:

$$8\pi p_r' = -2\lambda' e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2\Phi'}{r}\right) + e^{-2\lambda}\left[-\frac{2}{r^3} + 2\frac{\Phi'' r - \Phi'}{r^2}\right] + \frac{2}{r^3} \tag{A.108}$$

$$= e^{-2\lambda}\left(-2\frac{\lambda'}{r^2} - 4\frac{\Phi'\lambda'}{r} - \frac{2}{r^3} + 2\frac{\phi''}{r} - 2\frac{\Phi'}{r^2}\right) + \frac{2}{r^3}. \tag{A.109}$$

From the components of the field equations it is also possible to express the relation

$$8\pi\Phi' (\varepsilon + p_r) = e^{-2\lambda}\left(2\frac{\Phi'\lambda'}{r} + 2\frac{\Phi'^2}{r}\right). \tag{A.110}$$

So

$$8\pi\left[p_r' + \Phi' (\varepsilon + p_r)\right] = e^{-2\lambda}\left(-2\frac{\lambda'}{r^2} - 2\frac{\Phi'\lambda'}{r} - \frac{2}{r^3} + 2\frac{\Phi''}{r} - 2\frac{\Phi'}{r^2} + 2\frac{\Phi'^2}{r}\right) + \frac{2}{r^3}. \tag{A.111}$$

Now observe that

$$8\pi\left[\frac{2}{r}(p_\perp - p_r)\right] = e^{-2\lambda}\left(2\frac{\Phi''}{r} + 2\frac{\Phi'^2}{r} - 2\frac{\lambda'\Phi'}{r} - \frac{2\Phi'}{r^2} - \frac{2\lambda'}{r^2} - \frac{2}{r^3}\right) + \frac{2}{r^3}. \tag{A.112}$$

Noticing that the two last results are equal it follows that

$$p_r' = -(\varepsilon + p_r)\,\Phi' + \frac{2\Delta}{r}, \tag{A.113}$$

Finally, observing that Φ' is computed by combining the (t,t) and (r,r) components of the field equations, which preserve the same form of the perfect fluid case, it follows that:

$$p_r' = -\frac{(\varepsilon + p_r)\left(m + 4\pi r^3 p_r\right)}{r(r-2m)} + \frac{2(p_\perp - p_r)}{r} \tag{A.114}$$

Obviously the isotropic case is recovered when $p_\perp = p_r = p$. The radius of the star is determined by the condition $p_r(R) = 0$. It is not necessary to have a vanishing tangential pressure [26]. Moreover, the last term in the above equation can be seen as representing a "force", which is directed outward when $(p_\perp - p_r) > 0$ and inward when $(p_\perp - p_r) < 0$. Therefore one should expect more massive configurations in the former case and less massive in the latter [184]. Besides that, the introduction of the tangential pressure requires the specification of an additional equation of state, such as $p_\perp(\varepsilon)$ [26]. Since a detailed microscopic description of anisotropy is not available, several authors often use ansatzes to model scenarios with different principal stresses. Usually the presence of anisotropy affects the compactness and the surface redshift of the configurations [26].

Appendix B

Spacetime of a Relativistic Star

In this appendix a heuristic demonstration of the spacetime of a static and spherically symmetric relativistic star, sufficient for our purposes, is presented[74]. Consider, as a starting point, the spherically symmetric line element in special relativity

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{B.1}$$

The idea is to modify the above expression to allow curvature induced by the gravitational influence of the star, while preserving spherical symmetry. The simplest proposal is to vary the metric components that are non-zero already in (B.1), that is,

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{B.2}$$

where θ is the polar angle, ϕ is the azimuthal angle and the radial coordinate r is defined such that the circumference of a circle about the origin at that space location is $2\pi r$ [79]. Note that because of the underlying symmetries the functions Φ, λ and R depend only on r, measured from the star's origin[75] [21]. One could object that (B.2) is not the most general metric possible under our assumptions and propose, for example

$$ds^2 = -a^2dt^2 - 2abdrdt + c^2dr + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{B.3}$$

However this expression is not more general in any physical sense. Consider a time coordinate transformation defined by

$$e^{\Phi}dt' = adt + bdr \tag{B.4}$$

[74] For a more rigorous proof see [54, 57].

[75] The static assumption imposes that $\frac{\partial g_{\mu\nu}}{\partial t} = 0$ must be satisfied.

Inserting the above equation in (B.3) yields

$$ds^2 = -a^2\left(\frac{e^{\Phi}dt' - bdr}{a}\right)^2 - 2ab\left(\frac{e^{\Phi}dt' - bdr}{a}\right)dr + c^2dr + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{B.5}$$

$$= -\left(e^{2\Phi}dt'^2 + b^2dr^2 - 2be^{\Phi}dt'dr\right) - 2be^{\Phi}drdt' + 2b^2dr^2 + c^2dr^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{B.6}$$

$$= -e^{2\Phi}dt'^2 + \left(b^2 + c^2\right)dr^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{B.7}$$

Defining $e^{2\lambda} = b^2 + c^2$ allow us to rewrite the line element as

$$ds^2 = -e^{2\Phi}dt'^2 + e^{2\lambda}dr^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{B.8}$$

which apart from prime on the t is the equation (B.2)[76]. Therefore the reason why $g_{rt} = 0$ (B.2) comes from an adequate choice of time coordinate. An advantageous choice can also be made for r coordinate, as long as the spherical symmetry is respected. So it is possible to define

$$r = R(r). \tag{B.9}$$

Inserting this new coordinate in (B.8) and dropping the primes results

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{B.10}$$

which is the line element used to derive the structure equations for the stellar configurations considered throughout this work.

[76] Equation (B.4) is successful in defining a new time coordinate as long as t' is integrable as a differential equation for t'. By choosing the integration factor e^{Φ} to be just $e^{\Phi} = a(r)$ implies that $t' = t + \int \frac{b(r)}{a(r)}dr$ is the integral of (B.4). Therefore the new time coordinate always exists, no matter the forms that the functions a, b, c and R may assume.

