UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
ESCOLA DE ENGENHARIA
PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA ELÉTRICA

FÉLIX EDUARDO MAPURUNGA DE MELO

IDENTIFIABILITY AND ACCURACY OF DYNAMIC NETWORKS

Porto Alegre
2022

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Thesis presented to Programa de Pós-Graduação em Engenharia Elétrica of Universidade Federal do Rio Grande do Sul in partial fulfillment of the requirements for the degree of Doctor in Electrical Engineering.
Area: Automation and Control

ADVISOR: Prof. Dr. Alexandre Sanfelice Bazanella

Porto Alegre
2022

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Porto Alegre, June 2022.

To Nagirlene, Maria, and Clara.

ACKNOWLEDGMENTS

It has been an incredible adventure and a delightful experience to finish this work. As usual, every adventure has many people involved that contributed to a successful outcome. Here, I would like to express my gratitude to all these people. First of all, I would like to express my most sincere gratitude to my advisor, Alexandre S. Bazanella. It has been a great pleasure to work with you. Not only your guidance was fundamental to the success of this work, but it came with such doses of style. I am, however, not able to translate all contributions into a single paragraph with the same class. Your holistic vision, sharp thinking, and scientific endeavor were truly cornerstones for the development of this work. They served me as both a beacon and compass in this adventure, always indicating the right path to go.

A very special thanks goes to Michel Gevers, who has made many crucial contributions to this work. It has been such a fun to work with you. I am particularly impressed that Michel Gevers had, in such little space-time, made a colossal difference to the quality of this work. Certainly, this work would not be the same without your contributions. I truly appreciate everything you have done, not only to the improvement of this work, but also to my personal development.

I would like to thank the professors from the Data-Driven Control Group: Lucíola Campestrini and Diego Eckhard. They have contributed in many ways to the improvement of this work, from their advice in the seminars to reviewing this work throughout its completion. In the same spirit, I would like to thank professor Luis Antonio Aguirre for such careful analysis of this work, and for the so many improvements and comments you have made in a prior version of this work. I would like to thank professor Gustavo Henrique da Costa Oliveira for the evaluation of this work and his many suggestions, which improved the quality of this work.

I would like to thank the friends that I have made during this period. Of course, I would not dare to write in a couple of paragraphs how much you have contributed to different spheres of my life. A special thanks goes to my friends Arthur Scolari, Daniel Huff, Eduardo Varriale, Emerson Boeira, Everton Viana, and Rodrigo Binotto. You made me feel at home.

I am grateful to Miriam Rosek for always been really helpful. She is always available

to guarantee that everything is running smoothly as they should for the whole program.

In difficult times, it is always good to know who is supporting you in your decisive moments. I have been blessed with the best family I could wish. They not only understand my issues as they always maintained their full support throughout the challenges of life. I am eternally indebted to the women of my life: Nagirlene, Maria and Clara. A truly special thanks go to my stepfather Cesare Dal Molin, for his unswerving continuous support.

Last but not least, I would like to thank CAPES for the financial support, which made this work possible. In such obscure times, it is even more important to acknowledge the importance of these organizations to the development of science.

ABSTRACT

This work addresses the problem of identification in linear dynamic networks. These networks are defined as a set of node signals that are related by causal transfer functions. Each node signal may be subject to additive external signals to the network. These signals could be known to the user, in which case they are used as inputs, or they could be unknown to the user, in which case they act as disturbances in the network. The identification problem consists of identifying the causal transfer functions relating the node signals on the basis of network data. More specifically, this work deals with the problem of characterizing whether or not a network model can be uniquely recovered from data. If this is possible, we say that the network model is identifiable. Identifiability of a dynamic network is characterized by the experimental setting under which the network is submitted. We address this issue by determining all experimental conditions, here called Excitation and Measurement Patterns (EMP), for a number of networks with different topologies. Necessary and sufficient conditions on the identifiability are provided for some network topologies, specifically for isolated cycles and acyclic networks with parallel interconnections. These conditions took the form of indicative conditions on the nodes that need to be excited and the ones that need to be measured. They characterize a set of candidate EMPs that render a network identifiable. A new problem to select the EMP that yields the most accurate estimates is posed, and a framework for which different EMPs can be compared with respect to the precision of the estimates is introduced. A structural property – how the excitations and measurements are distributed in the EMP – is shown to be a key factor in the selection of the most accurate EMP. Furthermore, a guideline for the selection of the most accurate EMP is developed for two network topologies – branches and isolated cycles.

Keywords: Network Identification, Identifiability, Complex Systems, Dynamic Networks.

RESUMO

Este trabalho aborda o problema de identificação em redes dinâmicas lineares. Essas redes são definidas por um conjunto sinais de nós que estão relacionados através de funções de transferências causais. Cada sinal de nó pode ser influenciado pro sinais externos à rede. Tais sinais externos podem ser conhecidos ao usuário, nesse caso, eles são utilizados como entradas da rede, e alguns esses sinais podem ser desconhecidos ao usuário, nesse caso, esses sinais atuam como distúrbios na rede. O problema de identificação é o de identificar as funções de transferência que estão relacionados com os sinais de nós da rede com base nos dados coletados da rede dinâmica. Especificamente, esse trabalho trata do problema de caracterizar se um modelo de rede pode ou não ser unicamente identificado com base nos dados provindos da rede. Se isso for possível, dizemos que o modelo da rede é identificável. A identificabilide de uma rede dinâmica é caracterizada pela configuração experimental em que a rede está submetida. Nós endereçamos esse problema determinando todas as condições das configurações experimentais, aqui denominadas de Padrões de Excitação e Medição (EMP do inglês), para várias redes dinâmicas com diferentes topologias. Condições necessárias e suficientes para a identifificabilidade de redes dinâmicas são fornecidas para algumas topologias de rede, especificamente para ciclos isoladas e redes acíclicas com conexões em paralelo. Essas condições possuem a forma de condições indicativas dos nós que precisam ser excitados e os que precisam ser medidos. Elas caracterizam um conjunto de EMPs candidatos que tornam uma rede identificável. Um novo problema para selecionar o EMP que determina as estimativas mais precisas é proposto, e uma estrutura para a qual diferentes EMPs podem ser comparados com relação a precisão das estimativas é apresentada. Uma propriedade estrutural – em relação a distribuição de excitações e medições na rede – se mostra um fator chave na seleção do EMP que gera as estimativas mais precisas. Além disso, uma diretriz com os principais fatores na seleção do EMP com estimativas mais acuradas é desenvolvido para duas topologias de rede.

Palavras-chave: Identificação de Redes, Identificabilidade, Sistemas Complexos, Redes Dinâmicas.

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LIST OF ABBREVIATIONS

EMP	Excitation and Measurement Pattern
FIR	Finite Impulse Response
LTI	Linear Time-Invariant
MIMO	Multiple-Input Multiple-Output
PEM	Prediction Error Method
SISO	Single-Input Single-Output
SNR	Signal-to-Noise Ratio

LIST OF SYMBOLS

$G^0(q)$	Network matrix
$G^0_{ji}(q)$	Network's module
q	Forward shift operator
n	Number of nodes of a network
M_j	Information matrix associated with EMP$_j$
P_j	Asymptotic covariance matrix associated with EMP$_j$
$T^0(q)$	Input-output matrix of the network
$r_j(t)$	j-th external input signal of the network
$w_j(t)$	j-th internal signal of the network
$y_j(t)$	j-th output signal of the network
$\hat{y}_j(t\|t-1,\theta)$	Output predictor
$\varepsilon_j(t,\theta)$	Prediction error
$e_j(t)$	j-th Gaussian distributed corrupting noise
$\Psi_v(\omega)$	Spectrum of signal $v(t)$.
N	Number of data samples.
$N(\mu, P)$	Gaussian distribution with mean μ and covariance P.
V_N	Cost function.
Z_N	Data set.
λ_j	j-th noise variance
σ_i^2	i-th input variance
$\psi(t)$	Gradient of the one step optimal predictor
θ	Unknown parameter
ν	Cardinality of an EMP
$\mathcal{B}$	The set of excited nodes
$\mathcal{C}$	The set of measured nodes
$\mathcal{D}_\theta$	Feasible set for the parameter θ

$\mathcal{G}$	A (directed) graph
$\mathcal{E}_{\mathcal{G}}$	Set of edges of a (directed) graph $\mathcal{G}$
$\mathcal{F}$	The set of source nodes
$\mathcal{I}$	Set of nodes that are neither sinks nor sources
$\mathcal{M}(\theta)$	A model with a specific structure
$\mathcal{M}^*$	Set of all models with a specific structure
$\mathcal{S}$	The set of sink nodes
$\mathcal{W}$	The set of all nodes from a dynamic network
$\mathcal{V}_{\mathcal{G}}$	Set of vertices of a (directed) graph $\mathcal{G}$
$\mathcal{U}(a, b)$	Uniform distribution between a and b
$\mathfrak{S}$	The data generating system
$\mathbb{C}$	Set of complex numbers
$\mathbb{E}$	Mathematical expectation operator
$\mathbb{P}$	A generic property
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_2$	Set of binary numbers $\{0, 1\}$

CONTENTS

1 INTRODUCTION

We live in an interconnected world. From the air we breathe to charging our phone in the electric power grid, every element interacts with many other elements that together form a complex system. The behavior of such a system is composed of the behavior of its individual elements and the behavior of their interaction. This interaction scheme among different elements in a complex system is what we call a network, which represents the interconnections among the systems. Examples of networks are brains, social systems, the internet, power distribution systems, and Earth system – the conjunction of all material processes – to name a few. A traditional way in the scientific method to study these complex systems is to isolate them from their neighborhood and study it as a single part separated from the assembled system. However, in reality the elements are interconnected and each element is influenced by others, such that the general behavior of such a complex system is not simply the sum of the behaviors of its individual elements isolated. The structure of these interactions plays a major role in the behavior of such complex phenomena (LATORA; NICOSIA; RUSSO, 2017; WATTS; STROGATZ, 1998). In order to fully understand this complex behavior of such system, the knowledge of the isolated elements alone is usually not enough and the complex system must be analyzed as a whole.

The focus of this work is on networks whose elements represent scalar time signals with dynamic relationships among them, the so-called dynamic networks. These signals may represent a wide range of variables of interest. Examples are voltage and current in electrical machines, flow and pressure in hydraulic systems, stock prices and currencies quote in financial systems. We will refer to those internal signals of the network as nodes signals, with each node of the network representing a scalar time signal.

The interconnection links may represent a wide range of objects, for example, transmission lines, pipes and financial transactions. In this work, connections represent dynamic relationships among the nodes of the network. These interconnections are defined by discrete-time linear time-invariant (LTI) systems and will be referred to as modules.

There are external variables to the network that can influence its behavior. Some of these external variables may affect the node signals and they can be manipulated by the user, for this reason, we call them network inputs. On the other hand, there are also external

variables that are unknown to the user and they are not in control of the user. These external signals are typically disturbances or noise processes that randomly affect the behavior of the network.

This kind of network can represent many classes of complex structures, that ranges from physical systems to networks of protein interactions in biology and the formation of consensus networks. Therefore, this research theme has an intrinsic multidisciplinary character among many fields of science and has a potential to be beneficial for the advancement of many others.

This work addresses the subject of system identification in the dynamic networks. Identification methods are an important tool for building mathematical models from data. These methods are adequate for large scale systems, since they use only data to generate the models and usually require little expert knowledge. A module representation and a framework for the study of network identification was proposed in VAN DEN HOF *et al.* (2013). In the literature we can distinguish among three major areas of research for dynamic network identification. The first area of research addresses the problem of identifying a single module embedded in a dynamic network (VAN DEN HOF *et al.*, 2013; GEVERS; BAZANELLA; DA SILVA, 2018). The second concerns the identification of all modules of the network (BAZANELLA; GEVERS; HENDRICKS, 2019; HENDRICKX; GEVERS; BAZANELLA, 2019; FONKEN; RAMASWAMY; VAN DEN HOF, 2022). Finally, a third area concerns the identification of the interconnection structure, i.e. topology of the network (DIMOVSKA; MATERASSI, 2020; INNOCENTI; MATERASSI, 2012; MATERASSI; SALAPAKA, 2010; VAN WAARDE; TESI; CAMLIBEL, 2019). This work addresses the problem of identifying all modules of a dynamic network.

For identification of a single module several methods have been proposed in the literature (WEERTS *et al.*, 2020; VAN DEN HOF; DANKERS; WEERTS, 2018; JAHANDARI; MATERASSI, 2021). Many of them are extensions of the methods applied in the closed-loop approach of classical system identification. Another important feature that characterizes this research area is the selection of variables that must be part of the identification in order to comply with certain objectives, such as to obtain consistent estimates (DANKERS *et al.*, 2016). This problem is particularly challenging due to the parallel interconnections and feedback loops that induce correlation among the node signals. The choice of which variables must be part of the identification process is also crucial from an experiment design perspective. For some choices of variables it may not be even possible to uniquely recover some modules of the network, i.e. some modules will be not identifiable.

In the following, some examples of dynamic networks are presented that can be applied to model complex phenomena. Afterwards, we provide a brief overview of the methods presented in the literature of network identification. We conclude this chapter formally introducing the research problems investigated in this thesis and providing an overview of the contents of this work.

1.1 Examples

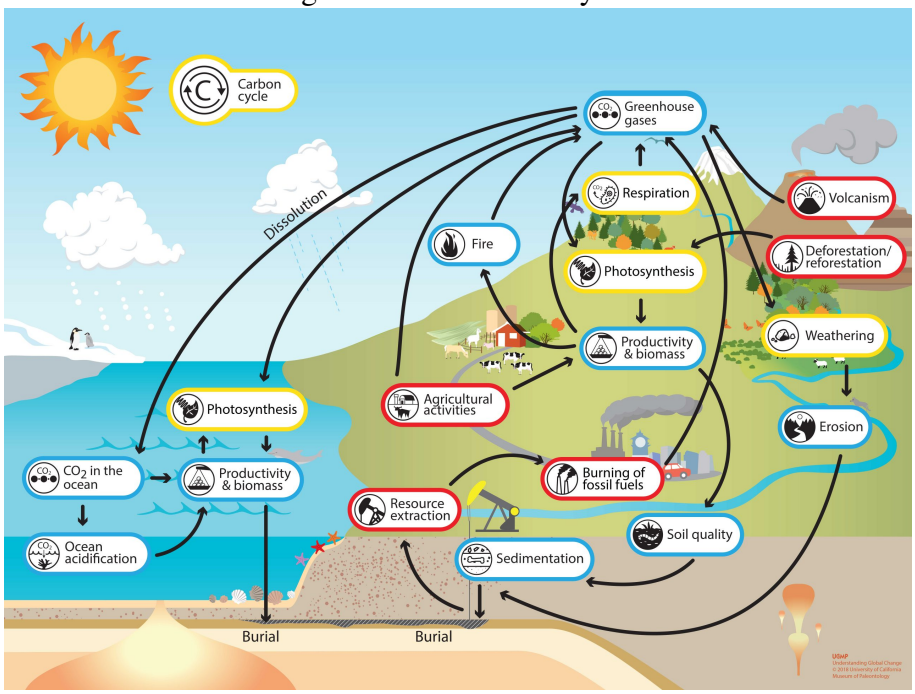
The study of complex networks is crucial in the scientific investigation and many challenges lie ahead. Here we provide some examples that illustrate the importance of the study of dynamic networks in a broader context.

The climate crisis is one of the major (if not the biggest) challenges of humanity in the years to come (MASSON-DELMOTTE *et al.*, 2021). Mitigation actions for climate change is an important issue to be addressed in order to cope with the many difficulties of a rapid (from a geological perspective) changing world. Several models have been proposed in the literature to explain some key variables in Earth science that together define how the earth atmosphere behave. However, such a system is full of interconnections and feedbacks among many different systems that compose the earth climate: different layers of the atmosphere, cloud circulation, carbon feedbacks, the thermodynamic of the oceans, tectonics plates, and the interaction with large ecosystems and economic systems that influence the behavior of such systems (SKINNER; MURCK, 2011). Modeling the Earth system – the geosphere, hydrosphere, atmosphere, biosphere, and anthroposphere – is a true multi-disciplinary field which involves a number of subsystems and their interconnection structure.