Appendix C

Hydrostatic Equilibrium in Semiclassical Gravity

C.1 The Semiclassical TOV Equation

This appendix is dedicated to outline the procedure for obtaining and analyzing the semiclassical structure equations, assuming the results of the Appendix A. The idea is to solve the semiclassical field equations expressed as

$$G_{\mu\nu} = 8\pi \left(T_{\mu\nu} + \hbar N Q_{\mu\nu}\right), \tag{C.1}$$

with respect to the line element presented in Chapter 4, namely

$$ds^2 = ds^2_{(2)} + r^2 d\Omega^2 = g_{ab}(y) dy^a dy^b + r^2(y) d\Omega^2(\theta, \phi). \tag{C.2}$$

Taking advantage that in $(1+1)$ dimensions all metrics are conformal to the flat metric [46], that is

$$g_{\mu\nu} = C(x)\eta_{\mu\nu}. \tag{C.3}$$

The line element can be written in null coordinates as [129]

$$ds^2_{(2)} = -C(u,v) du dv, \tag{C.4}$$

which makes the computations much easier. The Christoffel symbols of the "second kind" are given by

$$\Gamma^{\tau}_{\mu\beta} = \frac{1}{2} g^{\alpha\tau} \left(\frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}}\right). \tag{C.5}$$

Its non-zero components are

$$\Gamma^{u}_{uu} = \frac{1}{2} g^{vu} \left(2 \frac{\partial g_{uv}}{\partial u}\right) \tag{C.6}$$

$$= \frac{1}{C} \partial_u C; \tag{C.7}$$

and

$$\Gamma^{v}_{vv} = \frac{1}{2} g^{uv} \left(2 \frac{\partial g_{uv}}{\partial v} \right) \tag{C.8}$$
$$= \frac{1}{C} \partial_v C. \tag{C.9}$$

From the Christoffel symbols the components of the Ricci tensor can be obtained using

$$\mathcal{R}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta nu}\Gamma^{\beta}_{\mu\alpha}, \tag{C.10}$$

which yields

$$\mathcal{R}_{uv} = \mathcal{R}_{vu} = -\Gamma^{u}_{uu,v} = -\partial_v \left(\frac{1}{C} \partial_u C \right). \tag{C.11}$$

Therefore the scalar of curvature is

$$\mathcal{R} = \mathcal{R}_{\mu\nu} g^{\mu\nu} \tag{C.12}$$
$$= 2\mathcal{R}_{uv} g^{uv} \tag{C.13}$$
$$= -2 g^{\mu\nu} \partial_v \left(\frac{1}{C} \partial_u C \right) \tag{C.14}$$
$$= -2 g^{\mu\nu} \partial_v \partial_u \ln C \tag{C.15}$$
$$= -2 \Box \ln C(u,v). \tag{C.16}$$

As discussed in Chapter 4, the evaluation of the renormalized stress-energy tensor is a crucial problem in semiclassical gravity, which usually can not be obtained analytically. However when the space-time has some degree of symmetry it can be calculated explicitly [129]. Fortunately, considering the s-wave Polyakov approximation under spherical symmetry, the computations are rather simple because the components can be expressed in a tensorial way as [39, 129]:

$$Q_{\mu\nu} = \frac{\delta^a_\mu \delta^b_\nu}{4\pi r^2} Q^{(2)}_{ab}; \tag{C.17}$$
$$Q^{(2)}_{ab} = \frac{1}{48\pi} \left(\frac{1}{2} \mathcal{R}^{(2)} g_{ab} + A_{ab} - \frac{1}{2} g_{ab} A \right). \tag{C.18}$$

In the above expression A_{ab} is given by

$$A_{ab} \coloneqq -4 |\xi| \nabla_a \nabla_b |\xi|^{-1}, \tag{C.19}$$

where $|\xi| = \sqrt{C}$. The components of (C.18) are

$$Q^{(2)}_{uu} = \frac{1}{48\pi} A_{uu} = -\frac{1}{12\pi} C^{\frac{1}{2}} \partial_u^2 C^{-\frac{1}{2}} = \frac{1}{24\pi} \left(-\frac{3}{2C^2} \left(\partial_u C \right)^2 + \frac{1}{C} \partial_u^2 C \right); \tag{C.20}$$

$$Q^{(2)}_{vv} = \frac{1}{48\pi} A_{vv} = -\frac{1}{12\pi} C^{\frac{1}{2}} \partial_v^2 C^{-\frac{1}{2}} = \frac{1}{24\pi} \left(-\frac{3}{2C^2} \left(\partial_v C \right)^2 + \frac{1}{C} \partial_v^2 C \right); \tag{C.21}$$

$$Q^{(2)}_{uv} = Q^{(2)}_{vu} = \frac{1}{96\pi} \mathcal{R}^{(2)} g_{uv} = -\frac{\mathcal{R} C}{96\pi} . \tag{C.22}$$

Now it is useful to rewrite these components in (t, r) coordinates associated with the line element

$$ds^2_{(2)} = -C(r) dt^2 + \frac{dr^2}{1 - 2m(r)/r}, \tag{C.23}$$

which yields [39]

$$Q^{(2)}_{rr} = -\frac{1}{96\pi} \left(\frac{C'}{C} \right)^2; \tag{C.24}$$

$$Q^{(2)}_{tt} = \frac{1}{24\pi} \left[\left(1 - \frac{2m}{r} \right) C'' - C' \left(\frac{m}{r} \right)' - \frac{3}{4} \left(1 - \frac{2m}{r} \right) \frac{C'^2}{C} \right]; \tag{C.25}$$

$$Q^{(2)}_{tr} = Q^{(2)}_{rt} = 0. \tag{C.26}$$

Through these components, it can be checked that the semiclassical source is identically conserved[77] [39].

Having in mind the results presented in Appendix A it is immediate to obtain

$$e^{2\Phi} = C \implies \Phi' = \frac{C'}{2C}; \tag{C.27}$$

$$p' = -\left(\varepsilon + p \right) \Phi'; \tag{C.28}$$

$$p' = -\frac{C'}{2C} \left(\varepsilon + p \right). \tag{C.29}$$

Now, returning to the whole line element (C.2) and taking into account that the Einstein tensor of the semiclassical field equations and that both the classical and semiclassical sources are diagonal, this amounts in principle to five differential equations: the diagonal components (t, t), (r, r), (θ, θ), (ϕ, ϕ) of the semiclassical field equations, plus the equation (C.29) [39]. Nevertheless, as in general relativity, only three of these five equations are independent. So it is sufficient to proceed the equation (C.29) and the (t, t) and (r, r) components, where the latter two are given respectively by [39]:

$$\frac{2m'}{r^2} = 8\pi\varepsilon + \frac{\ell_P^2}{r^2} \left[\left(1 - \frac{2m}{r} \right) \frac{C''}{C} - \frac{C'}{C} \left(\frac{m}{r} \right)' - \frac{3}{4} \left(1 - \frac{2m}{r} \right) \left(\frac{C'}{C} \right)^2 \right]; \tag{C.30}$$

$$\frac{C'}{rC} - \frac{2m}{r^2 \left(r - 2m \right)} = \frac{8\pi p}{1 - \frac{2m}{r}} - \frac{\ell_P^2}{4} \left(\frac{C'}{rC} \right)^2. \tag{C.31}$$

[77] A commentary is useful to avoid possible confusions. The 4-dimensional conservation law for the energy-momentum tensor, $\nabla^\mu \langle \hat{T}_{\mu\nu} \rangle = 0$ can be expressed in terms of the two dimensional energy-momentum tensor as $\nabla^a \langle \hat{T}_{ab} \rangle^{(2)} - \partial_a \langle \hat{T}^\theta{}_\theta \rangle = 0$, which violates in general the naive 2-dimensional conservation law $\nabla^a \langle \hat{T}_{ab} \rangle^{(2)} = 0$ [142].