The current setting for scientific assessment of the climate is composed of a myriad of models that are jointly evaluated and compared under the umbrella of the *Coupled Model Intercomparison Project* (CMIP) (EYRING *et al.*, 2016). Therefore, advances on how these models are coupled together can be of paramount importance to the scientific understanding of the complex network of systems that make the Earth system. The interaction of these large systems shape the complex behavior of Earth, which should be interpreted as a complex system and evolving network. Such interactions interfere in a wide range of other subsystems. As an example, consider that tectonic activity and the resulting motion of continental plates is an important trigger for the formation of ocean currents, which transport energy in terms of heat that is transferred to the atmosphere and thus determine the climate on large scale (SALTZMAN, 2001). The presence of ice sheets (which are influenced by the oceanic currents) in high latitudes is known to influence both the dynamics of the Earth body via pressure forces and the climate system (through the well-known ice-albedo feedback related to the reflection of solar irradiation) (DONNER *et al.*, 2009).

Another subsystem that is of paramount importance to regulate the climate on Earth is the carbon cycle. Figure 1 depicts a conceptual interaction among many different factors that contribute to the carbon cycle. Notice that this is one of the subsystems that are part of the Earth system, and therefore it interacts with many other subsystems. The carbon cycle is the process in which carbon flows from an environment to another. This is a natural process and it has slow and fast actuation mechanisms.

Figure 1 – The carbon cycle.



Source: (Carbon Cycle, March, 2022)

A series of chemical reactions and tectonic activity make carbon move, over a period of 100-200 million years, between rocks, ocean and atmosphere. The cycle from atmosphere to rocks happens through the formation of carbonic acid, a combination of water and carbon, which drops from atmosphere along with the rains. Once this carbonic acid enters in contact with the rocks, a series of chemical reactions occurs that will finally land carbon at the ocean, in which it will be used as a source for organisms (corals and plankton). These organisms will ultimately die and sink into the seafloor. Over time, layers of sediment will form and these layers will store carbon into the stones. The carbon finally returns to the atmosphere through volcanoes. This cycle is then repeated between the ocean, land, and atmosphere. The ocean is also responsible for the fast carbon cycle. In the ocean surface carbon dioxide gas dissolves in and ventilates out of the ocean in a steady change with atmosphere. This cycle is responsible for the degree of acidification in the oceans. Over the last century, anthropogenic activities have been disturbing the carbon cycle under a point that might profoundly modify the climate beyond any unseen level before (MASSON-DELMOTTE *et al.*, 2021).

Another class of problems that can be represented by dynamic networks comes from the field of social systems. The interaction between humans and how we form our beliefs and opinions can be modeled as a dynamic network (ANDERSON *et al.*, 2020). Indeed, we have many interactions with our closest friends, neighbors, acquaintances, and from many people over the internet - by itself another network - which can have a particular influence over our behavior (BOND *et al.*, 2012; MARENGO *et al.*, 2021) . This interactive network

can have profound significance over our life, not only in how we think, vote, but also in our emotional states and even dietary patterns (FLETCHER; BONELL; SORHAINDO, 2011; KRAMER; GUILLORY; HANCOCK, 2014). In fact, a single behavior in a social media can tell with high probability when we are near to have a new romantic relationship with just little information as the pattern in which we post on social media (DIUK, 2014; KRAMER; GUILLORY; HANCOCK, 2014). The pattern of our likes in such social networks can tell much of our opinions, in such a way that algorithms from social media can predict some of our traits better than our family (YOUYOU; KOSINSKI; STILLWELL, 2015). Understanding how people reach consensus is very important to propose effective communication strategies to target a number of necessary policies. This is remarkably important not only for an efficient communication program but also for decision-making in important sectors from environmental issues to health policies, such as vaccinations campaigns (D'I TREEN; WILLIAMS; O'NEILL, 2020; BALL; MAXMEN, 2020).

Let us consider a very simple mathematical network model that can represent a number of network systems, such as opinion in social networks and wireless communication systems. Our opinion can be modeled as a probabilistic distribution over a certain topic. We update our opinions according to the information available from our social circle. Let individual i have its own subjective pdf $p_i(k)$ at instant k in the social network of n individuals. This individual is a node in the social network and the edges represent the social interactions with the neighborhood. Each individual will associate to other individuals a weighting factor that will represent the confidence on their opinion. Now, the individuals will revise their own opinion based on their neighborhood as $p_i(k+1) = \sum_{j=1}^{n} a_{ij} p_j(k)$, where a_{ij} denotes the weight that individual i assigns to the pdf of individual j. Each weight a_{ij} must be non-negative and they must obey $\sum_{j=1}^{n} a_{ij} = 1$. This update rule is known as the French-Harary-DeGroot model of opinion dynamics (BULLO, 2018). It can be represented in matrix form as $p(k+1) = Ap(k)$. This model represents an averaging system in which the opinion of each individual will be the average of its neighborhood. The same principle can be used for measuring variables in a network of wireless sensors.

Our final example is inspired by the biological field: gene regulatory networks (GRN). A gene is a section of nucleotides in the deoxyribonucleic acid (DNA) and it is a basic unit of heredity in evolution. These networks are a collection of molecular regulators that interact with each other and also with other substances in the cell to govern gene expression levels of mRNA and proteins (ALBERTS *et al.*, 2015). A gene expression is the process in which a cell converts a nucleotide sequence of genes into a sequence of RNA molecules and then into amino acids sequence of proteins, which in turn, will perform a number of functions in the cell.

There are a number of functions in a cell that are determined by the interaction of these molecules. The regulators can be DNA, RNA, proteins, and a complex collection

of these. The nodes of such network can be these molecules, while the edges correspond to individual biochemical reactions through which the products of these molecules affect those of another. These networks are responsible for a number of different processes in a cell, which can globally affect an organism. For instance, they are involved in regulating complex biological oscillators such as the circadian cycle (rhythm) (YAN *et al.*, 2008). The circadian cycle is a biological process that regulates the sleep-wake stages of an organism, roughly for a period of twenty four hours according to its environment.

We have presented a number of examples in which network models can be a key factor in the analysis of complex phenomena. These examples are just a small subset of all possible processes for which dynamic networks can be used to analyze and understand these behaviors. The applications for dynamic networks are multidisciplinary and they cover a wide range of different disciplines, from social networks to engineering systems.

1.2 Literature review

System Identification is a research field that seeks to obtain mathematical models from input-output data. The extension of the framework of closed-loop systems to identification of general dynamic networks was originally proposed in DANKERS *et al.* (2012); VAN DEN HOF *et al.* (2012, 2013). The framework for the study of network identification was proposed in these works. The objective of these contributions was to obtain consistent estimates of a module embedded in a dynamic network. Their approach was to extend the classic methods from identification of closed-loop systems - direct, indirect, and joint input-output approaches - to dynamic networks. This approach took advantage of the interpretation that dynamic networks can be seen as a general extension of closed-loop systems (VAN DEN HOF; DANKERS; WEERTS, 2017).

Identification in dynamic networks have a number of additional challenges with respect to the classical system identification setting of Single Input Single Output (SISO) systems. This is due to a network being full of feedbacks and parallel connections, which increase the difficulty because of possible correlation among the signals involved. Furthermore, the number of signals involved may increase the complexity of the problem at hand. This gives rise to the necessity of choosing the set of signals that can be used to estimate a module within the network. Under this scenario, some contributions addressed the problem of selecting the input predictors for consistent estimates for the direct method in DANKERS; VAN DEN HOF; HEUBERGER (2013) and for the two stage identification approach in DANKERS *et al.* (2013). Sufficient conditions for the problem of prediction input selection were derived in DANKERS *et al.* (2016).

In the same spirit, some methods from classic identification literature were extended to the dynamic network framework. Instrumental variables methods were also employed for the estimation of continuous time modules in dynamic networks in DANKERS; VAN DEN

HOF; BOMBOIS (2014). This method was also explored in the identification of dynamic networks subject to process and measurement noises sources in DANKERS *et al.* (2015). Nonparametric identification was addressed in DANKERS; VAN DEN HOF (2015).

A common assumption in all these works was that all nodes are measured and one would only need to deal with input selection. In order to consistently estimate a module, it is necessary to guarantee that the signals involved are persistently exciting (GEVERS *et al.*, 2009; BAZANELLA; BOMBOIS; GEVERS, 2012; LJUNG, 1999). Sufficient conditions on the richness of input signals were derived for the consistent estimates of a module in GEVERS; BAZANELLA (2015). Path based conditions that depend on the topology of the network for data informativity of a single module were given in RAMASWAMY; VAN DEN HOF (2021). A practical method for the identification of one module was proposed in GEVERS; BAZANELLA; DA SILVA (2018), that was based on the selection of which variables should be measured and excited in order to identify the desired module. This problem of local identification was further explored in the contributions VAN DEN HOF *et al.* (2019); RAMASWAMY; VAN DEN HOF; DANKERS (2019); DANKERS; VAN DEN HOF (2015). An overview of identification methods in dynamic networks can be found in VAN DEN HOF; DANKERS; WEERTS (2018).

The majority of works presented so far were primarily interested in obtaining consistent estimates for a single module embedded in a network. In WAHLBERG; HJALMARSSON; MÅRTENSSON (2009) the authors analyzed the effects of common dynamics in the variance of the modules for the particular class of cascade networks. Further works that analyzed accuracy aspects for specific cascade structures followed in HÄGG; WAHLBERG; SANDBERG (2011). A strategy inspired by these results was introduced in GUNES; DANKERS; VAN DEN HOF (2014), where a slight modification in the cost function of the two state identification method was used to reduce the variance of the estimates.

A Bayesian setting for identification of a module embedded in a dynamic network is presented in EVERITT *et al.* (2016); EVERITT; BOTTEGAL; HJALMARSSON (2018). The application of more recent kernel based identification results for the local identification of a module was addressed in RAMASWAMY; BOTTEGAL; VAN DEN HOF (2018).

For the identification of the whole network several methods have been proposed based on methods from the system identification literature. The weighted null space fitting (GALRINHO; ROJAS; HJALMARSSON, 2019) was used for identification of the modules in a cascade network in GALRINHO *et al.* (2018); FERIZBEGOVIC; GALRINHO; HJALMARSSON (2018). This identification scheme was further applied for identification of ARMAX dynamic networks in WEERTS *et al.* (2018).

One of the key challenges in the identification of dynamic networks is the correlation among the signals. In VAN DEN HOF *et al.* (2019) dynamic networks with correlated noise were analyzed for identification of one module within the network. A prediction error identification for rank reduced noise which led to maximum likelihood properties

was addressed in WEERTS; VAN DEN HOF; DANKERS (2018a) for the identification of the whole network. Obtaining maximum likelihood properties for local identification techniques with correlated noise was a major concern in RAMASWAMY; BOTTEGAL; VAN DEN HOF (2021).

Whether the modules could be uniquely recovered from network data, a problem known as network identifiability, was investigated in a number of works. In GEVERS; BAZANELLA; PARRAGA (2017) it was shown that network identifiability heavily depends on prior knowledge about the structure of the network. Single module identifiability was addressed in WEERTS; VAN DEN HOF; DANKERS (2018b). The main conclusions of these works was that identifiability would depend on the excitation and noise structure of the network. Identifiability with singular noise spectra was addressed in WEERTS; VAN DEN HOF; DANKERS (2018c). In GEVERS; BAZANELLA; PIMENTEL (2019) it was shown that for the case of rank reduced noise the conditions on identifiability could be slightly modified and remained similar to the full rank case.

One of the first works to consider network identification with partial measurement was BAZANELLA *et al.* (2017), which addressed conditions on the measurement of some nodes in order to render a subset of modules identifiable, and eventually the whole network, from network data. This lead to the introduction of the concept of generic identifiability in HENDRICKX; GEVERS; BAZANELLA (2019), which was based on path-based conditions that depend upon the network topology rather than the rank of certain transfer matrices. This work was fundamental as allowed checkable (in polynomial time) path-based conditions to determine identifiability of a given network, and inspired a flurry of contributions on this theme.

Generic identifiability, briefly speaking, is a definition that says that a network is identifiable for *almost all* network parameters, with exception of a thin set in the parameter space. It inspired many works to characterize identifiability from a graph theoretic perspective. Graphical conditions for identifiability were given in VAN WAARDE; TESI; CAMLIBEL (2018a,b). For the partial measurement case a graphical algorithm was proposed in VAN WAARDE; TESI; CAMLIBEL (2019) to deal with identifiability for *all* networks parameters. In CHENG; SHI; VAN DEN HOF (2019, 2022) the algebraic conditions for generic identifiability were used to introduce the concept of pseudo-trees, which could allocate the inputs in a network in order to guarantee generic identifiability. In SHI; CHENG; VAN DEN HOF (2020, 2022) similar conditions were derived for allocation of inputs for single module identifiability.

Identification of single module under partial measurement was also addressed in (MATERASSI; SALAPAKA, 2015). Sufficient and necessary graphical conditions were presented for the local identification of a module in a dynamic network in JAHANDARI; MATERASSI (2021).

The first work to address network identification with partial measurement and partial

excitation was BAZANELLA; GEVERS; HENDRICKS (2019), where conditions on the excitation and measurement of some nodes were given. Conditions for local identifiability under partial excitation and measurement were formulated in LEGAT; HENDRICKX (2020). In SHI; CHENG; VAN DEN HOF (2021) single module identifiability was addressed under this scenario.

For topology identification, a number of approaches have been proposed in the literature. The main objective of this research area is to recover the structure of the network. This area can be seen as a part of the network identification problem. Here, we provide, by no means exhaustive, references on the literature of topology identification. In MATERASSI; INNOCENTI (2010); MATERASSI; SALAPAKA (2010) is presented an algorithm for topology detection in dynamic networks, which is based on a distance measure from different structures. An ordinary least squares approach is used to identify the topology of the network in MATERASSI *et al.* (2011). Methods based on Wiener filtering theory for topology identification were proposed in INNOCENTI; MATERASSI (2012). A more recent result for continuous time networks is presented in VAN WAARDE; TESI; CAMLIBEL (2019), that is based on constrained Lyapunov equations. Conditions for topology identifiability were given in VAN WAARDE; TESI; CAMLIBEL (2019) along with a reconstruction scheme based on subspace methods. Compressing methods for topology identification were analyzed in JAHANDARI; MATERASSI (2018); SANANDAJI; VINCENT; WAKIN (2011).

1.3 Research problems

In this thesis we will focus on two issues regarding the area of network identification. We focus on the problems of obtaining identifiability conditions for dynamic networks and in obtaining the most accurate parameter estimates for the identification method. We formulate a new problem regarding the selection of network inputs and outputs.

We have seen from the literature review that an appropriate selection of which nodes to measure and which nodes to excite is a crucial problem in network identification. Typically, identifiability conditions will depend on which variables are available for measurement and which are known to the user. Our first problem is to provide a framework to study which experimental settings render a network identifiable, that is, the allocation of inputs and measurements in the network. Our objective is to determine conditions on the nodes of a network that need to be excited and on the nodes that need to be measured. Ultimately, these conditions will depend on the topology of the network. We therefore analyze some general classes of networks in order to provide conditions on the experimental setting such that we can uniquely recover all modules from a given network. The first research problem can be stated as follows.

Problem 1. *Given a dynamic network with a known topology. Determine the allocation of*

excitations and measurements for a dynamic network which renders the network modules identifiable from network data.

The answer to this problem will allow us to characterize all experimental scenarios under which a given network can be uniquely identified. Naturally, there will be a number of such experimental scenarios from which a user will have to choose. A second problem we face is how to determine the experimental setting based on a quality criterion to use in the identification of the whole network. We introduce a framework for comparison among the different experimental scenarios with the objective of choosing the one that yields the most accurate estimates. We investigate what are the ingredients that make an experimental setting more attractive than other candidates. In analyzing this problem we try to answer the following questions. Is there any structural property based on the topology of the network that makes a particular experimental setting preferable? Are there any factors that contribute to the selection of inputs and outputs? Is there any equivalence between exciting or measuring a particular node?

This problem is fundamental from an experiment design perspective, as it allows the users to choose an appropriate experimental setting according to their objectives. We can state this problem as follows.

Problem 2. *Given a dynamic network with a known topology and a set of identifiable experimental scenarios. Determine which experimental setting yields the most accurate parameter estimates.*

These two problems are intimately connected by the experimental settings applied in a network. By answering them we can formulate experimental strategies for dynamic network identification. This will result in more accurate models using less resources from the network.

The next chapter will introduce the framework for the study of identification of dynamic networks and the basic tools to apply identification methods. In Chapter 3 we focus on Problem 1 and we present a number of results for some classes of dynamic networks. The problem of whether the modules of a network can be uniquely identified is reframed as to determine the experimental settings in which the network is identifiable. In Chapter 4 we focus on Problem 2 for the selection of the most accurate experimental setting for dynamic networks. Finally, we present our mains conclusions for this work in Chapter 5.

Associated with Chapters 3 and 4 a number of contributions were submitted to journals and conferences. They are listed below.

Chapter 3

1. E. Mapurunga and A. S. Bazanella. Identifiability of Dynamic Networks from Structure, IFAC-PapersOnLine, vol. 54, no 7, p. 55–60, jan. 2021. Presented at the 19th IFAC Symposium on System Identification.

2. E. Mapurunga, M. Gevers, and A. S. Bazanella. Necessary and Sufficient Conditions for the Identifiability of Isolated Loops, IEEE Control Systems Letters, vol. 6, p. 2276–2280, 2022.

3. E. Mapurunga, M. Gevers, and A. S. Bazanella. Excitation and Measurement Patterns for the Identifiability of Directed Acyclic Graphs, *Accepted to the 61st IEEE Conference on Decision and Control, 2022*.

Chapter 4

1. E. Mapurunga and A. S. Bazanella. Optimal Allocation of Excitation and Measurement for Identification of Dynamic Networks, IFAC-PapersOnLine, vol. 54, no 7, p. 43–48. Presented at the 19th IFAC Symposium on System Identification.

2. E. Mapurunga and A. S. Bazanella. Optimal excitation and measurement pattern for cascade networks, *Submitted to Automatica*, set. 2021.

3. E. Mapurunga and A. S. Bazanella. Optimal excitation and measurement pattern for isolated cycles, *To be submitted*.

2 NETWORK IDENTIFICATION

This chapter presents the necessary tools used for identification of dynamic networks. Here, the basic setup is presented for models that are able to represent interconnected systems. We start by providing a brief overview of the prediction error method in the general context of system identification. We proceed by presenting the mathematical background of dynamic networks and some preliminary concepts from graph theory. Afterwards, we show how the prediction error method can be applied for identification in dynamic networks. The conclusions of this chapter are presented at the end.

2.1 Prediction error method

The prediction error method (PEM) is a method and a general framework used in the literature of system identification (LJUNG, 1999; SÖDERSTRÖM; STOICA, 1989). Not only the approach of using the prediction error is very general, but also PEM encompasses a number of other methods as special cases under some mild assumptions (LJUNG, 1999). The framework adopted in PEM requires a model parametrization for the system to be identified. In order to present all ingredients of the prediction error identification, consider a discrete-time single-input single-output (SISO) system described as

$$\mathfrak{S} : y(t) = G^0(q)u(t) + v^0(t), \tag{1}$$

where $y(t) \in \mathbb{R}$ is the output of the system, $u(t) \in \mathbb{R}$ is the input of the system, $v^0(t) \in \mathbb{R}$ is the corrupting noise, and q is the forward shift operator, i.e. $qu(t) = u(t+1)$. The corrupting noise $\{v^0(t)\}$ is assumed to be a stationary random process with rational spectral density $\Phi_{v^0}(\omega)$, that is, this process can be equivalently modeled as (LJUNG, 1999):

$$v^0(t) = H^0(q)e^0(t), \tag{2}$$

where $H^0(q)$ is a monic, proper rational transfer function, stable and inversely stable, and $\{e^0(t)\}$ is a stationary white noise process with zero mean and variance λ^0.

The problem in system identification is to obtain a representation of the system that better fits an input-output data set $Z_N = \{u(t), y(t)\}_{t=1}^N$ collected from the system $\mathfrak{S}$.

There are many ways in which a model can be fitted to a particular data set, the purpose of this section is to deal with parametric methods for which PEM is one of these methods. In order to apply the prediction error method, we first need to define a parametric model of system $\mathfrak{S}$:

$$\mathcal{M}(\theta) : y(t) = G(q,\theta)u(t) + H(q,\theta)e(t),\ \theta \in \mathcal{D}_\theta \subseteq \mathbb{R}^d, \tag{3}$$

with θ the parameter vector that describes the model through the parametrized transfer functions $G(q,\theta)$ and $H(q,\theta)$. The signal $\{e(t)\}$ is a stationary white noise process with zero mean and variance λ. The idea of the prediction error method is to select among the model candidates (3) in the model set:

$$\mathcal{M}^* \triangleq \{\mathcal{M}(\theta)|\theta \in \mathcal{D}_\theta \subseteq \mathbb{R}^d\}, \tag{4}$$

the one that minimizes a quality criterion based on the prediction error, which is defined as:

$$\varepsilon(t,\theta) \triangleq y(t) - \hat{y}(t|t-1,\theta), \tag{5}$$

where $\hat{y}(t|t-1,\theta)$ is the optimal one-step ahead predictor. The problem of finding the model that best fits the data in the set of candidate models is then translated into finding a parameter vector that provides a model that better explains the data. This problem can be recast as an optimization problem. That is, the problem is to find the parameter vector that minimizes a criterion based on the prediction error:

$$\hat{\theta}_N = \arg\min_\theta V_N(\theta), \tag{6}$$

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N} l\left(\varepsilon(t,\theta)\right). \tag{7}$$

With $l(\cdot)$ an appropriate norm function. A standard criterion for system identification is the quadratic cost criterion:

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon(t,\theta)^2. \tag{8}$$

The optimal one-step ahead predictor can be shown to be the conditional expectation:

$$\hat{y}(t|t-1,\theta) = \mathbb{E}[y(t)|t-1;\theta] = W_y(q)y(t) + W_u(q)u(t), \tag{9}$$

$$W_y(q) = -(1 - H(q,\theta)^{-1}), \tag{10}$$

$$W_u(q) = H(q,\theta)^{-1}G(q,\theta), \tag{11}$$

where $t-1$ stands for input-output data up to time $t-1$, i.e. $Z_{t-1} = \{u(k), y(k)\}_{k=-\infty}^{t-1}$, and with $\mathbb{E}[\cdot]$ denoting the mathematical expectation operator.

Now, there are a number of desirable properties that an estimator should have. These properties will depend on whether there exists a "true" representation of the system at hand. In other words, if there exists a parameter vector θ^0 such that $G(q,\theta^0) = G^0(q)$ and $H(q,\theta^0) = H^0(q)$. In this case, we say that the system $\mathfrak{S}$ is contained in the model set: $\mathfrak{S} \in \mathcal{M}^*$. For the analysis of estimation methods, such an assumption is usually required, otherwise little could be said about the precision obtained by the identification method. Among the desirable properties an estimator should have is consistency. Under some technical conditions on the data set, the estimator (6) converges as the number of data samples (N) tends to infinity with probability 1 to (LJUNG, 1999):

$$\theta^* \triangleq \arg\min_{\theta} \overline{V}(\theta), \tag{12}$$

$$\overline{V}(\theta) \triangleq \overline{\mathbb{E}}[\varepsilon(t,\theta)^2], \tag{13}$$

where $\overline{\mathbb{E}}[\cdot] \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathbb{E}[\cdot]$. Consistency is the property that asserts that an estimate converges to a "true" representation θ^0 as the number of data goes to infinity. If $\mathfrak{S} \in \mathcal{M}^*$, the model structure is identifiable at θ^0, and the data set is sufficiently rich, then the prediction error method is a consistent estimator. This means that the estimate $\hat{\theta} \to \theta^0$ as the number of data go to infinity $N \to \infty$. Furthermore, it can be shown that as the number of data tends to infinity the parameter error converges to a Gaussian random variable:

$$\sqrt{N}\left(\hat{\theta}_N - \theta^0\right) \to N(0, P_{\theta^0}), \tag{14}$$

where

$$P_{\theta^0} = M_{\theta^0}^{-1}, \tag{15}$$

$$M_{\theta^0} = \overline{\mathbb{E}}\psi(t,\theta)[\lambda^0]^{-1}\psi^T(t,\theta)\Big|_{\theta=\theta^0}, \tag{16}$$

$$\psi(t,\theta) = \frac{\partial \hat{y}(t|t-1,\theta)}{\partial\theta}. \tag{17}$$

The matrix P_{θ_0} is the normalized asymptotic covariance matrix of the parameter estimates, which is also the inverse of the information matrix defined in (16). The information matrix can be computed by using the gradient of the optimal predictor (9). Another important property that an estimator should ideally have is to produce estimators that have a degree of dispersion as small as possible. That is, if the identification experiment is performed a number of times it is expected that the different estimates are close enough of each other. A natural measure for this dispersion is the covariance matrix. For any consistent estimator there is a lower bound for the covariance matrix, known as Cramer-Rao lower bound (LJUNG, 1999):

$$\mathrm{cov}(\hat{\theta}_N) \succeq M_{\theta^0}^{-1}. \tag{18}$$

Notice that this lower bound on the accuracy of the estimates can be computed as the inverse of the information matrix. The prediction error method achieves the Cramer-Rao lower bound under mild conditions on the data set. (LJUNG, 1999; SÖDERSTRÖM; STOICA, 1989). An unbiased estimator that achieves the Cramer-Rao lower bound is said to be an efficient estimator. If there is any estimator that achieves the Cramer-Rao lower bound, then this estimator is also the maximum likelihood estimator (LEHMANN; CASELLA, 1998).

In conclusion, we have presented the main ingredients for the prediction error identification. The prediction error method is a parametric method that requires a parametrized model. PEM is also a consistent and efficient estimator provided the "true" system is in the model set and under some weak conditions on the data set. We are now ready to apply the prediction error method for identification in dynamic networks.

2.2 Dynamic networks setting

In this work, our focus is to deal with discrete-time systems, that is, systems that depend on the discrete time variable $t \in \mathbb{Z}$. The setup adopted here to represent dynamic networks was first introduced in the seminal work VAN DEN HOF *et al.* (2013) for identification in dynamic networks. A network is composed of the following basic entities that together represent a dynamic network.

- **Nodes** – the nodes in a network represent scalar *measurable signals*, also called node signals. These signals represent quantities of interest in a particular system, for instance, voltages and currents across some electrical components, pressure and flow rate in a hydraulic system, or temperature in a thermodynamic system. A node signal is the sum of all signals that enter a node.

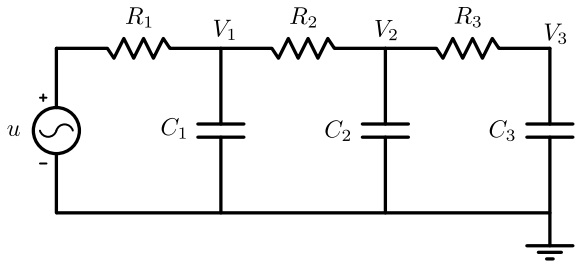
- **Modules** – they represent the dynamics among the nodes of a network. The modules are modeled as Single-Input Single-Output (SISO) discrete-time proper rational transfer functions.

- **External signals** – They represent external signals that act over the network. Some of these signals are known to the user, which we refer to them as inputs. Some external signals are unknown disturbances that affect the network. The inputs of the network are signals that can be directly manipulated by the user, while the disturbances are not in control of the user.

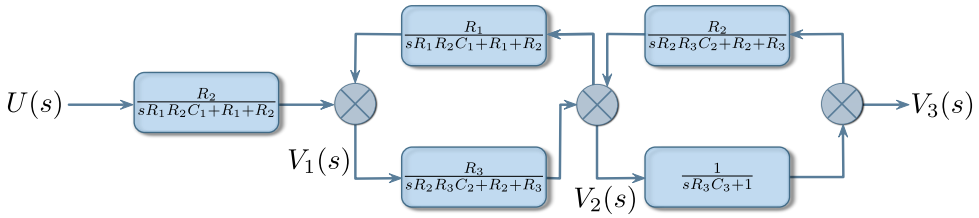
Figure 2 depicts an RC electrical circuit that can be represented as a dynamic network. The node signals of the system are the voltages across the capacitors V_1, V_2, V_3, while the external signal is the voltage supply u. Each node of the circuit in Figure 2-a) represents a

corresponding voltage and the interconnections are the electrical components, i.e. voltage supply, resistors and capacitors. Whereas in the dynamic network representation, as depicted in Figure 2-b), the nodes still represent the capacitor voltages, but the interconnections are now transfer functions. Notice that the transfer functions in Figure 2-b) are in the continuous-time domain, but they can be transformed to a discrete-time domain to be in the adopted framework for dynamic networks.

Figure 2 – An RC circuit and its dynamic network representation.
a) An RC circuit with three RC stages.



b) A block diagram that represents the RC circuit as a dynamic network. The nodes are represented by the summing points, while the modules are the transfer functions associated with the node signals.



Source: The Author.

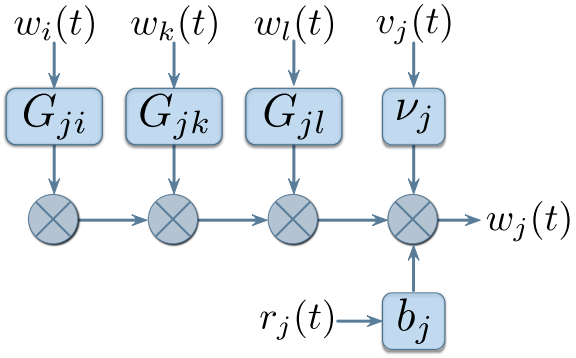
A dynamic network is characterized by its number of nodes, by its modules and by the interconnection structure of their elements. Each scalar node signal is denoted as $\{w_j(t)\}, j = 1, 2, \ldots, n$ and represents a signal associated with a particular node. To every node we denote and refer to it through a label, typically a numeric label $j \in \mathbb{Z}$. We denote by $\mathcal{W} \triangleq \{1, 2, \ldots, n\}$ the set of node labels of a dynamic network. A given node signal may have the influence of other node signals $\{w_i\}, j \neq i \in \mathcal{W}$, external excitation signal $\{r_j\}, j \in \mathcal{W}$, and/or unknown disturbance $\{v_j\}, j \in \mathcal{W}$. This relationship is mathematically described by the following equation.

$$w_j(t) = \sum_{k \in \mathcal{N}_j^-} G_{jk}^0(q) w_k(t) + b_j r_j(t) + \nu_j v_j(t). \tag{19}$$

The variable $b_j \in \mathbb{Z}_2 \triangleq \{0, 1\}$ is a variable that indicates whether $\{r_j(t)\}$ is applied to node j or not. Similarly, $\nu_j \in \mathbb{Z}_2$ defines whether the unknown noise process affects node

j. The set $\mathcal{N}_j^-$ collects the node signals that enter node j. Figure 3 depicts the relationship among the signals affecting the node signal j.