Using (C.29) the above equation can be rewritten as

$$\frac{1}{r}\left(\frac{-2p'}{\varepsilon+p}\right)-\frac{2m}{r^{2}\left(r-2m\right)}=\frac{8\pi p}{1-\frac{2m}{r}}-\frac{\ell_{P}^{2}}{4r^{2}}\left(\frac{4p'^{2}}{\left(\varepsilon+p\right)^{2}}\right). \tag{C.32}$$

Rearranging this equation it follows

$$p'\left(1-\frac{\ell_{P}^{2}}{2r}\frac{p'}{\varepsilon+p}\right)=-\left(\varepsilon+p\right)\frac{\left(m+4\pi r^{3}p\right)}{r\left(r-2m\right)}, \tag{C.33}$$

which is the semiclassical Tolman-Oppenheimer-Volkoff equation. The right hand side is the same obtained in general relativity, but the left hand side contains a new contribution related to the modifications due to semiclassical effects [39].

Regarding the exterior geometry, since the g_{rr} metric function, as expressed in equation (C.23) is the same found in general relativity, equation (A.98) remains valid in the semiclassical case. The g_{tt} metric function can be analyzed by rewriting the semiclassical TOV equation in terms of $\Phi(r)$. In this case, considering $r \geqslant R$, one is lead to

$$\Phi'\left(1+\frac{\ell_{P}^{2}}{2r}\Phi'\right)=\frac{M}{r\left(r-2M\right)}. \tag{C.34}$$

In the limit $\hbar \to 0$ (reasonable since the quantum stress-energy tensor outside a static star is extremely weak for a distant observer) the above expression recovers equation (A.99), therefore in this case the exterior geometry matches the exterior Schwarzschild solution found in general relativity [142].

C.2 Evaluating Classicality

This subsection is dedicated to what could be seen as the classical counterparts of the structure equations in semiclassical gravity. Such limits are employed in the γ-CFL presented in Section 5.2. Here, it will be useful to restore some constants and work only with $c = 1$. In many approaches to modify general relativity, the gravitational constant G associated with different sources is allowed to differ [168]. With this in mind, here the gravitational constant coming from classical energy-momentum tensor and the one coming from the quantum expectation value the first will be denoted by G and Θ respectively.

The semiclassical hydrostatic equilibrium equation, being quadratic with respect to the pressure gradient, has two roots given by

$$\hbar\Theta p'_{\pm}=\frac{r\left(\varepsilon+p\right)}{N}\left(1\pm\sqrt{1+\frac{G}{r^{3}}\frac{\hbar\Theta N}{6\pi}\frac{m+4\pi r^{3}p}{1-2Gm/r}}\right). \tag{C.35}$$

It is interesting to analyze the classical limit of the semiclassical structure equations. First, for better visualization, consider the p'_- solution expanded in a taylor series as

$$\Theta\hbar p'_- = \frac{1}{2}\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right)\left(\frac{\Theta N}{6\pi r\left(\varepsilon+p\right)}\Xi\right)\hbar + \frac{1}{8}\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right)\left(\frac{\Theta N}{6\pi r\left(\varepsilon+p\right)}\Xi\right)^2\hbar^2 \tag{C.36}$$

$$+\frac{1}{16}\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right)\left(\frac{\Theta N}{6\pi r\left(\varepsilon+p\right)}\Xi\right)^3\hbar^3 + \mathcal{O}\left(\hbar^4\right), \tag{C.37}$$

where, for simplicity, it was defined

$$\Xi(r) \coloneqq -\frac{G}{r}\frac{\left(\varepsilon+p\right)\left(m+4\pi r^3 p\right)}{r-2Gm}. \tag{C.38}$$

Matching both sides in powers of $\hbar$ and taking the limit $\hbar \to 0$ yields [39]

$$p'_- = -\frac{G}{r}\frac{\left(\varepsilon+p\right)\left(m+4\pi r^3 p\right)}{r-2Gm} \tag{C.39}$$

Therefore, at the classical limit the general relativistic picture for hydrostatic equilibrium is recovered [39]. Additionally, since the semiclassical mass function adds a term proportional to $\hbar$ to the expression known from general relativity, it also recovers the classical expression when $\hbar \to 0$.

On the other hand, the $\hbar \to 0$ limit is not suitable to evaluate classicality in the p'_+ solution, in which the semiclassical effects are dominant [39]. In this case, due to the first term in the right hand side, the limit $\hbar \to 0$ is not sufficient to satisfy the conditions demanded by the matching. Nevertheless it would be interesting for astrophysical applications to obtain some expression that operationally resembles a classical limit. As mentioned in Chapter 4, the $N \to \infty$ can be used as an alternative measure of classicality. Let is see how this applies to both solutions for the pressure gradient. For example, looking at the factors multiplying each power of $\hbar$ in the Taylor expansion, it is clear that the (C.39) result could be obtained using the limit $N \to \infty$ with $N\Theta = \text{constant}$ (the semiclassical coupling is rescaled in such a way that $N\Theta$ is finite even when N goes to infinity [42]).

Similar to what was done with the previous root, the Taylor expansion gives for p'_+ can be written as:

$$\begin{aligned}\Theta\hbar p'_+ =& 2\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right) - \frac{1}{2}\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right)\left(\frac{\Theta N}{6\pi r\left(\varepsilon+p\right)}\Xi\right)\hbar \\ &- \frac{1}{8}\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right)\left(\frac{\Theta N}{6\pi r\left(\varepsilon+p\right)}\Xi\right)^2\hbar^2 \\ &- \frac{1}{16}\left(\frac{12\pi r\left(\varepsilon+p\right)}{N}\right)\left(\frac{\Theta N}{6\pi r\left(\varepsilon+p\right)}\Xi\right)^3\hbar^3 + \mathcal{O}\left(\hbar^4\right)\end{aligned} \tag{C.40}$$

Matching both sides and taking the limit $N \to \infty$ with $N\Theta = \text{constant}$ gives

$$p'_+ = \frac{G}{r}\frac{\left(\varepsilon+p\right)\left(m+4\pi r^3 p\right)}{r-2Gm}. \tag{C.41}$$

which is the result obtained in general relativity, except for a sign change. So, a formal approach able to deal with all semiclassical structure equations at once is to take the $N \to \infty$ limit followed by the usual limit $\hbar \to 0$.