Figure 3 – The dynamics in node j. Node j represents the sum of filtered node signals $i, k, l \in \mathcal{N}_j^-$, an input $r_j(t)$, and an unknown $v_j(t)$ disturbance.



Source: The author.

Collecting all equations (19) for $j \in \mathcal{W}$ one obtains the dynamic relationship for all nodes in matrix notation as:

$$w(t) = G^0(q)w(t) + Br(t) + Vv(t), \tag{20}$$

where $w(t) = \begin{bmatrix} w_1(t) & w_2(t) & \dots & w_n(t) \end{bmatrix}^T$ is the vector of nodes signals, $r(t) \in \mathbb{R}^{n_r}$ is a column vector of stacked inputs, $v(t) \in \mathbb{R}^{n_v}$ is the column vector of stacked unknown noise processes, and $B \in \mathbb{Z}_2^{n \times n_r}$ and $V \in \mathbb{Z}_2^{n \times n_v}$ are binary matrices responsible for selecting which external signals affect the nodes of the network. They have in common the property that each row has at most one 1 and they are full column rank. Equation (20) represents the dynamics of the network, where the modules of the network are collected in the *network matrix* $G^0(q)$. The general form of the network matrix is as follows:

$$G^0(q) = \begin{bmatrix} 0 & G_{12}^0(q) & \cdots & G_{1n}^0(q) \\ G_{21}^0(q) & 0 & \ddots & \cdots \\ \vdots & \ddots & \ddots & G_{1,n-1}^0(q) \\ G_{n1}^0(q) & \cdots & G_{n,n-1}^0(q) & 0 \end{bmatrix}. \tag{21}$$

As we consider that not all node signals are available for measurement, the following equation describes the output of the network with matrix notation:

$$y(t) = Cw(t) + e(t), \tag{22}$$

where $y(t) \in \mathbb{R}^{n_y}$ is a vector containing the output of the networks, $e(t)$ is a stationary stochastic process that represents measurement noise, and $C \in \mathbb{Z}_2^{n_y \times n}$ is a binary matrix responsible for selecting which node signals are available for measurement. This matrix has at most one 1 at each row and is full row rank. Associated with the binary selection matrices B and C are the corresponding set of excited $\mathcal{B}$ and measured $\mathcal{C}$ nodes.

In this thesis we will be concerned with the problem of identifying the modules of the network from input-output data $\{r(t), y(t)\}$. For this purpose, it is useful to rewrite the equations describing the dynamics of the network in the input-output form

$$y(t) = CT^0(q)Br(t) + \overline{v}(t), \tag{23}$$

with $\overline{v}(t) \triangleq CT^0(q)Vv(t) + e(t)$ and

$$T^0(q) \triangleq (I_n - G^0(q))^{-1}. \tag{24}$$

The following assumption will hold throughout this work regarding the inputs, disturbances and the modules of the dynamic network.

Assumption 2.1.

1. *Each module $G^0_{ji}(q)$ is represented by a rational proper transfer function;*
2. *There are no self-loops in the network, i.e. $G^0_{jj}(q) = 0, \forall j \in \mathcal{W}$. We refer to network matrices with this property as hollow;*
3. *The input signals $\{r(t)\}$ are statistically independent of the unknown disturbance processes $\{v(t)\}$;*
4. *The input signals $\{r(t)\}$ are statistically independent of the unknown measurement noise $\{e(t)\}$;*
5. *The unknown disturbance $\{v(t)\}$ is a stationary random process with rational spectral density $\Phi_v(\omega)$;*
6. *The network is stable, i.e. all entries of $T^0(q)$ are stable transfer functions;*

These assumptions are standard from network identification literature. As not every node is affected by all other nodes in (21), there are some known zeros in the network matrix. This zero/nonzero pattern in the network matrix characterizes the topology of the network. Regarding the topology of the network, we adopt the following assumption, which is standard in the field of dynamic network identification.

Assumption 2.2. *The topology of the dynamic network is known.*

In order to identify the modules using the prediction error method we need to provide a parametrized model of the dynamic network. This can be done as in (3) for each module of the network, or more generally for the network matrix. The parametrized input-output model is

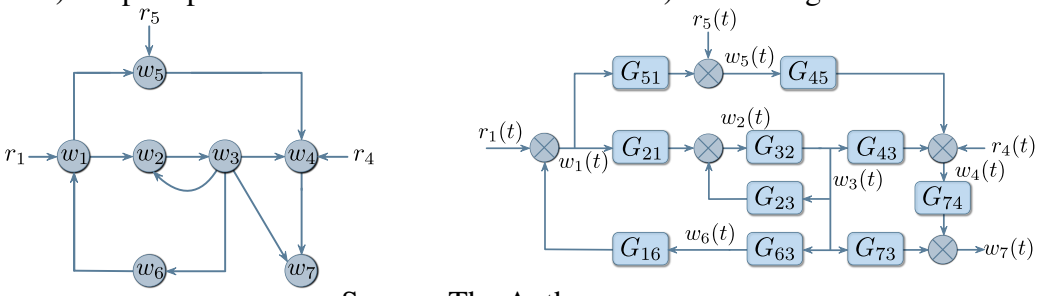
$$\mathcal{M} : y(t) = CT(q, \theta)Br(t) + \overline{v}(t),\ \theta \in \mathcal{D}_\theta \subset \mathbb{R}^d. \tag{25}$$

with $T(q,\theta) \triangleq (I - G(q,\theta))^{-1}$ and $\overline{v}(t) = CT(q,\theta)Vv(t) + e(t)$. The prediction error method can consistently identify the Multiple-Input Multiple-Output (MIMO) transfer matrix $CT(q,\theta)B$ from input-output data. If $\overline{v}(t)$ has a rational spectrum $\Phi_{\overline{v}}(z)$, then by the spectral factorization theorem (SÖDERSTRÖM; STOICA, 1989; LJUNG, 1999), we can rewrite $\overline{v}(t) = \tilde{H}(q,\theta)e(t)$, with $\tilde{H}(q,\theta)$, monic, proper, real, rational, stable and inversely stable transfer function. Hence, we can define the one-step ahead optimal predictor as in (9), from which we can recover the modules $G_{ji}(q,\theta)$. Therefore, the identification problem is to recover $G(q,\theta)$ from the input-output data.

Figure 4 depicts an example of a dynamic network. In the left side the network is depicted in the graph representation, while in the right side the same network is represented in the more usual way of control systems through a block diagram. As it will become clear in the next section, the graph representation of the network has a close relationship to graph objects from graph theory.

Figure 4 – Two representations of the same network system.

a) Graph representation. b) Block diagram.



Source: The Author.

2.3 Digraphs

In the study of dynamic networks we will find useful to study some topological properties of the interconnection structure. We will associate the structure of the network to a graph object using the tools provided from graph theory. In this section we formally introduce the concept of digraphs, terminology, notation, and other useful tools for these objects. These concepts and terminologies are standard in graph theory and can be found in textbooks such as BANG-JENSEN; GUTIN (2008); DIESTEL (2006).

A graph is defined as a tuple $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a non-empty finite set of elements called vertices (nodes) of the graph $\mathcal{G}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of pairs of elements called edges, which connect the vertices of $\mathcal{G}$. The edges are also called arcs and we refer to $\mathcal{E}$ as edge set. The edges in a graph are unordered pairs. Thus, an edge (j,i) is the same as (i,j) for $j,i \in \mathcal{V}$. The *size* of a graph is the number of vertices in the vertex set $\mathcal{V}$, which is equivalent to the cardinality $|\mathcal{G}| \triangleq |\mathcal{V}|$.

A directed graph (or digraph) is a generalization of the concept of graph, which is also

defined as a tuple $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$. The vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$ are defined similarly as for a graph. However, in a digraph the order of the elements of the edges matter, i.e. $(j, i) \neq (i, j)$ for $j, i \in \mathcal{V}$. Hence, the edges in $\mathcal{E}$ of a digraph are ordered pairs.

For an edge (j, i), we say that this edge leaves vertex i and enters vertex j. We also say that j is an *out-neighbor* of node i and that i is an *in-neighbor* of node j if $(j, i) \in \mathcal{E}$. We define the set of all out-neighbors of node i as $\mathcal{N}_i^+ \triangleq \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. Similarly, let the set of all in-neighbors of node j be defined as $\mathcal{N}_j^- \triangleq \{i \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The set $\mathcal{N}_i^-$ is called the in-neighborhood of i, while the set $\mathcal{N}_j^+$ is called the out-neighborhood of vertex j. A vertex i which has no in-neighbors ($\mathcal{N}_i^- = \emptyset$) is called a *source*, while a vertex j that has no out-neighbors ($\mathcal{N}_j^+ = \emptyset$) is called a *sink*. We denote by $\mathcal{F}$ the set of all sources of a digraph: $\mathcal{F} \triangleq \{i \in \mathcal{V} | \mathcal{N}_i^- = \emptyset\}$. The set of all sinks from a digraph is denoted as $\mathcal{S} \triangleq \{j \in \mathcal{V} | \mathcal{N}_j^+ = \emptyset\}$. Vertices that are neither a source nor a sink are called *internal* vertices. We collect all internal vertices from a digraph in the set $\mathcal{I} \triangleq \{i \in \mathcal{V} | \mathcal{N}_i^+ \neq \emptyset \text{ and } \mathcal{N}_i^- \neq \emptyset\}$.

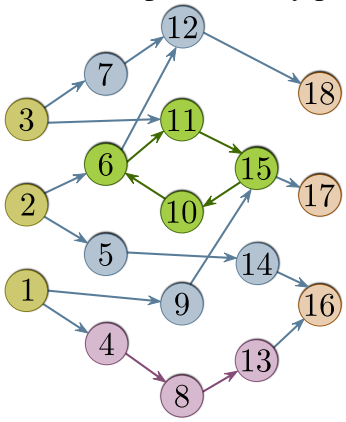
To the network matrix $G^0(q)$ we can associate a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \mathcal{W}$ representing the nodes of the dynamic network and the edges in $\mathcal{E}$ representing the modules of the network. An edge (j, i) is contained in $\mathcal{E}$ if and only if $G_{ji}^0(q) \neq 0$. The vertices of a digraph are to the nodes of a dynamic network as the edges are to the modules. Therefore, in some occasions, we may refer to the identification of an edge when we are actually referring to the identification of a module whose transfer function is associated with this edge.

Now, let us introduce some concepts from graph theory that will be useful in the analysis of dynamic networks. A *path* $\mathcal{P}\,(\mathcal{V}_\mathcal{P}, \mathcal{E}_\mathcal{P})$ is a digraph defined by a sequence of vertices $\mathcal{V}_\mathcal{P} = \{v_1, v_2, v_3, \ldots, v_{k-1}, v_k\}$, such that all vertices are distinct, and the set of edges is $\mathcal{E}_\mathcal{P} = \{(v_2, v_1), (v_3, v_2), \ldots, (v_k, v_{k-1})\}$. A *cycle* is a path in which the last vertex is equal to the first vertex $v_k = v_1$. A pair of paths $\mathcal{P}_1, \mathcal{P}_2$ are said to be mutually vertex-disjoint paths if the two paths share no common vertices: $\mathcal{V}_{\mathcal{P}_1} \cap \mathcal{V}_{\mathcal{P}_2} = \emptyset$. A digraph $\mathcal{G}_s\,(\mathcal{V}_s, \mathcal{E}_s)$ is said to be a *subdigraph* of $\mathcal{G}\,(\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_s \subseteq \mathcal{V}$, and $\mathcal{E}_s \subseteq \mathcal{E}$. That is, all vertices and edges from a subdigraph $\mathcal{G}_s$ must be present in the digraph $\mathcal{G}$. This concept will be particularly useful in the analysis of large networks, for which we can decompose the network matrix using subdigraphs. A digraph is said to be *weakly* connected if there is a path connecting every vertex $i \in \mathcal{V}$ to any other vertex $j \in \mathcal{V}$. A digraph is called *disconnected* if there is a node that is not reachable from any node of the network and has no paths to any other nodes, i.e. a node without neighborhood. All dynamic networks in this thesis will have network matrices with corresponding graphs that are weakly connected.

In order to clarify the terminology used in this thesis, Figure 5 depicts an example of a digraph. In this figure, there are three nodes that are sources: 1, 2, and 3, since they do not have in-neighbors. We collect the sources in the set $\mathcal{F} = \{1, 2, 3\}$. On the other hand, nodes 16, 17 and 18 are sinks, as they do not have any out-neighbors. Similarly,

the nodes classified as sinks are collected in the set $\mathcal{S} = \{16, 17, 18\}$. The set of internal nodes for this digraph is $\mathcal{I} = \{x | x \in \mathbb{N}, 3 < x < 16\}$. The subdigraph formed by the nodes $\mathcal{V}_{\text{path}} = \{4, 8, 13\}$ with the set of edges $\mathcal{E}_{\text{path}} = \{(8, 4), (13, 8)\}$ is a path. A cycle is represented by the subdigraph formed by the set of nodes $\mathcal{V}_{\text{cycle}} = \{6, 10, 11, 15\}$ and the corresponding set of edges $\mathcal{E}_{\text{cycle}} = \{(11, 6), (15, 11), (10, 15), (6, 10)\}$.

Figure 5 – An example of digraph. The set of sources is $\mathcal{F} = \{1, 2, 3\}$, whose nodes are depicted in yellow. The set of sinks is $\mathcal{S} = \{16, 17, 18\}$, whose nodes are depicted in brown. A cycle formed by the nodes 6, 10, 11, and 15 is depicted in the color green. A path formed by the nodes 4, 8 and 13 is represented by pink nodes.



Source: The Author.

In this section, we have presented a number of definitions and terminology from graph theory that will be useful later on when dealing with identifiability of dynamic networks in Chapter 3.

2.4 Conclusions

In this chapter we have presented the mathematical objects that we are going to use in the analysis of dynamic networks. The identification problem for dynamic networks was introduced and the framework for identification of the modules of a dynamic network was presented. A brief overview of the prediction error method was given and the application of this method to dynamic networks was presented. We concluded this chapter with the terminology and concepts from graph theory, which will be particularly useful in the analysis of dynamic networks.

3 IDENTIFIABILITY OF DYNAMIC NETWORKS

In this chapter we address the problem of identifiability in dynamic networks. We start by introducing the concept of identifiability and how this concept is applied for dynamic networks. We proceed by presenting the concept of *generic* identifiability for dynamic networks, on which the analysis of this chapter is based. Afterwards, we formally introduce the problem of generic identifiability for dynamic networks. Then, we pursue an approach for identifiability that is based on the experimental setting applied at the network. We then present necessary and sufficient conditions for generic identifiability of some classes of networks.

3.1 Introduction

Identifiability is a property related to the identification task. Briefly speaking, this property is related to the ability to distinguish between different models that could produce the same data. If two different models produce exactly the same data, we therefore could not hope to distinguish these two models based solely on data. This ability to discern different models is on the core of the identifiability property.

There are many definitions for identifiability in the literature and some of them evolved in different fields of research. In the control community, identifiability was first conceived as the property of an estimator to yield consistent estimates (LJUNG, 1976; BAZANELLA; GEVERS; MIŠKOVIC, 2010). This consistency-based identifiability asserted that a model would be identifiable if it is possible to consistently recover the "true" system. As the field gained maturity and system identification started to be perceived as a method to obtain approximate models from data, the search for a "true" system lost its prominence. The identifiability concept then acquired a uniqueness oriented feature. The question of interest was not anymore to search for the "true" system, rather the objective was to guarantee that no two models could generate the same data. Thus, a model would be unique with respect to a data set. An interesting account of the evolution of the concept of identifiability and an insightful overview were presented in BAZANELLA; GEVERS; MIŠKOVIC (2010).

The original concept of identifiability was separated into two distinct concepts that

allow us to distinguish different models from data: identifiability of the model structure and data informativity. Identifiability of the model structure is related to the question of whether there exists a single parameter vector that uniquely describes a particular model, i.e. two different parameter vectors cannot produce the same model. Notice that this uniqueness-oriented definition of identifiability concerns only the model structure. It is related to the injectivity of the mapping from the parameter space to the model structure. In other words, it is a concept independent of the data. On the other hand, data informativity is related to the experimental conditions on the data set (GEVERS *et al.*, 2009). If any two models within a given model structure do not generate the same predictor (9) – consequently the same prediction error – for a given data set, then we say that data is informative with respect to the given model structure.

For an identification method to obtain consistent estimates it is necessary that the employed model structure is identifiable. Therefore, it is crucial to know whether some model structure is identifiable, since this property is independent of the data. In this chapter, our main interest is to determine under what conditions we can guarantee that a given dynamic network is identifiable.

3.2 Dynamic network identifiability

In this section we introduce the concept of identifiability for dynamic networks as presented in GEVERS; BAZANELLA; PARRAGA (2017). The identifiability problem for dynamic networks consists in testing whether there is a unique network model structure able to represent the "true" network. Let us recall the network identification problem introduced in Section 2.2. We are going to first present the case where there is no measurement noise ($e(t) \equiv 0$) and all nodes are measured ($y(t) = w(t)$). Notice that networks with measurement error can be equivalently represented by a single noise source using the spectral factorization theorem. Hence, the "true" network is described by the following matrix equation:

$$w(t) = G^0(q)w(t) + B^0(q)r(t) + v(t). \tag{26}$$

with $B^0(q)$ an unknown transfer matrix of dimension $n \times n_r$. In this thesis we assume that $B^0(q)$ is a known binary selection matrix. However, in this section, we consider that $B^0(q)$ is an unknown transfer matrix and it is part of the identification problem. As the process noise $v(t)$ has rational spectral matrix, it can be equivalently modeled as:

$$v(t) = H^0(q)e(t). \tag{27}$$

The corresponding input-output representation is given by:

$$\begin{aligned} y(t) &= \underbrace{(I - G^0(q))^{-1}}_{T^0(q)} B^0(q)r(t) + (I - G^0(q))^{-1}H^0(q)v(t) \\ &= T^0(q)B^0(q)r(t) + T^0(q)H^0(q)e(t). \end{aligned} \tag{28}$$

From the prediction error method, we know that this is a MIMO input-output model for which PEM can consistently identify the transfer matrices $T^0(q)B^0(q)$ and $T^0(q)H^0(q)$. A corresponding parametrized input-output model is defined as

$$\mathcal{M}_{io}(\theta) \triangleq (T(q,\theta)B(q,\theta), T(q,\theta)H(q,\theta)) . \tag{29}$$

In this way, the transfer matrices $T^0(q)B^0(q)$ and $T^0(q)H^0(q)$ are uniquely recovered from PEM, and therefore we assume that they are known. Recall that the objective is to estimate the network matrix $G^0(q)$, $B^0(q)$, and $H^0(q)$. The network model is characterized by a triple $\mathcal{M}(\theta) = (G(q,\theta), B(q,\theta), H(q,\theta))$.

The identifiability problem of dynamic networks can be recast into the problem of whether or not we can uniquely recover the network matrix $G^0(q)$, $B^0(q)$, and $H^0(q)$ from knowledge of $T^0(q)B^0(q)$ and $T^0(q)H^0(q)$. This can be solved by a two step procedure. In the first step we consistently estimate the input-output representation from (possibly asymptotic) input-output data. Afterwards, we recover the matrices $G^0(q), B^0(q)$, and $H^0(q)$ from the input-output estimates. We formally give a definition for the identifiability originally introduced in GEVERS; BAZANELLA; PARRAGA (2017).

Definition 3.1. *(GEVERS; BAZANELLA; PARRAGA, 2017) Consider a true network (26) defined by the tuple* $\mathfrak{S} = (G^0(q), B^0(q), H^0(q))$ *and a parametrized model structure* $\mathcal{M}^* = \{\mathcal{M}(\theta) = G(q,\theta), B(q,\theta), H(q,\theta), \theta \in D_\theta\}$*, such that* $\mathcal{M}(\theta^0) = \mathfrak{S}$ *for some* $\theta^0 \in \mathcal{D}_\theta$*. Let* $\mathfrak{S}_{io} = (T^0(q)B^0(q), T^0(q)H^0(q))$ *be the corresponding true input-output network system defined in (28). We say that* $\mathcal{M}(\theta^0)$ *is network identifiable if there exists no other network model structure* $\tilde{\mathcal{M}}^* = \{\tilde{M}(\tilde{\theta}) = (\tilde{G}(q,\tilde{\theta}), \tilde{B}(q,\tilde{\theta}, \tilde{H}(q,\tilde{\theta})), \tilde{\theta} \in \tilde{\mathcal{D}}_{\tilde{\theta}}\}$ *such that* $C(I - \tilde{G}(q,\tilde{\theta}^0))^{-1}\tilde{B}(q,\tilde{\theta}^0) = T^0(q)B^0(q)$ *and* $C(I - \tilde{G}(q,\tilde{\theta}^0))^{-1}\tilde{H}(q,\tilde{\theta}^0) = T^0(q)H^0(q)$ *for some* $\tilde{\theta}^0 \in \tilde{\mathcal{D}}_{\tilde{\theta}}$ *with* $(\tilde{G}(q,\tilde{\theta}), \tilde{B}(q,\tilde{\theta}), \tilde{H}(q,\tilde{\theta})) \neq (G^0(q), B^0(q), H^0(q))$.

We first notice that this definition is related to the mapping of a transfer function space $(G^0(q), B^0(q), H^0(q))$ to another transfer function space $(T^0(q)B^0(q), T^0(q)H^0(q))$. This definition is based on the injectivity between these two spaces of transfer functions. The usual definition of identifiability in LTI system identification is related to the mapping between the parameter space and the transfer function space. Moreover, this definition says that for a network model to be identifiable is required that there are no other models in any other network model structure resulting in the same input-output relationship.

A first key result for identifiability of dynamic networks is that without prior knowledge about the network structure, any compatible network matrix $\tilde{G}(q)$ could generate the same input-output data, see Theorem 5.1 in GEVERS; BAZANELLA; PARRAGA (2017). This means that if two networks $(\tilde{G}(q), \tilde{K}(q), \tilde{H}(q))$ and $(G^0(q), K^0(q), H^0(q))$ are driven by the same signals $\{r(t), e(t)\}$ they would generate the same node signal $\{w(t)\}$. Therefore, these two networks are indistinguishable by using input-output data $\{r(t), w(t)\}$. Hence, a dynamic network (26) is not identifiable without some prior knowledge from the network structure.

An important remark is that Definition 3.1 is independent of any identification method. It depends only on the injectivity of the mapping between the two transfer function spaces. From now on, we are going to assume that $B^0(q)$ is a known binary selection matrix. In the next section we introduce the concept of *generic* identifiability, which will be our main definition for the identifiability analysis.

3.3 Generic identifiability

In this section we formally introduce the concept of generic identifiability that will be used in the analysis of dynamic networks. The concept of generic identifiability was introduced in HENDRICKX; GEVERS; BAZANELLA (2019). It was inspired by the field of structural systems where some properties of the system hold for almost all systems within the same structure, regardless of the numerical values that describe them. A structural system is defined by a set of parameters that describe the physical quantities of the system and its zero/nonzero pattern (DION; COMMAULT; VAN DER WOUDE, 2003). This zero/nonzero pattern is the knowledge associated with how the variables that describe a particular system are related to each other. For a dynamic network, this zero/nonzero pattern corresponds to the topology of the network. To the network topology we can naturally associate a direct graph, as pointed out in Sections 2.2 and 2.3. Let us now formally introduce the concept of a generic property.

Definition 3.2. *A property $\mathbb{P}$ is called **generic** if it holds for almost all elements of a set $\mathcal{A}$, except for those lying on a subset of measure zero.*

Once $\mathbb{P}$ is satisfied for a system, then property $\mathbb{P}$ is true for almost all other systems within the same class. This simplifies the analysis since one can simply verify if a property holds for a single system within the class. Notice that this definition allows a probabilistic interpretation. If $\mathbb{P}$ holds generically and we randomly select an element $a \in \mathcal{A}$, then property $\mathbb{P}$ holds true for a almost surely, i.e. it holds with probability one.

In this thesis we are interested in properties associated with dynamic networks. The concept of a generic property is related to the concept of structural properties in the field of structural systems, first introduced in LIN (1974) for the analysis of controlability in linear systems. Not all properties of a system are generic. One system theoretic property that is not generic is stability. There are other relevant system theoretic properties that are generic, such as controllability and observability (DION; COMMAULT; VAN DER WOUDE, 2003; MONTANARI; AGUIRRE, 2020; MONTANARI *et al.*, 2022). Generic properties are important in the analysis of structured systems because one can perform the analysis entirely based on the topology (structure) instead of the specific numeric values of the system. This means that the analysis can be carried out in the associated digraph that defines the topology. Moreover, there are many graph theoretic tools that can be used to verify whether a generic property holds. The property that we are interested here is

identifiability, which has also been shown to be a generic property for a given network topology in HENDRICKX; GEVERS; BAZANELLA (2019).

In HENDRICKX; GEVERS; BAZANELLA (2019) the problem of identifiability in dynamic networks has been given an algebraic reformulation. It was shown that the problem of identifiability of dynamic networks with partial measurement is equivalent to solving a linear system of transfer functions. Thus, the identifiability problem of finding a unique solution for the modules of the network is then translated into checking the rank of a sub matrix of the input-output relationship. This fundamental link allowed a characterization of the identifiability problem in terms of the topological properties of the associated graph corresponding to the dynamic network.

This new framework was made possible due to an algebraic characterization of the identifiability problem. Under the partial measurement case, we have that

$$\begin{aligned} w(t) &= G^0(q)w(t) + Br(t) + v(t) \\ y(t) &= Cw(t) \end{aligned}, \tag{30}$$

with $B = I$ and C chosen arbitrarily. In this case, the problem is to find a unique $G(q)$ from input-output data such that:

$$CT^0(q)(I - G(q)) = C. \tag{31}$$

Recall that $CT^0(q)$ can be consistently identified from input-output data, the problem of identifiability is thus whether we can find a unique solution $G(q) = G^0(q)$ for (31). In other words, the identifiability problem for dynamic networks consists in uniquely recovering the network matrix $G^0(q)$ from the input-output representation $CT^0(q)$. The problem of obtaining a unique solution for a particular module was characterized in the following lemma.

Lemma 3.1. *(HENDRICKX; GEVERS; BAZANELLA, 2019) Let $\mathcal{N}_i^+$ be the out-neighbors of node i. Let $T^0_{C,\mathcal{N}_i^+}$ denote the restriction of T^0 to the rows selected by C and to the columns corresponding to $\mathcal{N}_i^+$. Let $\Delta \triangleq G(q) - G^0(q)$ and $\Delta_{\mathcal{N}_i^+,i}$ denote a restriction of the i-th column of Δ to the rows selected by $\mathcal{N}_i^+$. Then, G^0_{ji} is identifiable from CT^0 if and only if*

$$T^0_{C,\mathcal{N}_i^+}\Delta_{\mathcal{N}_i^+,i} = 0 \implies \Delta_{ji} = 0. \tag{32}$$

Proof. See HENDRICKX; GEVERS; BAZANELLA (2019). □

The implication in (32) only holds if $T^0_{C,\mathcal{N}_i^+}$ is full row rank. A first result that characterized the rank of a rational transfer matrix in terms of graph theoretical properties was given in VAN DER WOUDE (1991). The author has shown that the rank of a transfer

matrix is associated with the number of vertex-disjoint paths in an associated graph – see Section 2.3. Building up on this concept, the result for general dynamic networks was further extended in HENDRICKX; GEVERS; BAZANELLA (2019). To properly introduce this notion we need to give a parametrization to every module in the network. This can be done as follows

$$G_{ji}(q,\theta) = \theta_1^{ji} q^{-k_{ji}} \frac{q^{n_b^{ji}} + q^{n_b^{ji}-1}\theta_{n_b+n_a+1}^{ji} + \cdots + \theta_{n_a+2}^{ji}}{q^{n_a^{ji}} + q^{n_a^{ji}-1}\theta_{n_a+1}^{ji} + \cdots + \theta_2^{ji}}. \tag{33}$$

for real parameters $\{\theta_k^{ji}\}$. Let us collect all parameters in the vector $\theta \in \mathbb{R}^d$ and denote $G(q,\theta)$ the transfer function obtained by a specific parameter. For a dynamic network, a property is said to hold generically for $G^0(q)$ if, for any network matrix $G(q,\theta)$ consistent with the graph associated with $G^0(q)$, the property holds for $G(q,\theta)$ for all parameters θ except possibly those lying on a zero measure set on $\mathbb{R}^d$. In a similar fashion, we say that a property is generic for $T^0(q) = (I - G^0(q))^{-1}$ if for any parametrization $G(q,\theta)$ consistent with the graph associated to $G^0(q)$ it holds for $T(q,\theta) = (I - G(q,\theta))^{-1}$ for almost all θ. We are now ready to define the concept of generic identifiability for a dynamic network. First, the following lemma shows that for any parametrization consistent with $G^0(q)$ if there exists a parametrization such that a property holds, then this property holds for almost all parametrizations.

Lemma 3.2. *(HENDRICKX; GEVERS; BAZANELLA, 2019). Let $Q(\cdot) : \mathbb{C}^{n\times n} \to \mathbb{C}$ be an analytic function and consider a network matrix $G^0(z)$. If there exists a matrix $A \in \mathbb{C}^{n\times n}$ consistent with the graph associated to $G^0(z)$ such that $Q(A) \neq 0$, then $Q(G^0(z))$ is generically not identically zero as a function of z (for polynomial or rational $Q(\cdot)$, it then has finitely many roots). Otherwise, $Q(G(z)) \equiv 0$ for every $G(z)$ consistent with the directed graph.*

When we use this lemma with the determinant of $(I - G(\theta,z))^{-1}$ as the function $Q(\cdot)$ we see that the rank of $T^0(z)$ is a generic property. Since identifiability is dependent on the rank of a subset of rows and columns of $T^0(q)$ as shown in Lemma 3.1, we have that identifiability of dynamic networks is a generic property for a given network topology. Before introducing the concept of generic identifiability, recall that $\mathcal{B}$ denote the set of excited nodes and $\mathcal{C}$ denote the set of measured nodes. We can now state the concept of generic identifiability for dynamic networks as posed in HENDRICKX; GEVERS; BAZANELLA (2019).

Definition 3.3. *(HENDRICKX; GEVERS; BAZANELLA, 2019). A network matrix $G^0(q)$ is* generically *identifiable from a set of measured nodes defined by $\mathcal{C}$ if, for any rational transfer matrix parametrization $G(\theta,q)$ consistent with the directed graph associated to $G^0(q)$, there holds*

$$C(I - G(\theta,q))^{-1} = C(I - G^0(q))^{-1} \implies G(\theta,q) = G^0(q), \tag{34}$$

for almost all parameters θ except possibly those lying on a zero measure set in $\mathbb{R}^d$.

This definition given in HENDRICKX; GEVERS; BAZANELLA (2019) was motivated by the problem of partial measurement. That work was the first to address identifiability using either partial measurement or the dual problem of partial excitation. Definition 3.3 can be naturally extended to the case of *partial excitation and partial measurement*. An extension was formally introduced in BAZANELLA; GEVERS; HENDRICKS (2019) and is defined in the following.

Definition 3.4. *(BAZANELLA; GEVERS; HENDRICKS, 2019). The network matrix $G^0(q)$ is generically identifiable from excitation signals applied in $\mathcal{B}$ and measurements made at $\mathcal{C}$ if, for any rational transfer matrix parametrization $G(\theta, q)$ consistent with the directed graph associated to $G^0(q)$, there holds*

$$C(I - G(\theta, q))^{-1}B = C(I - G^0(q))^{-1}B \implies G(\theta, q) = G^0(q). \tag{35}$$

for all parameters θ except possibly those lying on a zero measure set in $\mathbb{R}^d$, where $G(q, \theta)$ is any network matrix consistent with the graph.