C.3 Anisotropic Case

The anisotropic version of equation (C.35) can be easily obtained. Equation (A.113) still applies and can be rewritten as:

$$\Phi' = \frac{1}{\varepsilon + p_r}\left(\frac{2\Delta}{r} - p_r'\right) \tag{C.42}$$

Substituting this into equation (C.31) having in mind equation (C.27) yields

$$\frac{2}{r(\varepsilon + p_r)}\left(\frac{2\Delta}{r} - p_r'\right) - \frac{2m}{r^2(r-2m)} = \frac{8\pi p_r}{1 - \frac{2m}{r}} - \frac{\ell_P^2}{4}\left(\frac{2}{r(\varepsilon + p_r)}\left(\frac{2\Delta}{r} - p_r'\right)\right)^2. \tag{C.43}$$

Solving for p_r'

$$p_r' = \frac{2\Delta}{r} + \frac{r(\varepsilon + p_r)}{\ell_P^2}\left(1 \pm \sqrt{1 + \frac{2\ell_P^2}{r^3}\frac{m + 4\pi r^3 p_r}{1 - 2m/r}}\right) \tag{C.44}$$

which obviously recovers equation (C.35) when $p_r = p_\perp$.

Appendix D

Mathematical Supplementary Material

This appendix is a supplementary material to some mathematical notions exposed throughout the text. A good starting point is the definition of an affine space, the structure that describes the spacetimes in Newtonian physics and special relativity [52].

■ **Definition 1.** *An affine space A is a subspace of a linear space V whose elements may be written in the form $a = k + v_0$, with k in a linear subspace of V and v_0 a fixed point of V.*

Therefore, roughly speaking, it is a vector space V in which the origin (the zero vector) is not fixed, or at least not relevant. In that sense vectors are differences between points in the affine space. Physics relies on affine spaces whenever invariance under translations holds, cases where the true position of the origin is irrelevant. [52].

As mentioned in the main text, in general relativity the general notion of a manifold, not reduced to affine spaces, has to be invoked. A topological manifold (which will be called simply manifold) can be described by the following conditions [60]:

■ **Definition 2.** *A manifold of dimension n (n is an integer equal or greater to one) is a topological space $\mathcal{M}$ obeying the following properties*

1. *$\mathcal{M}$ is a separated space (also called Hausdorff space): Any two distinct points of $\mathcal{M}$ admit disjoint open neighbourhoods.*
2. *$\mathcal{M}$ has a countable base: there exists a countable family $(\mathcal{U}_k)_{k\in\mathbb{N}}$ of open sets of $\mathcal{M}$ such that any open set of $\mathcal{M}$ can be written as the union (possibly infinite) of some members of the above family.*

3. *Around each point of $\mathcal{M}$, there exists a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.*

The first property excludes manifolds with "forks" (it allows to distinguish between two points even after a small perturbation) which are unpleasant from the physics point of view. Property 2 permits to establish a theory of integration on manifolds by excluding "too large" manifolds. Property 3 contains the essence of a manifold. They can be, at least in a domain around each one of its points, approximated by Euclidean spaces. They are, consequently, spaces on which coordinates make sense [52].

As mentioned in Chapter 2, general relativity unites space and time through a manifold $\mathcal{M}$ endowed with a pseudo-Riemannian metric g of Lorentzian signature. So the next natural mathematical object to be discussed is the metric, which can be seen intuitively as the "infinitesimal squared distance " associated with an "infinitesimal displacement" [2]. Mathematically, the notion of an "infinitesimal displacement " can be translated into the concept of a tangent vector, seen as a directional derivative[78] [2]. Since "infinitesimal squared distance" should be quadratic in the displacement, it is reasonable to infer that the metric g should be a linear map from $\mathcal{M}_p \times \mathcal{M}_p$ (where $\mathcal{M}_p$ is the tangent space at point p of a manifold) into numbers, that is, a tensor of type $(0, 2)$. Besides that, the metric is also required to be symmetric and nondegenerate. Symmetric means that for all v_1 and $v_2 \in \mathcal{M}_p$ we have $\mathbf{g}\,(v_1, v_2) = \mathbf{g}\,(v_2, v_1)$. Nondegenerate means that the only case in which $\mathbf{g}\,(v, v_1) = 0$ for all $v \in \mathcal{M}_p$ is the case $v_1 = 0$ [2]. To summarize, a metric g on a manifold $\mathcal{M}$ is a symmetric, nondegenerate tensor field of type $(0, 2)$. Alternatively, a metric can be viewed as a (not necessarily positive definite) inner product on the tangent space at each point.

In a coordinate basis the metric is usually expanded in terms of its components,

$$\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{D.1}$$

Sometimes the quadratic form ds^2 is used in place of g to represent the metric tensor, in which case we write equation as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{D.2}$$

The next fundamental concept is curvature. Intuitively this notion arises mainly from two-dimensional surfaces that are embedded in ordinary Euclidean space, associating curvature with the way it

[78] In non-relativistic physics it is assumed that space has the natural three-dimensional vector space once one has chosen the origin. The rules for adding and scalar multiplying then satisfy the vector space axioms. In special relativity the situation is similar and one still has a natural structure of a four-dimensional vector space. In curved geometries the situation is different and this vector space structure is lost. In that case there is no natural notion to "add " two points on a sphere, resulting in a third point on the sphere. Still, the vector space structure can be recovered in the limit of "infinitesimal displacements" about a point. It is upon the notion of "infinitesimal displacement", or tangent vector that the calculus on manifolds is based [2].

bends in $\mathbb{R}^3$. This *extrinsic* curvature will not be our primary interest here. Rather, the goal is to investigate the curvature of spacetime, which is not naturally embedded (as far as we know) in a higher dimensional space. Thus it is useful to construct an *intrinsic* notion of curvature that can be applied to a manifold without any reference to a higher dimensional space in which it might be embedded [2]. The first step is to discuss the meaning of a derivative when applied to a manifold. Partial derivatives acting on components of tensor fields can not be intrinsically defined on a manifold [3]. An intrinsic derivation is possible by endowing the manifold with a structure called a connection. This operator is called a covariant derivative and usually denoted by ∇. A covariant derivative maps differentiate vector fields into tensor fields obeying to the following rules [3]:

- Linearity: $\nabla(v+w) = \nabla v + \nabla w$
- Leibnitz Rule: $\nabla(fv) = f\nabla v + df \otimes v$

In general relativity the derivative operator is also torsionless, that is, the second derivatives of scalar functions must commute, namely

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f. \tag{D.3}$$

The disagreement between two derivative operators on dual vector fields can be measured through a tensor field C^c_{ab} such that [2]

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C^c_{ab}\omega_c. \tag{D.4}$$

A symmetry property of C^c_{ab} follows from the torsionless condition D.3. Considering $\omega_b = \nabla_b f = \tilde{\nabla}_b f$ yields

$$\nabla_a \nabla_b f = \tilde{\nabla}_a \tilde{\nabla}_b f - C^c_{ab}\nabla_c f. \tag{D.5}$$

Since equation D.3 has to be satisfied for both derivative operators, it follows that

$$C^c_{ab} = C^c_{ba}. \tag{D.6}$$

Obviously the above condition will not hold in general if the torsionless condition is dropped [2]. In Chapter 2 it was discussed that given a metric, there exists a unique torsionless connection which preserves it, that is, $\nabla_a g_{bc} = 0$. In order to see this observe that:

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d_{ab} g_{dc} - C^d_{ac} g_{bd}, \tag{D.7}$$

which implies that

$$C_{cab} + C_{bac} = \tilde{\nabla}_a g_{bc}, \tag{D.8}$$

or equivalently through index substitution

$$C_{cba} + C_{abc} = \tilde{\nabla}_b g_{ac}; \tag{D.9}$$

$$C_{bca} + C_{acb} = \tilde{\nabla}_c g_{ab}. \tag{D.10}$$

By adding equations D.8 and D.9 and then subtracting D.10, alongside with the symmetric property D.6 it is found that

$$2C_{cab} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab}, \tag{D.11}$$

that can be rewritten as

$$C^a_{bc} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} \right), \tag{D.12}$$

which is obviously unique.

Therefore the metric is sufficient to determine the derivative operator [2]. Specifically, in terms of an ordinary derivative operator D.12 can be written as

$$\Gamma^\tau_{\mu\beta} = \frac{1}{2} g^{\alpha\tau} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\beta}}{\partial x^\alpha} \right), \tag{D.13}$$

which is called the Levi-Civita connection. This is the curvature applied in general relativity [52]. Mathematically, curvature signals the non-commutativity of covariant derivatives [3]. In non-infinitesimal terms it indicates the non-identity of a vector and the vector parallelly transport along a closed loop. The definition goes as follows. Consider the commutation of two covariant derivatives applied to some dual vector field v_γ, $\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) v_\gamma$. there exist coefficients $\mathcal{R}_{\alpha\beta\gamma}{}^\delta$ such that for all dual fields v_γ:

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) v_\gamma = \mathcal{R}_{\alpha\beta\gamma}{}^\delta v_\delta, \tag{D.14}$$

which are the components of the Riemann curvature tensor. In a coordinate component method the components can be expressed as [2]

$$\mathcal{R}^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\nu\beta,\mu} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\nu\beta} - \Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\mu\beta}. \tag{D.15}$$

It can be proved that the vanishing of the Riemann tensor on a domain of a manifold implies that the metric is locally flat in this region [3].

Below it is listed the main properties of the Riemann tensor:

1. $\mathcal{R}_{\beta\gamma\delta}{}^\alpha = -\mathcal{R}_{\gamma\beta\delta}{}^\alpha$;
2. $\mathcal{R}_{[\beta\gamma\delta]}{}^\alpha = 0$;
3. For the Levi Civita connection holds: $\mathcal{R}_{\alpha\beta\gamma\delta} = -\mathcal{R}_{\alpha\beta\delta\gamma}$;
4. Bianchi Identity: $\nabla_{[\alpha}\mathcal{R}_{\beta\gamma]\delta}{}^\epsilon = 0$.

In the above equations it was used a common notation for totally symmetric and totally antisymmetric tensors, that is [2]

$$T_{(a_1 \ldots a_l)} = \frac{1}{l!} \sum_{\pi} a_{\pi(1)} \ldots a_{\pi(l)}; \tag{D.16}$$

$$T_{[a_1 \ldots a_l]} = \frac{1}{l!} \sum_{\pi} \delta_{\pi} a_{\pi(1)} \ldots a_{\pi(l)}. \tag{D.17}$$

For example, in the case of a tensor of type $(0, 2)$ the above equations assume the form:

$$T_{(\alpha\beta)} = \frac{1}{2} \left(T_{\alpha\beta} + T_{\beta\alpha} \right); \tag{D.18}$$

$$T_{[\alpha\beta]} = \frac{1}{2} \left(T_{\alpha\beta} - T_{\beta\alpha} \right). \tag{D.19}$$

It follows from properties 1, 2, and 3 that the Riemann tensor also satisfies another symmetry property, namely

$$\mathcal{R}_{\alpha\beta\gamma\delta} = \mathcal{R}_{\gamma\delta\alpha\beta}. \tag{D.20}$$

It is useful to decompose the Riemann tensor into a "trace part " and a "trace-free part ". Note that, by the antisymmetry properties (1) and (3), the trace of the Riemann tensor over its first two or last two indices vanishes [2]. At the other hand, its trace over the second and fourth (or the first and third), indices defines the Ricci tensor, with components:

$$\mathcal{R}_{\alpha\gamma} = \mathcal{R}_{\alpha\beta\gamma}{}^{\beta}. \tag{D.21}$$

By equation (D.20) its components satisfy the symmetry property

$$\mathcal{R}_{\alpha\gamma} = \mathcal{R}_{\gamma\alpha}. \tag{D.22}$$

The scalar curvature, $\mathcal{R}$, is defined as the trace of the Ricci tensor:

$$\mathcal{R}^{\alpha}_{\alpha} = \mathcal{R} \tag{D.23}$$

Another important tensor is the Weyl tensor or conformal tensor. In some sense, it is considered that this tensor embodies the non-Newtonian properties of the gravitational field, in particular its radiation properties. The reason for this is the fact that the equations for massless fields, at least in four-dimensional spacetimes, are conformally invariant [3]. The Weyl tensor is also used in global existence proofs, for instance, the proof of the non-linear stability of the Minkowski spacetime. The Weyl tensor has the same symmetries as the Riemann tensor and they are equal in Ricci-flat space. In addition it also has zero trace. Its components, $C_{\alpha\beta\gamma\delta}$, for manifolds with dimension $n \geqslant 3$ are defined by:

$$\mathcal{R}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{2}{2-n} \left(g_{\alpha[\gamma} \mathcal{R}_{\delta]\beta} - g_{\beta[\gamma} \mathcal{R}_{\delta]\alpha} \right) - \frac{2}{(n-1)(n-2)} \mathcal{R} g_{\alpha[\gamma} g_{\delta]\beta}. \tag{D.24}$$