Recall that $CT^0(q)B$ can be consistently identified from MIMO identification techniques using input-output data provided the model structure is general enough to contain the "true" network and if a sufficiently rich input signal is applied (LJUNG, 1999). Hence, for any pair of selections matrices B and C we can recover the input-output relationship of the network. The problem is then recast into an algebraic problem to verify whether or not we can uniquely recover the modules in the network matrix $G^0(q)$ from knowledge of $CT^0(q)B$. From now on, we are going to assume that $CT^0(q)B$ is known, since this relationship can be consistently identified from input-output data. Definition 3.4 of identifiability will be adopted as the main concept of identifiability in dynamic networks throughout this thesis. An important remark is that the noise process can contribute to the identifiability of the modules of the network through the spectrum $\Phi_v(z)$. In this chapter, the focus will be on identifiability as stated in Definition 3.4.

Now, let us give an example that illustrates how these concepts can be applied and what *almost all* parameters in the definition of generic identifiability means.

Example 3.1. *(Adapted from HENDRICKX; GEVERS; BAZANELLA (2019)). Consider the following network matrix:*

$$G^0(q) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ G^0_{21}(q) & 0 & 0 & 0 & 0 \\ G^0_{31}(q) & 0 & 0 & 0 & 0 \\ 0 & G^0_{42}(q) & G^0_{43}(q) & 0 & 0 \\ 0 & G^0_{52}(q) & G^0_{53}(q) & 0 & 0 \end{bmatrix}.$$

Suppose that nodes 1, 2 and 3 are excited, while nodes 4 and 5 are measured. This means that $\mathcal{B} = \{1, 2, 3\}$ *and* $\mathcal{C} = \{4, 5\}$*, from which the associated matrices* B *and* C *can be formed. We need to verify whether the implication in (35) is satisfied. For this purpose, consider any parametrization* $G(\theta, q) \triangleq G$ *consistent with the directed graph associated with* $G^0(q)$*. Using Definition 3.4 we can write omitting the arguments:*

$$C(I - G)^{-1}B = CT^0B$$
$$\begin{bmatrix} G_{42}G_{21} + G_{43}G_{31} & G_{42} & G_{43} \\ G_{52}G_{21} + G_{53}G_{31} & G_{52} & G_{53} \end{bmatrix} = \begin{bmatrix} T^0_{41} & T^0_{42} & T^0_{43} \\ T^0_{51} & T^0_{52} & T^0_{53} \end{bmatrix}, \tag{36}$$

from which we can uniquely recover

$$G_{42} = T^0_{42} = G^0_{42},$$
$$G_{43} = T^0_{43} = G^0_{43},$$
$$G_{52} = T^0_{52} = G^0_{52},$$
$$G_{53} = T^0_{53} = G^0_{53}.$$

Therefore, equation (36) implies that $G_{42} = G^0_{42}$*,* $G_{43} = G^0_{43}$*,* $G_{52} = G^0_{52}$*, and* $G_{53} = G^0_{53}$*. It only remains to verify under what conditions the remaining modules* (G_{21}, G_{31}) *can be uniquely recovered. This can be done by using the information obtained from the other relationships* (T^0_{41}, T^0_{51}) *in (36). The remaining unknown modules can be identified from the following linear system:*

$$\begin{bmatrix} G^0_{42} & G^0_{43} \\ G^0_{52} & G^0_{53} \end{bmatrix} \begin{bmatrix} G_{21} \\ G_{31} \end{bmatrix} = \begin{bmatrix} T^0_{41} \\ T^0_{51} \end{bmatrix}.$$

The remaining modules can be uniquely recovered from this linear system if and only if the transfer matrix in the left hand side is full rank, or equivalently if

$$G_{42}(q, \theta_0)G_{53}(q, \theta_0) \neq G_{52}(q, \theta_0)G_{43}(q, \theta_0). \tag{37}$$

This means that $G^0(q)$ *is identifiable from measurements made at nodes* $\{4, 5\}$ *and excitations applied at nodes* $\{1, 2, 3\}$ *for almost all parameters, except for those that obey* $G^0_{42}G^0_{53} = G^0_{52}G^0_{43}$ *. Hence,* $G^0(q)$ *is generically identifiable.*

What this example shows is that for some networks we can uniquely recover the modules of a network, except for a subset that obeys very particular equalities. The use of the concept of generic identifiability is very useful, as it is linked to concepts from graph theoretic tools. The first major advantage is that one does not need to be concerned with the particular numeric values of θ. Once any particular θ is shown to yield identifiability, then this property holds for almost all parameters values. Secondly, the verification of identifiability is translated from verifying the rank of some matrices to verifying the

number of vertex-disjoint paths from the associated directed graph of the network matrix (HENDRICKX; GEVERS; BAZANELLA, 2019). This is particularly useful for large scale networks as verification can be performed in polynomial time. Several graph-theoretic conditions are given in HENDRICKX; GEVERS; BAZANELLA (2019) for the case where all nodes are excited but only a subset of nodes are measured and for the dual problem of full measurement and partial excitation. Within the same framework, in VAN WAARDE; TESI; CAMLIBEL (2019) a graph simplification process is introduced and, based on it, necessary and sufficient topological conditions were presented for the stronger notion of *global identifiability*. Briefly speaking, this definition of identifiability concerns the ability to recover the network matrix *for all* parametrizations as opposed to the concept of generic identifiability where the property holds for almost all parametrizations. The concept of generic identifiability was further extended to cope with the other definitions from literature. See for instance SHI; CHENG; VAN DEN HOF (2022). The main objective of these works is to give topological conditions under which the network matrix is generically identifiable.

An alternative approach to the identifiability problem for dynamic networks is to determine what conditions on the experimental setting are necessary to guarantee generic identifiability of the dynamic network. These conditions do not take the form of topological conditions, rather the focus is to determine which nodes need to be excited and which nodes need to measured such that the experiment render the network identifiable. That is, the conditions for identifiability take the form of excitation/measurement conditions on the nodes of the network. Thus, this approach consists of solving a synthesis problem: how to allocate excitation signals and measurements that render a network generic identifiable. This approach is specially useful from an experiment design perspective, since one must decide which nodes to measure and which ones to excite before performing the identification experiment. This new look at the identifiability problem for dynamic networks was formally stated in MAPURUNGA; BAZANELLA (2021a).

From now on, we are going to focus on this alternative approach. When dealing with the problem of identifiability, we are going to determine what are the experimental conditions on the nodes that need to be excited and on the nodes that need to be measured such that the network can be uniquely identified. For this purpose, let us introduce the following definition that characterizes an excitation and measurement pattern (EMP) for a dynamic network.

Definition 3.5. *(MAPURUNGA; BAZANELLA, 2021a). Let $G^0(q)$ be a network matrix, for which different selection matrices are considered. A pair of selection matrices B and C, with its corresponding node sets $\mathcal{B}$ and $\mathcal{C}$, is called an* ***excitation and measurement pattern*** *- EMP, for short. An EMP is said to be a* ***valid*** *EMP if it is such that the network (30) is generically identifiable. Define $\mid EMP \mid = |\mathcal{B}| + |\mathcal{C}|$[1] as the* cardinality *of an EMP.*

[1] $|\cdot|$ standing for the cardinality of a set.

*A given EMP is said to be a **minimal EMP** if it is valid and there is no valid EMP with smaller cardinality.*

This definition establishes the framework for which we are going to tackle the problem of identifiability in dynamic networks. One possible approach is to characterize all minimal EMPs for a given network. This approach is also useful from an experiment design perspective, since one can identify the network modules using the minimum number of external excitation signals and measurements combined. We are going to first focus on determining conditions for the minimal EMPs, and from that we are going to discuss how to further extend these conditions such that all valid EMPs are characterized. Usually, the extension from minimal EMPs to valid EMPs is straightforward because a valid EMP can be obtained from minimal EMPs by adding an excitation or measurement at a particular node.

In fact the requirement for excitation or measurement of some nodes depends on the local topology of the node, i.e. its neighborhood. In order to render a network generically identifiable, some nodes will need to be excited, while other nodes will need to be measured. The next result presents a necessary condition for two types of nodes: sources and sinks.

Theorem 3.1. *(BAZANELLA et al., 2017) The network matrix $G^0(q)$ is generically identifiable from excitation signals applied to $\mathcal{B}$ and measurements made at $\mathcal{C}$, only if $\mathcal{F} \subseteq \mathcal{B}$ and $\mathcal{S} \subseteq \mathcal{C}$, i.e. all sources are excited and all sinks are measured.*

This theorem gives a necessary condition for generic identifiability of a dynamic network. If a source node is not excited, then one can not uniquely recover the out-neighbors of this source node. A dual argument is valid for sinks, if they are not measured, then we can not uniquely recover their in-neighbors. In other words, all sources from a network must be excited, while all sinks need to be measured. Hence, in order for an EMP to be valid it must obey these excitation and measurement conditions for sources and sinks. The excitation of a sink provides no useful information from which we can recover the modules from the available equations. The dual argument is valid for sources, measuring a source provides no useful information about the modules.

We have seen that sources need to be excited and sinks need to be measured. What more can we say about the necessity of exciting or measuring a given particular node? When any particular node is considered, the following theorem gives a necessary condition for the identifiability of the modules that enter and leave any particular node.

Theorem 3.2. *(BAZANELLA; GEVERS; HENDRICKS, 2019) If a node is neither excited nor measured, then none of its corresponding incoming and outgoing modules are identifiable.*

This theorem presented a necessary condition for identifiability of the in(out)-neighbors of any node in a network. This means that every node must participate in the EMP if

the objective is to render the network identifiable. In other words, each node of the network must be either excited or measured to guarantee that the corresponding modules are identifiable. Hence, the cardinality of a minimal EMP must be at least equal to the number of nodes in the network, otherwise this EMP would not allow the recovery of some modules of the network.

To summarize the findings of this section. We have given an overview of some key previously known results for the identifiability problem for dynamic networks. Particularly, we have presented that the identifiability problem of a dynamic network can be recast into an algebraic problem, for which the solution depends on the rank of certain submatrices. Moreover, we have presented results that show that identifiability is a generic property for a given network topology, that is, it is valid for almost all parametrizations consistent with the digraph associated to the network matrix. This concept was formalized by the definition of generic identifiability. An attractive feature of this concept is that allowed it to provide conditions based on the topology of the network, instead of the rank of certain matrices. We have formulated an alternative approach that consists in determining the identifiability conditions based on the experimental setting – EMPs – of the network. Based on this approach, we have formally introduced the concept of valid EMP: one that renders the network generic identifiable; and minimal EMP: a valid EMP with minimum cardinality among all valid EMPs. We have presented results for excitation and measurement of a network. Any valid EMP must obey the following conditions:

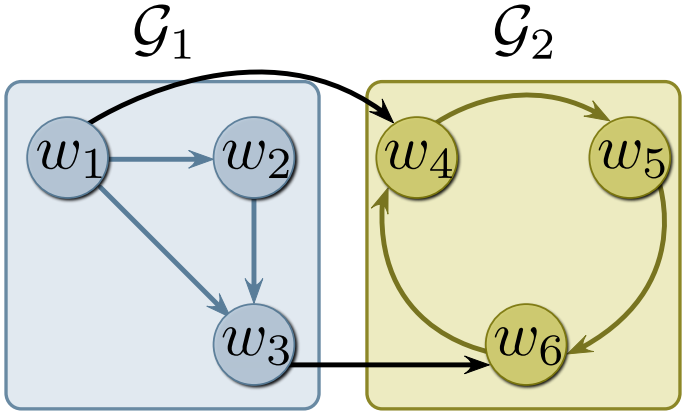
- Any source must be excited: $\mathcal{F} \subseteq \mathcal{B}$;
- Any sink must be measured: $\mathcal{S} \subseteq \mathcal{C}$;
- Every node must be either excited or measured: $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$.

3.4 Combining structures

In this section we present conditions to infer whether a dynamic network is generically identifiable by separating the network into well-defined internal structures with known identifiability conditions and their interconnections. The question of interest is which additional conditions may be required for the generic identifiability of the whole network given that it can be subdivided into known structures, each of which is generically identifiable.

We consider that a dynamic network can be split up into two particular structures, each being associated to a directed graph $\mathcal{G}_j = (\mathcal{W}_j, \mathcal{E}_j)$, for $j = 1, 2$. We assume that these two structures have at least two nodes each, that they share no nodes between them ($\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$), that there may exist edges connecting $\mathcal{G}_1$ to $\mathcal{G}_2$, but none from $\mathcal{G}_2$ to $\mathcal{G}_1$. Figure 6 depicts an example of network subdivided into two subnetworks.

Figure 6 – An example of a dynamic network divided into two subnetworks with no shared nodes.



Source: The author.

Thus, the network matrix can be partitioned as:

$$G^0(q) = \begin{bmatrix} G_1^0(q) & 0 \\ G_{21}^0(q) & G_2^0(q) \end{bmatrix}, \tag{38}$$

where $G_i^0(q) \in \mathbb{C}^{n_i \times n_i}$ are the network matrices associated with the graphs $\mathcal{G}_i$ for $i = 1, 2$. The matrix $G_{21}^0(q) \in \mathbb{C}^{n_2 \times n_1}$ represents the edges that connect the first part of the network to the second part.

From the network matrix (38) we have that:

$$T^0(q) = \begin{bmatrix} T_1^0(q) & 0 \\ T_{21}^0(q) & T_2^0(q) \end{bmatrix} = \left(I_{n_1+n_2} - G^0(q)\right)^{-1} = \\ \begin{bmatrix} \left(I_{n_1} - G_1^0(q)\right)^{-1} & 0 \\ \left(I_{n_2} - G_2^0(q)\right)^{-1} G_{21}^0(q) \left(I_{n_1} - G_1^0(q)\right)^{-1} & \left(I_{n_2} - G_2^0(q)\right)^{-1} \end{bmatrix}, \tag{39}$$

where $T_1^0(q)$ and $T_2^0(q)$ are nonsingular. The IO representation of this network is given by:

$$\begin{aligned} CT^0(q)B &= \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} T_1^0(q) & 0 \\ T_{21}^0(q) & T_2^0(q) \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 T_1^0(q) B_1 & 0 \\ C_2 T_{21}^0(q) B_1 & C_2 T_2^0(q) B_2 \end{bmatrix}, \end{aligned} \tag{40}$$

where $B_1 \in \mathbb{Z}_2^{n_1 \times m_1}$, $C_1 \in \mathbb{Z}_2^{p_1 \times n_1}$, $B_2 \in \mathbb{Z}_2^{n_2 \times m_2}$, and $C_2 \in \mathbb{Z}_2^{p_2 \times n_2}$ are the selection matrices corresponding to each part of the network.

By hypothesis, it is possible to recover all modules of $G_1^0(q)$ and $G_2^0(q)$, since we assume that both parts are generically identifiable, that is, we can recover $G_1^0(q)$ from $C_1 T_1^0(q) B_1$ and $G_2^0(q)$ from $C_2 T_2^0(q) B_2$. Knowledge of $G_{21}^0(q)$ can only be recovered

from $C_2T_{21}^0(q)B_1$ where $T_{21}^0(q)$ is given by the bottom-left part of (39). From the IO model (40) and (39) we have:

$$C_2T_2^0(q)G_{21}^0(q)T_1^0(q)B_1 = C_2T_{21}^0(q)B_1. \tag{41}$$

Therefore, the problem is whether we can identify $G_{21}^0(q)$ from (41) given that $C_2T_2^0(q)$, $T_1^0(q)B_1$ and $C_2T_{21}^0(q)B_1$ are known, by knowledge of $G_1^0(q), G_2^0(q)$, and hence $T_1^0(q)$, $T_2^0(q)$. Whether $G_{21}^0(q)$ is generically identifiable depends on the topology defined by the interconnection between the two subnetworks. Therefore, the problem we investigate in this section is equivalent to determining which additional measurements may need to be made at the nodes of $\mathcal{G}_2$ and which additional nodes of $\mathcal{G}_1$ may need to be excited in order to guarantee that $G_{21}^0(q)$ is generically identifiable using (41).

The next theorem states a number of results regarding the conditions under which $G_{21}^0(q)$ is generically identifiable from the excitations in $\mathcal{G}_1$ and measurements at $\mathcal{G}_2$. Before presenting the results, we will introduce two concepts that will be useful in the following.

Definition 3.6. *The Kruskal rank of a matrix $A \in \mathbb{C}^{I\times L}$ denoted by k_A is the maximum value of k such that any k columns of A are linearly independent.*

Notice that $k_A = 1$ if A has at least two colinear columns and $k_A = 0$ if and only if it has a zero column. Finally, we call two edges $G_{ji}^0(q)$, $G_{kl}^0(q)$ *vertex-disjoint* if $j \neq k$ and $i \neq l$. Notice that this is a particular case of vertex-disjoint paths, see Section 2.3. Now, we are ready to state the identifiability results when the unknowns in $G_{21}^0(q)$ have a specific structure.

Theorem 3.3. *Consider a dynamic network with network matrix (38) divided into two substructures that are generically identifiable with a given EMP, and let d_k be the number of edges connecting them, i.e. the number of non-zero elements of $G_{21}^0(q)$. The following conditions hold.*

(a) The edges of $G_{21}^0(q)$ can be recovered from (41) only if $m_1p_2 \geq d_k$;

(b) If all nodes of $\mathcal{G}_1$ are connected to all nodes of $\mathcal{G}_2$, then all edges of $G_{21}^0(q)$ are generically identifiable if and only if all nodes of $\mathcal{G}_1$ are excited and all nodes of $\mathcal{G}_2$ are measured;

(c) If only one node j in $\mathcal{G}_1$ is connected to $\mathcal{G}_2$, and it is connected to all of its nodes, then $G_{21}^0(q)$ is generically identifiable if and only if all nodes of $\mathcal{G}_2$ are measured and the corresponding jth row of $T_1^0(q)B_1$ is not identically zero;

(d) If all nodes of $\mathcal{G}_1$are connected to node j in $\mathcal{G}_2$ and the nodes of $\mathcal{G}_1$ are not connected to any other node of $\mathcal{G}_2$, then $G_{21}^0(q)$ is generically identifiable if and only if all nodes of $\mathcal{G}_1$ are excited and the corresponding jth column of $C_2T_2^0(q)$ is not identically zero;

(e) Let all d_k edges from $\mathcal{G}_1$ to $\mathcal{G}_2$ be vertex-disjoint, then $G_{21}^0(q)$ is generically identifiable if $m_1 p_2 \geq d_k$ and $k_{(T_1^0(q)B_1)^T} + k_{C_2T_2} \geq d_k + 1$.

Proof. We can rewrite (41) using the Kronecker product:

$$\left([T_1^0(q)B_1]^T \otimes C_2T_2^0(q)\right) \mathrm{vec}(G_{21}^0(q)) = \mathrm{vec}(C_2T_{21}^0(q)B_1), \tag{42}$$

where $\mathrm{vec}(\cdot)$ is the operation that stacks the columns of a matrix into a single column. A restriction on the columns of $B_1^T T_1^{0T}(q) \otimes C_2T_2^0(q)$ is defined by removing the columns corresponding to the $n_1n_2 - d_k$ known zeros of $G_{21}^0(q)$. The corresponding linear system can be written as $Ax = b$ with A a rational matrix of dimension $m_1p_2 \times d_k$. A unique solution is obtained only if A is generically full column rank. Thus, if $m_1p_2 < d_k$ no unique solution is obtained, which proves item (a).

The scenario stated in item (b) implies $d_k = n_1n_2$. In this case, a unique solution is obtained if and only if $\mathrm{rank}\left(B_1^T T_1^{0T}(q) \otimes C_2T_2^0(q)\right) = n_1n_2$. Note that, by a property of the Kronecker product, we have

$$\mathrm{rank}\left(B_1^T T_1^{0T}(q) \otimes C_2T_2^0(q)\right) = \mathrm{rank}(T_1^0(q)B_1)\mathrm{rank}(C_2T_2^0(q)) = m_1p_2,$$

because $T_1^0(q)$ and $T_2^0(q)$ are nonsingular. Thus, if all elements of $G_{21}^0(q)$ are unknown, they can be recovered if and only if $m_1 = n_1$ and $p_2 = n_2$, which means that $B_1 = I_{n_1}$ and $C_2 = I_{n_2}$, i.e. all nodes of $\mathcal{G}_1$ are excited and all nodes of $\mathcal{G}_2$ are measured.

Now, let us prove item (c). Since node j in $\mathcal{G}_1$ is the only node connected to $\mathcal{G}_2$ and since it is connected to all its nodes, it means that only the jth column of $G_{21}^0(q)$ is nonzero with all its elements being nonzero. Hence, there are exactly $d_k = n_2$ unknowns in $G_{21}^0(q)$. We can rewrite the expression $(T_1^0(q)B_1)^T \otimes C_2T_2^0(q)$ in (42) as a function of its columns:

$$\begin{aligned}&\Big[[(T_1^0(q)B_1)^T]_1 \otimes [C_2T_2^0(q)]_1 \quad \cdots \quad [(T_1^0(q)B_1)^T]_1 \otimes [C_2T_2^0(q)]_{n_2} \quad \cdots \\ &\cdots \quad [(T_1^0(q)B_1)^T]_{n_1} \otimes [C_2T_2^0(q)]_{n_2}\Big],\end{aligned} \tag{43}$$

where $[\cdot]_i$ is used to denote the ith column. Removing the columns with known zeros in $G_{21}^0(q)$ we obtain from (43) the following:

$$\begin{aligned}&\Big[[(T_1^0(q)B_1)^T]_j \otimes [C_2T_2^0(q)]_1 \quad \cdots [(T_1^0(q)B_1)^T]_j \otimes [C_2T_2^0(q)]_{n_2}\Big] = \\ &[(T_1^0(q)B_1)^T]_j \otimes C_2T_2^0(q).\end{aligned} \tag{44}$$

Using the Kronecker rank property, a unique solution is obtained if and only if

$$\mathrm{rank}([(T_1^0(q)B_1)^T]_j)\,\mathrm{rank}(C_2T_2^0(q)) = n_2. \tag{45}$$

This means, equivalently, that $[(T_1^0(q)B_1)^T]_j$ is not identically zero and that $C_2 = I_{n_2}$, because $T_2^0(q)$ is nonsingular. Hence, $G_{21}^0(q)$ is generically identifiable if and only if $G_2^0(q)$

is fully measured and column $[(T_1^0(q)B_1)^T]_j$ is not identically zero. The proof of item (d) follows from duality with result (c).

Finally, consider item (e). It follows from the assumptions that each row of $G_{21}^0(q)$ contains at most one non-zero element. By a relabeling of the nodes, $G_{21}^0(q)$, of size $n_2 \times n_1$, can always be rewritten in a form such that the upper-left block is a diagonal matrix of dimension d_k, with $d_k \leq \min(n_1, n_2)$, while all other elements are zero. Eliminating the columns corresponding to the known zeros of $G_{21}^0(q)$ in (43) yields:

$$\begin{bmatrix} [B_1^T T_1^{0T}(q)]_1 \otimes [C_2 T_2^0(q)]_1 & [B_1^T T_1^{0T}(q)]_2 \otimes [C_2 T_2^0(q)]_2 & \cdots \\ [B_1^T T_1^{0T}(q)]_{d_k} \otimes [C_2 T_2^0(q)]_{d_k} \end{bmatrix} = [B_1^T T_1^{0T}(q)]_{1:d_k} \odot [C_2 T_2^0(q)]_{1:d_k}, \tag{46}$$

where $\odot$ represents the column-wise Khatri-Rao product (BREWER, 1978). A useful result for this kind of operation was given in Lemma 1 of SIDIROPOULOS; BRO; GIANNAKIS (2000), which states that $A \odot B$ is full column rank if $k_A + k_B \geq R + 1$ for $A \in \mathbb{K}^{S \times R}$ and $B \in \mathbb{K}^{L \times R}$. Applying this result in (46) we have that the resulting linear system is generically full column rank under the conditions stated. □

It is interesting to note that condition (e) in Theorem 3.3 depends on the structure of each part of the network. The results from Theorem 3.3 give a motivation for searching identifiability conditions for particular structures, as one can combine them to identify some modules and eventually the whole network. The next results present sufficient conditions for the case where no additional conditions for generic identifiability are required for the network.

Corollary 3.1. *Let a network be divided into two subnetworks as in (38), with $G_{21}^0(q)$ containing only vertex-disjoint edges. The whole network is generically identifiable if a valid EMP is applied at each subnetwork and all nodes of subnetwork $\mathcal{G}_1$ are excited or all nodes of subnetwork $\mathcal{G}_2$ are measured.*

Proof. All modules within each subnetwork can be recovered since a valid EMP is applied at each subnetwork. It remains to identify the interconnection structure between the subnetworks. Under the assumptions stated, the interconnection $G_{21}^0(q)$ has $d_k = \min(n_1, n_2)$ unknowns. If all nodes of $G_1^0(q)$ are excited or all nodes of $G_2^0(q)$ are measured, then item (e) from Theorem 3.3 is satisfied. □

This result shows that for subnetworks interconnected through vertex-disjoint edges, in addition to applying a valid EMP in each part of the network, it is sufficient to either excite all nodes of subnetwork $\mathcal{G}_1$ or to measure all nodes from subnetwork $\mathcal{G}_2$. The next corollary shows that when there is only one edge connecting the two subnetworks, then it is enough to apply a valid EMP in each subnetwork.

Corollary 3.2. *Let a network be divided into two subnetworks as in (38), with $G_{21}^{0}(q)$ containing only one nonzero element. The whole network is generically identifiable if each subnetwork is subjected to a valid EMP.*

Proof. Since each valid EMP must have at least one excitation and one measurement, the result follows directly from item (e) from Theorem 3.3 with $d_k = 1$. □

These results can be used within a divide and conquer strategy: divide the networks into subnetworks with known identifiability conditions, and from that determine the additional requirements. One may combine known identifiability results, and repeating the procedure for each known subnetwork one can identify the whole network. We have shown that the fundamental problem for identifiability when we apply valid EMPs to the substructures is to recover the interconnection structure between the two subnetworks. This indicates that if the network matrix can be written in a lower triangular form, then by applying a valid EMP in each subnetwork, the only edges that we need to identify are the local edges that connect the two subnetworks. The identifiability conditions on the experimental setting depend upon the nodes that are excited in the first subnetwork and the nodes that are measured in the second subnetwork. In the next section, we are going to investigate identifiability conditions for particular classes of acyclic networks.

3.5 Identifiability of acyclic dynamic networks

In this section, we investigate the identifiability conditions based on the experimental setting approach introduced in the previous section for some classes of networks, namely trees, parallel networks, and directed acyclic graphs (DAGs). Some conditions are derived for these classes of networks based on the concept of valid EMPs. The approach adopted here is to pursue identifiability conditions for specific classes of network in order to be used in more general networks. In this way, one can build upon existing results for structures within a more complex dynamic network.

3.5.1 Trees

In this section, we consider the class of dynamic networks whose corresponding graph is a tree. A tree is a connected directed graph which contains no cycles, even if the order of the edges are not considered, and every node must be connected to another node by at most one path. We are going to assume that the nodes have been labeled sequentially, such that there is an edge (j, i) in the associated digraph only if $j > i$. Notice that the nodes of any tree can be labeled sequentially through a sorting algorithm (KAHN, 1962). An example of a tree is depicted in Figure 7. A formal definition is given as follows.

Definition 3.7. *A tree is a weakly connected graph, with the following properties: there are no cycles, every node is connected to any other node by at most one path, and it is*

minimally connected – removal of any edge makes the tree to be disconnected.

As a tree is minimally connected, there are exactly $n-1$ edges in a tree with n nodes. The following theorem states necessary and sufficient conditions for generic identifiability of a tree.

Theorem 3.4. *(BAZANELLA; GEVERS; HENDRICKS, 2019). A tree is generically identifiable if and only if the following conditions hold:*

- $\mathcal{F} \subseteq \mathcal{B}$*: all sources are excited;*
- $\mathcal{S} \subseteq \mathcal{C}$*: all sinks are measured;*
- $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$*: every node is either excited or measured;*

This theorem states that for a tree the necessary conditions on the EMP for any network are also sufficient. Therefore, a minimal EMP for a tree has cardinality n and all valid EMPs can be obtained from the minimal EMPs by simply adding more excitations and/or measurements. A tree is composed of *branches*, which are trees with only one source and one sink. The associated digraph of a branch is also called line or path. A formal definition of a branch is given in the following.

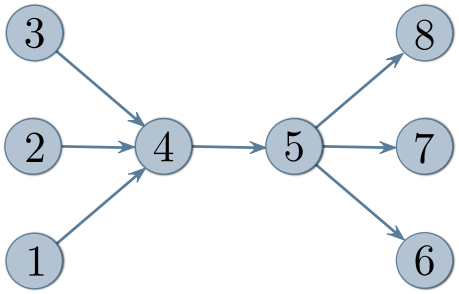
Definition 3.8. *A branch is a tree with only one source and one sink.*

A direct application of Theorem 3.4 to the case of branches results in the following corollary.

Corollary 3.3. *(BAZANELLA; GEVERS; HENDRICKS, 2019). A branch is generically identifiable if and only if the following conditions hold:*

- *the source is excited;*
- *the sink is measured;*
- *every node is either excited or measured.*

Figure 7 – An example of tree network.



Source: The Author.

In the next chapter we will analyze the accuracy of the estimates for networks with the topology of a branch. This corollary characterizes all minimal EMPs for a branch. These results produce the basic building blocks for minimal EMPs in dynamic networks with tree/branch structure. For a tree network, only the sinks and sources need a specific requirement of being measured and excited, respectively. In order to determine the number of minimal EMPs we are going to recall the concept of internal node – see Section 2.3.

Definition 3.9. *An **internal** node is one in which there is at least one in-neighbor and one out-neighbor.*

An internal node is any node that is neither a source nor a sink. Let m be the total number of internal nodes from a tree: $m \triangleq |\mathcal{W}| - |\mathcal{F}| - |\mathcal{S}|$. The number of minimal EMPs for a tree is 2^m, and specifically for a branch is 2^{n-2}.

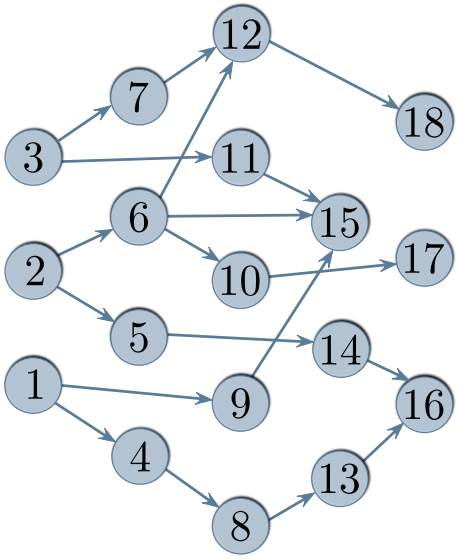
A more general network topology is a multitree, defined as follows.

Definition 3.10. *A multitree is a weakly connected graph in which there are no cycles and any two nodes are connected by at most one path.*

A multitree is a more general graph for which a tree is special case. The key difference between these two structures is that a multitree is not restricted to be minimally connected as a tree. In Figure 8 is shown an example of multitree network.