Now consider the contraction of the Bianchi identity,

$$\nabla_\alpha \mathcal{R}_{\beta\gamma\delta}{}^{\alpha} + \nabla_\beta \mathcal{R}_{\gamma\delta} - \nabla_\gamma \mathcal{R}_{\beta\delta} = 0. \tag{D.25}$$

Raising the index δ with the metric and contracting over β and δ yields

$$\nabla_\alpha \mathcal{R}^\alpha_\gamma + \nabla_\beta \mathcal{R}^\beta_\gamma - \nabla_\gamma \mathcal{R} = 0, \tag{D.26}$$

or,

$$\nabla^\alpha G_{\alpha\beta} = 0, \tag{D.27}$$

where

$$G_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2}\mathcal{R} g_{\alpha\beta}. \tag{D.28}$$

are the components of the Einstein tensor.

Cartan's structure equations, a technique used in Appendix A to derive the hydrostatic equilibrium equations, will be our final topic.

First, the metric tensor allows the use of observer fields, e_a, that is, an orthonormal tetrad satisfying [10]

$$g\left(e_a, e_b\right) = \eta_{ab}. \tag{D.29}$$

The dual elements of e_a, denoted by Θ^a, serve as a basis of the cotangent space and define the metric as

$$g = \eta_{ab} \Theta^a \otimes \Theta^b. \tag{D.30}$$

The antisymmetric character of both curvature and torsion tensors allow the natural definition of its corresponding two-forms:

$$\mathbf{T}\left(\mathbf{X}, \mathbf{Y}\right) = \mathcal{T}^a \left(\mathbf{X}, \mathbf{Y}\right) e_a, \tag{D.31}$$

$$\mathbf{R}\left(\mathbf{X}, \mathbf{Y}\right) e_b = \Omega^a_b \left(\mathbf{X}, \mathbf{Y}\right) e_a. \tag{D.32}$$

The Cartan's structure equations are

$$\Omega^a_b = d\omega^a_b + \omega^a_d \wedge \omega^d_b, \tag{D.33}$$

$$\mathcal{T}^a = d\Theta^a + \omega^a_b \wedge \Theta^b. \tag{D.34}$$

In the above expression ω^a_b represents the components of the connection form given by

$$de_b = e_a \omega^a_b. \tag{D.35}$$

In this language the compatibility between g and ∇ can be expressed as [54]

$$dg_{ab} = \omega_{ab} + \omega_{ba}. \tag{D.36}$$

For an orthonormal frame, $dg_{ab} = 0$, thus $\omega_{ab} = -\omega_{ba}$.

Locally the two-forms of the Cartan's equations can be written as:

$$\Omega^a_b = \frac{1}{2}\mathcal{R}^a_{bcd}\Theta^c \wedge \Theta^d; \tag{D.37}$$

$$\mathcal{T}^a = \frac{1}{2}T^a_{bc}\Theta^b \wedge \Theta^c. \tag{D.38}$$

Nevertheless, since general relativity is constructed without the notion of torsion the respective Cartan's equation can be written as:

$$-d\Theta^a = \omega^a_b \wedge \Theta^b. \tag{D.39}$$

Local expressions are useful to connect these mathematical objects with the ones traditionally used in general relativity. The Christoffel symbols can be related to the connection forms via [10]

$$\omega^\alpha_\beta = \Gamma^\alpha_{\mu\beta}dx^\mu. \tag{D.40}$$

Therefore

$$d\omega^\alpha_\beta = \Gamma^\alpha_{\mu\beta,\nu}dx^\nu \wedge dx^\mu \tag{D.41}$$

$$= \frac{1}{2}\left(\Gamma^\alpha_{\mu\beta,\nu} - \Gamma^\alpha_{\nu\beta,\mu}\right) dx^\nu \wedge dx^\mu. \tag{D.42}$$

Consider also

$$\omega^\alpha_\delta \wedge \omega^\delta_\beta = \Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\mu\beta}dx^\nu \wedge dx^\mu \tag{D.43}$$

$$= \frac{1}{2}\left(\Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\mu\beta} - \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\nu\beta}\right) dx^\nu \wedge dx^\mu. \tag{D.44}$$

The above relations express the fact that the Cartan's curvature equation is equivalent to the conventional definition of the Riemann tensor in local coordinates, given by (D.15).

Appendix E

Einstein Field Equations

This chapter discusses how the field equations of general relativity can be obtained through a variational principle, a program developed simultaneously by Einstein and Hilbert (although strictly speaking Hilbert was the first to obtain the correct result). This is also the approach in other theories, for instance, in the construction of the Standard Model, one does not start with the equations of motion but instead seeks for the simplest possible lagrangian with the desired field content and symmetries [185].

Specifically, for a gravitational theory, Hilbert proposed the following axioms [186]:

1. The gravitational field equations should follow from a variational principle with the components of the metric tensor acting as independent variables in the action integral.
2. The action functional should be a scalar.
3. The field equations should be second order differential equations with respect to the metric.

Some comments on these axioms are pertinent. Assuming a four-dimensional spacetime, the first axiom says that the action integral should have the form $\int_{\mathcal{V}} \mathcal{F} d^4x$, where $\mathcal{V}$ is some four-dimensional region in spacetime and $\mathcal{F}$ is a function that depends on the metric and its derivatives. From the second axiom, if $\int_{\mathcal{V}} \mathcal{F} d^4x$ should be a scalar, then $\mathcal{F}$ should be a scalar density of weight -1 (since d^4x is a scalar density of weight $+1$). The simplest option for a scalar density of weight -1 is $\sqrt{-g}$, therefore it seems reasonable to assume $\mathcal{F} = \sqrt{-g}F$, where F is a proper scalar. The only scalar that can be constructed without using higher derivatives of the metric, as specified in Axiom 3, is the Ricci scalar $\mathcal{R}$. All things considered, in the following sections it will be demonstrated how these insights can be used to obtain the Einstein field equations.

E.1 Einstein Field Equations in Vacuum

In the light of the previous comments, consider as a starting point the following action:

$$S_{EH} = \frac{1}{16\pi}\int_{\mathcal{V}} \mathcal{L}\left[g_{\mu\nu}\right]\sqrt{-g}d^4x = \frac{1}{16\pi}\int_{\mathcal{V}} \left(\mathcal{R} - 2\Lambda\right)\sqrt{-g}d^4x, \tag{E.1}$$

where a constant Λ, known as the cosmological constant, was included. Gravitational theories are said to be of the Einstein-Hilbert type whenever the underlying Lagrangian is linear on curvature [63].

Now, consider a variation of the metric tensor [187]

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{E.2}$$

where $\delta g_{\mu\nu}$ and its first derivative are assumed to vanish on the boundary $\partial\mathcal{V}$ of the region $\mathcal{V}$[79]. Therefore the variations of action integrals and the subsequent field equations are obtained from the requirement that $\delta S_{EH} = 0$ for arbitrary variations of the metric[80]. The variation of the action can be written as

$$\delta S_{EH} = \frac{1}{16\pi}\int_{\mathcal{V}} \left(g^{\mu\nu}\sqrt{-g}\delta\mathcal{R}_{\mu\nu} + \mathcal{R}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right) - 2\Lambda\delta\sqrt{-g}\right)d^4x. \tag{E.3}$$

Aiming to compute the variation $\delta\mathcal{R}_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$ it is useful to consider the variation of the connection coefficients [187]

$$\Gamma^{\sigma}_{\mu\nu} \to \Gamma^{\sigma}_{\mu\nu} + \delta\Gamma^{\sigma}_{\mu\nu}. \tag{E.4}$$

It is noteworthy to observe that the variation $\delta\Gamma^{\sigma}_{\mu\nu}$, being the difference between two connections, is a tensor [187]. As commonly done when proving tensor identities, it is useful to consider geodesic coordinates at some arbitrary point P where $\Gamma^{\sigma}_{\mu\nu}$ vanishes. It follows that

$$\delta\mathcal{R}^{\sigma}_{\mu\nu\rho} = \partial_\nu\left(\delta\Gamma^{\sigma}_{\mu\rho}\right) - \partial_\rho\left(\delta\Gamma^{\sigma}_{\mu\nu}\right). \tag{E.5}$$

Since partial and covariant derivatives coincide at P it is possible to rewrite the above expression as

$$\delta\mathcal{R}^{\sigma}_{\mu\nu\rho} = \nabla_\nu\left(\delta\Gamma^{\sigma}_{\mu\rho}\right) - \nabla_\rho\left(\delta\Gamma^{\sigma}_{\mu\nu}\right). \tag{E.6}$$

The above equation is valid not only when adopting geodesic coordinates at P, but in any coordinate system. That is a direct consequence of the fact that the quantities on the right hand side are tensors. In

[79] Einstein's field equations follow from (E.1), if and only if the surface term vanishes. Having in mind Axiom 3, an appropriate Dirichlet boundary value problem fixes the values of the metric at the boundary of the spacetime region. Nevertheless, the corresponding condition on the variations of the metric, $\delta g_{\mu\nu}|_{\partial\mathcal{V}} = 0$, is not sufficient in general to assure a vanishing surface term [63]. The exceptions are the asymptotically flat spacetimes associated with isolated matter configurations, which fortunately are the only class of physical systems considered in this work.

[80] Some might wonder if one could obtain the field equations via the subsequent Euler-Lagrange equations. Unfortunately this approach, albeit straightforward, demands cumbersome computations and will not be pursued in this work.

addition, remembering that P was chosen arbitrarily, the above result holds generally and is known as the Palatini equation [187]. The respective variation for the components of the Ricci tensor is obtained through contraction, namely

$$\delta\mathcal{R}_{\mu\nu} = \nabla_\nu\left(\delta\Gamma^\sigma_{\mu\sigma}\right) - \nabla_\sigma\left(\delta\Gamma^\sigma_{\mu\nu}\right). \tag{E.7}$$

Therefore

$$\begin{aligned}\frac{1}{16\pi}\int_{\mathcal{V}}\left(g^{\mu\nu}\sqrt{-g}\delta\mathcal{R}_{\mu\nu}\right)d^4x &= \frac{1}{16\pi}\int_{\mathcal{V}}\left(g^{\mu\nu}\sqrt{-g}\left(\nabla_\nu\left(\delta\Gamma^\sigma_{\mu\sigma}\right) - \nabla_\sigma\left(\delta\Gamma^\sigma_{\mu\nu}\right)\right)\right)d^4x \\ &= \frac{1}{16\pi}\int_{\mathcal{V}}\nabla_\nu\left(\left(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\sigma}\right) - g^{\mu\sigma}\left(\delta\Gamma^\nu_{\mu\sigma}\right)\right)\sqrt{-g}d^4x.\end{aligned} \tag{E.8}$$

This integral contains a total divergence, thus Stoke's theorem dictates that this term will only contribute as a surface integral over the boundary $\partial\mathcal{V}$ [70, 187]. The assumption that the variations of the metric and its first derivatives vanishes on the boundary implies that the connection also vanishes, yielding

$$\frac{1}{16\pi}\int_{\mathcal{V}}\left(g^{\mu\nu}\sqrt{-g}\delta\mathcal{R}_{\mu\nu}\right)d^4x = 0. \tag{E.9}$$

In order to proceed it is needed to evaluate the expression

$$\delta\left(\sqrt{-g}g^{\mu\nu}\right) = \sqrt{-g}\delta g^{\mu\nu} + g^{\mu\nu}\delta\sqrt{-g}. \tag{E.10}$$

First considering the identity $g^{\mu\alpha}g_{\alpha\beta} = \delta^\mu_\beta$, as well as the fact that the constant tensor δ^μ_β does not change under a variation it follows that [187]

$$\delta\left(g^{\mu\alpha}g_{\alpha\beta}\right) = 0. \tag{E.11}$$

The above equation implies that

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}. \tag{E.12}$$

Moreover, observe that

$$\delta\sqrt{-g} = \left(\frac{\partial\sqrt{-g}}{\partial g_{\alpha\beta}}\right)\delta g_{\alpha\beta} = -\frac{1}{2\sqrt{-g}}\left(\frac{\partial g}{\partial g_{\alpha\beta}}\right)\delta g_{\alpha\beta}. \tag{E.13}$$

The partial derivative above can be performed using the formula [70]

$$g = \frac{\text{Cof}^{\alpha\beta}}{g^{\alpha\beta}}, \tag{E.14}$$

where $\text{Cof}^{\alpha\beta}$ is the cofactor matrix of $g_{\alpha\beta}$ with respect to the matrix constructed using the components of the metric tensor. The above relation implies that

$$\frac{\partial g}{\partial g_{\alpha\beta}} = \text{Cof}^{\alpha\beta} = gg^{\alpha\beta}. \tag{E.15}$$

Therefore

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}. \tag{E.16}$$

Combining the above results equation (E.10) can be rewritten as

$$\begin{aligned}\delta\left(g^{\mu\nu}\sqrt{-g}\right) &= \sqrt{-g}\left(\delta g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right) \\ &= \sqrt{-g}\left(\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}\right).\end{aligned} \tag{E.17}$$

Inserting equations (E.16) and (E.17) into (E.3)

$$\delta S_{EH} = \frac{1}{16\pi}\int_{\mathcal{V}}\sqrt{-g}\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}d^4x. \tag{E.18}$$

The deduction is concluded from the requirement that the variation of the action must vanish for any metric variation, so [70]

$$\delta S_{EH} = 0 \implies G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{E.19}$$

These are called the vacuum field equations of general relativity.

E.2 Einstein Field Equations in the Presence of Matter and Energy

This section completes the deduction of the field equations by allowing the presence of non-gravitational fields. This generalization is achieved by simply adding an extra term to the action, namely [187]

$$S = S_{EH} + S_M = \frac{1}{16\pi}\int_{\mathcal{V}}\mathcal{L}\left[g_{\mu\nu}\right]\sqrt{-g}d^4x + \int_{\mathcal{V}}\mathcal{L}_M\left[g_{\mu\nu},\phi\right]\sqrt{-g}d^4x. \tag{E.20}$$

One observation is pertinent. The variation of the matter action with respect to the matter fields will provide covariant equations of motion of the matter fields [185]. However, if the goal is to derive the coupled gravity-matter equations from a variational principle, then the source contribution to the gravitational field equations must be given by variation of the matter action with respect to the metric [185]

$$\delta S = 0 \implies \frac{1}{16\pi}\delta\mathcal{L} + \delta\mathcal{L}_M = 0. \tag{E.21}$$

The energy-momentum tensor, that is, the source of the gravitational field equations, can be defined as

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{E.22}$$

It follows that

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{E.23}$$

which are the Einstein field equations. Setting the cosmological constant to zero the field equations assume the form expressed throughout the main text.

E.3 Energy-momentum Tensor for Perfect Fluids

Aiming to provide an example of energy-momentum tensor, motivated by (E.22), the perfect fluid seems to be a natural choice due to its importance throughout the main text. The derivation will be developed under the constraint that the rates of entropy and particle production are conserved under variation of the metric [70].

As a starting point consider the number flux vector density given by

$$n^{\mu} = n\sqrt{-g}u^{\mu}. \tag{E.24}$$

Consequently

$$n = \sqrt{\frac{g_{\mu\nu}n^{\mu}n^{\nu}}{g}}. \tag{E.25}$$

The Lagrangian density for a perfect fluid can be written in terms of its energy density through the relation

$$\mathcal{L}_M = -\varepsilon. \tag{E.26}$$

Before proceeding, observe that the Lagrangian does not contain any derivatives of the metric, therefore

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} = -2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} - \frac{\mathcal{L}_M}{g}\frac{\partial g}{\partial g^{\mu\nu}}. \tag{E.27}$$

Remembering equation (E.15) it is immediate to write

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g_{\alpha\mu}g_{\beta\nu}\frac{\partial g}{\partial g_{\alpha\beta}} = -gg_{\mu\nu}. \tag{E.28}$$

So the energy-momentum tensor in this case can be written as

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M. \tag{E.29}$$

The proposed constraints are:

$$\delta s = 0, \tag{E.30}$$

$$\delta n^{\mu} = 0. \tag{E.31}$$

The thermodynamical identity (where h is the specific enthalpy given by $\frac{\varepsilon+p}{n}$) [70]

$$\left(\frac{\partial\varepsilon}{\partial n}\right)_s = h, \tag{E.32}$$

provides that

$$\delta\varepsilon = h\delta n. \tag{E.33}$$

Combining equations (E.24), (E.25) and (E.31) it is possible to write

$$\delta n = \frac{1}{2n}\left(\frac{n^\mu n^\nu}{g}\delta g_{\mu\nu} - n^\mu n^\nu \frac{g_{\mu\nu}}{g}\delta g\right) = \frac{n}{2}\left(-u^\mu u^\nu \delta g_{\mu\nu} + \frac{u^\mu u_\mu}{g}\delta g\right). \tag{E.34}$$

Remembering that the four-velocity satisfies $u^\mu u_\mu = -1$ and using equations (E.15), (E.12) and (E.28)

$$\delta n = \frac{n}{2}\left(u_\mu u_\nu + g_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{E.35}$$

Moreover, from equations (E.26), (E.33) and (E.35) it follows that

$$\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} = -\frac{nh}{2}\left(u_\mu u_\nu + g_{\mu\nu}\right) = -\frac{1}{2}\left(\varepsilon + p\right)\left(u_\mu u_\nu + g_{\mu\nu}\right). \tag{E.36}$$

Substituting the expression for the specific enthalpy in the above relation and combining with equations (E.29), (E.26) yields

$$T_{\mu\nu} = \left(\varepsilon + p\right) u_\mu u_\nu + p g_{\mu\nu}, \tag{E.37}$$

which is the expression for the energy-momentum tensor of a perfect fluid as expressed in Chapter 2.

E.4 Uniqueness and the Lovelock Theorem

Now that the field equations are motivated, it is worthy to dedicate a few words on how uniquely they are determined. It is natural to postulate that the metric components $g_{\mu\nu}$ satisfy some equation of the form [57]

$$\mathcal{A}_{\mu\nu}[g] = T_{\mu\nu}, \tag{E.38}$$

where $A_{\mu\nu}[g]$ is a tensor field constructed from $g_{\mu\nu}$ and its first and second derivatives. In order to assure that $\nabla_\nu T^{\mu\nu} = 0$ as a consequence of the field equations, $A_{\mu\nu}$ must obey the equation

$$\nabla_\nu \mathcal{A}^{\mu\nu} = 0. \tag{E.39}$$

It is a fantastic result that in four spacetime dimensions the following theorem holds [57]:

▲ **Theorem 2.** *(Lovelock) A tensor $\mathcal{A}_{\mu\nu}[g]$ with the required properties is in four dimensions a linear combination of the metric and the Einstein tensor*

$$\mathcal{A}_{\mu\nu}[g] = aG_{\mu\nu} + bg_{\mu\nu}, \quad a, b \in \mathbb{R}. \tag{E.40}$$

The above theorem deeply constrains the possible forms of the field equations in general relativity. It is interesting to observe that this theorem makes it unnecessary to postulate linearity in the second derivative only in four dimensions [57]. It also does not require the symmetry of $\mathcal{A}_{\mu\nu}$, showing that a gravitational field which obeys equation (E.38) can not be coupled to some non-symmetric energy-momentum tensor.

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