It is straightforward to extend the results presented for trees in Theorem 3.4 to multitrees. In fact, there is no additional experimental requirements for mutitrees and the same procedure to identify trees can be applied to multitrees. We formally state necessary and sufficient conditions for identifiability of dynamic networks whose topology correspond to a multitree in the following.

Figure 8 – An example of multitree network.



Source: The author.

Theorem 3.5. *A dynamic network with topology of a multitree is generically identifiable if and only if* $\mathcal{F} \subseteq \mathcal{B}$*;* $\mathcal{S} \subseteq \mathcal{C}$*;* $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$*.*

Proof. The stated conditions were shown to be necessary for any dynamic network in Theorems 3.1 and 3.2. To show sufficiency, consider a source in the multitree and all paths that link it to sinks. This subnetwork corresponds to a tree and all its edges can be identified using Theorem 3.4. Apply the same reasoning to every source in the multitree and the result follows. □

This theorem shows that a multitree is generically identifiable under the same conditions in which a tree can be identified. These two structures are very general acyclic graphs, whose identifiability conditions will be useful later on to determine the identifiability of more general networks.

In this section we have presented necessary and sufficient conditions for two classes of networks: branches and trees. The former is a special case of the latter. It turns out that for these classes of networks the necessary conditions that apply for identifiability of any dynamic network are also sufficient. We have introduced the class of multitrees networks. Moreover, we have demonstrated that the conditions under which this network is generically identifiable are exactly the same as for trees. This means that the minimal EMPs for these classes of networks have cardinality equal to the number of nodes in the network.

3.5.2 Parallel networks

In this section we investigate the role which parallel paths play in the generic identifiability of dynamic networks. These results will be important because we can effectively combine previously known results for trees with new conditions for parallel paths. Combining these results, one can obtain general conditions for a wide class of dynamic networks. A formal definition of a parallel network is given in the following.

Definition 3.11. *A parallel network is a network composed of a single source and a single sink. There are* n_p *paths from the source to the sink. These paths share only two nodes: the source and the sink.*

This network represents a network whose two particular nodes are connected by an arbitrary number of paths. Before addressing the identifiability problem for this class of network, we will start the analysis with a simple parallel network in order to gain insight on the excitation and measurement requirements. A simple parallel network is defined as follows.

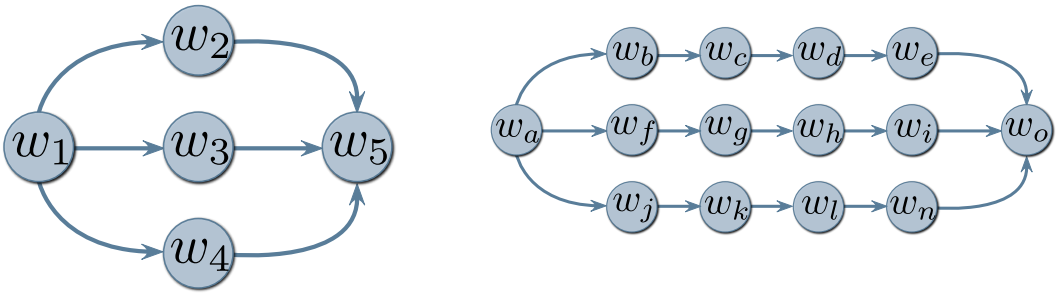
Definition 3.12. *A* ***simple*** *parallel network is a parallel network for which each path from the source node to the sink contains exactly three nodes.*

Examples of a simple parallel network and a parallel network are depicted in Figure 9. An n-node *simple* parallel dynamic network is defined by the following network matrix, assuming that the nodes have been labeled sequentially.

Figure 9 – Examples of parallel networks.

a) An example of simple parallel network with five nodes.

b) An example of parallel network.



Source: The author.

$$G^0(q) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ G^0_{21}(q) & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ G^0_{n-1,1}(q) & 0 & \cdots & \cdots & \vdots \\ 0 & G^0_{n2}(q), & \cdots & G^0_{n,n-1}(q) & 0 \end{bmatrix}. \tag{47}$$

The next theorem states necessary and sufficient conditions that characterize all valid EMPs for a simple parallel network.

Theorem 3.6. *(MAPURUNGA; BAZANELLA, 2021b). A parallel network with network matrix in (47) is generically identifiable if and only if the following conditions hold:* $\mathcal{F} \subset \mathcal{B}$, $\mathcal{S} \subset \mathcal{C}$, $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$, *and in addition:* $|\mathcal{B} \setminus \mathcal{F} \cap \mathcal{C} \setminus \mathcal{S}| = n - 3$, *i.e. at least* $n - 3$ *nodes, which must be neither a sink nor a source, are both excited and measured.*

Proof. The first three conditions are known to be necessary for any network. They imply that node 1 must be excited and node n must be measured. Now, from the structure of the network, one can form the input-output representation from

$$(I_n - G^0(q))^{-1} = \begin{bmatrix} G_E & \mathbf{0} \\ -G_M & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G_E^{-1} & \mathbf{0} \\ G_M G_E^{-1} & 1 \end{bmatrix},$$

where

$$G_E = \begin{bmatrix} 1 & \mathbf{0} \\ -G_s & I_{n-2} \end{bmatrix}, G_M = \begin{bmatrix} 0 & G_i \end{bmatrix},$$
$$G_s = \begin{bmatrix} G^0_{21}(q) & G^0_{31}(q) & \cdots & G^0_{n-1,1}(q) \end{bmatrix}^T,$$
$$G_i = \begin{bmatrix} G^0_{n2}(q) & G^0_{n3}(q) & \cdots & G^0_{n,n-1}(q) \end{bmatrix}.$$

Combining these expressions one can find that:

$$T^0(q) = \begin{bmatrix} 1 & \mathbf{0} & 0 \\ G_s & I_{n-2} & \mathbf{0} \\ \sum_{i\in\mathcal{I}} G^0_{ni}(q)G^0_{i1}(q) & G_i & 1 \end{bmatrix}. \tag{48}$$

The question is which elements of $T^0(q)$ must be identified in order to recover G_s and G_i from (48). Inspection of this equation shows that there is a total of $2n-4$ edges to be identified from $2n-3$ elements of $T^0(q)$ that depend on them, thus providing useful information. The modules $G^0_{j1}(q)$ ($G^0_{nj}(q)$) for $j \in \mathcal{I}$ can be recovered from either $T^0_{j1}(q)$ ($T^0_{nj}(q)$) or $T^0_{n1}(q)$. However, we can only recover at most one edge from $T^0_{n1}(q)$, leaving all other modules to be identified from $T^0_{j1}(q)$ and $T^0_{nj}(q)$. Assume that nodes $l, c \in \mathcal{I}$ are not excited, while all other nodes are both excited and measured. Since $T^0_{nl}(q)$ and $T^0_{nc}(q)$ are not available, it only remains $T^0_{n1}(q)$ to recover both $G^0_{nl}(q)$ and $G^0_{nc}(q)$. After identifying all modules, we are left with one equation and two remaining unknowns (argument q omitted):

$$G_{nl}T^0_{l1} + G_{nc}T^0_{c1} = T^0_{n1} - \sum_{i\in\mathcal{I}\setminus\{l,c\}} T^0_{ni}T^0_{i1}.$$

A similar argument can be made for the situation where l and c are not measured. Hence, it is not possible to identify some edges when more than two internal nodes are not excited and measured. To show sufficiency, assume that the conditions stated in the theorem hold. Suppose that node $c \in \mathcal{I}$ is either excited or measured. We can recover the transfer functions as:

$$G_{j1}(q) = T^0_{j1}(q), \text{ for } j \neq c, \tag{49}$$
$$G_{nj}(q) = T^0_{nj}(q), \text{ for } j \neq c, \tag{50}$$
$$G_{c1}(q) = \frac{-\sum_{i\in\mathcal{I}\setminus\{c\}} T^0_{i1}(q)T^0_{1i}(q)}{T^0_{nc}(q)}, G_{nc}(q) = T^0_{nc}(q), \text{ if } c \in \mathcal{B}, \tag{51}$$
$$G_{c1}(q) = T^0_{c1}, G_{nc} = \frac{-\sum_{i\in\mathcal{I}\setminus\{c\}} T^0_{i1}(q)T^0_{1i}(q)}{T^0_{c1}(q)}, \text{ if } c \in \mathcal{C}. \tag{52}$$

□

In this way, a network with parallel structure must have almost all nodes excited and measured, leaving just one to be relaxed from this condition. As a consequence, a *minimal*

EMP for this topology has cardinality $2n-3$, which corresponds to almost the worst case – exciting and measuring all nodes. This is a strong restriction, since a minimal EMP with this cardinality would be enough to identify a wide number of other networks. Now, we are going to generalize the results obtained for the simple parallel network to the class of parallel networks, in which the paths have length larger than one from the source to the sink. The next result states necessary and sufficient conditions for general parallel networks.

Theorem 3.7. *Consider a parallel network from Definition 3.11 with one source* $1 \in \mathcal{F}$ *connected through* $n_j \geq 2$ *paths* $\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{n_j}\right)$ *to one sink* $n \in \mathcal{S}$*. Let* $\mathcal{V}_{\mathcal{P}_i} \triangleq \{1, k_1^i, k_2^i, \ldots, k_{p_i}^i, n\}$ *be the set of nodes in path* $\mathcal{P}_i$*, for* $i = 1, \ldots, n_j$*. This class of network has the property that* $\mathcal{P}_{\mathcal{V}_1} \cap \mathcal{P}_{\mathcal{V}_2} \cap \cdots \cap \mathcal{P}_{\mathcal{V}_{n_j}} = \{1, n\}$*, i.e. all paths have in common just the first and last nodes. A parallel network is generically identifiable if and only if* $\mathcal{F} \subset \mathcal{B}$*,* $\mathcal{S} \subset \mathcal{C}$*,* $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$*, and in addition for each path among* $n_j - 1$ *paths, there exist at least* $k_j', k_j'' \in \mathcal{V}_{\mathcal{P}_j} \setminus \{1, n\}$*, such that* $k_j' \leq k_j''$*,* $k_j' \in \mathcal{B}$*,* $k_j'' \in \mathcal{C}$*.*

Proof. Necessity: The first three conditions are known to be necessary for any network. It remains to show the last condition, let us prove it by contradiction. Assume that two paths $\mathcal{P}_i$ and $\mathcal{P}_j$ do not obey the last condition, this implies that there are three possibilities: 1) all *internal* nodes of a particular path are excited, 2) all *internal* nodes of a path are measured, or 3) the first l *internal* nodes of a path are measured and the remaining are excited. Assume, without loss of generality, that each path $\mathcal{P}_k$ for $k = 1, 2, \ldots, n_j$, has $p_k + 2$ nodes. Since each path only shares the first and last nodes with any other path, this implies that there are $p_k + 1$ modules in path $\mathcal{P}_k$. Another consequence of this fact is that each module appears only in its respective path. Therefore, each module for each path could only be identified with information from its own path. If every node in $\mathcal{P}_i$ is excited (or measured), we would get exactly $p_k + 1$ useful equations from which we could recover its modules. This is the maximum number of useful equations that we could get from a path without obeying the last condition of the theorem. Hence, by either exciting or measuring all nodes in $\mathcal{P}_i$ and $\mathcal{P}_j$, we will have $p_j + p_i + 2$ unknowns for $p_j + p_i + 2$ useful equations. However, as both paths have the same start and end nodes, this means that one of the useful equations will be the same for both paths. Specifically, $T_{n1}^0(q)$ represents the same information for all paths. This implies that there is $p_j + p_i + 1$ different useful equations from which we need to recover $p_j + p_i + 2$ unknowns.

Sufficiency: Let us consider that path $\mathcal{P}_m$ does not obey the condition stated in the theorem. For each path $\mathcal{P}_l \neq \mathcal{P}_m$ we can identify its edges as follows. Let $\mathcal{V}_{\mathcal{P}_l} = \{1, k_1^l, k_2^l, \ldots, k_{p_k}^l, n\}$, and assume that node k_i^l is excited and node k_j^l is measured, with $k_j^l \geq k_i^l$. The transfer functions corresponding with the edges of $\mathcal{P}_l$ can be identified as

follows:

$$\text{if } k_1^l \in \mathcal{B},\ G_{k_1,1}^l(q) = \frac{T_{k_j^l,1}^0(q)}{T_{k_j^l,k_1^l}^0(q)}; \tag{53}$$

$$\text{if } k_1^l \in \mathcal{C},\ G_{k_1^l,1}(q) = T_{k_1^l,1}^0(q). \tag{54}$$

Consider now k_2^l. If $k_2^l \in \mathcal{C}$ then we can recover $G_{k_2^l,k_1^l}(q) = \frac{T_{k_2^l,1}^0(q)}{G_{k_1^l,1}(q)}$. Conversely, if $k_2^l \in \mathcal{B}$, then we can recover $G_{k_2^l,k_1^l}(q) = \frac{T_{k_j^l,1}^0(q)}{T_{k_2^l,1}^0(q)G_{k_1^l,1}(q)}$. We can apply the same reasoning for all nodes $k = k_1^l, k_2^l, \ldots, k_j^l$. In a similar fashion we can recover the remaining transfer functions as:

$$\text{if } k_j^l + 1 \in \mathcal{B},\ G_{k_j^l+1,k_j}(q) = \frac{T_{n,k_1^l}^0(q)}{T_{k_j^l,k_i^l}^0(q)T_{n,k_j^l+1}^0(q)}; \tag{55}$$

$$\text{if } k_j^l + 1 \in \mathcal{C},\ G_{k_j^l+1,k_j}(q) = \frac{T_{k_j^l+1,k_i^l}^0(q)}{T_{k_j^l,k_i^l}^0(q)}. \tag{56}$$

Applying this reasoning for the remaining nodes $k = k_j^l + 1, k_j^l + 2, \ldots n$ in $\mathcal{P}_l$ one can recover all transfer functions. The only remaining path is $\mathcal{P}_m$, whose set of nodes corresponds to $\mathcal{V}_{\mathcal{P}_m} = \{1, k_1^m, k_2^m, \ldots, k_{p_m}^m, n\}$. Notice that if $\mathcal{P}_m$ obeyed the condition stated in theorem, there would be nothing else to proof. For this reason, we assume that $\mathcal{P}_m$ does not obey the last condition of this theorem. We can recover all modules in a similar way as we did for path $\mathcal{P}_l$, except for one module in each case. Suppose that k_1^m is excited, in order to the last stated condition to be false, all other nodes from $\mathcal{P}_m$ need to be excited. In this situation, we can recover all transfer functions from $\mathcal{P}_m$, except $G_{k_1^m,1}^0(q)$. Suppose now that $k_{p_m}^m$ is measured, then $\mathcal{P}_m$ does not obey the last stated condition if all other nodes from $\mathcal{P}_m$ are measured. In a similar way, we can recover all transfer functions from $\mathcal{P}_m$, except $G_{n,k_{p_m}^m}^0(q)$. The last case is where the first $(k_1^m, k_2^m, \ldots, k_j^m)$ are measured and the remaining $(k_{j+1}^m, \ldots, k_{p_m}^m)$ are excited. For this case, we can recover all modules from $\mathcal{P}_m$, but $G_{k_{j+1}^m,k_j^m}^0(q)$. In all cases, there will be only one remaining unknown in the network. As everything else is known, the remaining unknown module can be successfully recovered from $T_{n,1}^0(q)$. □

Theorem 3.7 provides necessary and sufficient conditions for the class of parallel networks. The conditions state that there are a number of minimal EMPs with minimal cardinality – |EMP| $= n$. The main restriction is that in $n_j - 1$ paths, we must have a node excited followed by a measured node somewhere in each of these $n - j - 1$ paths. This restriction is less severe for the cardinality of minimal EMPs than exciting and measuring a node, since we can have minimal EMPs with the least achievable cardinality n. What we can conclude from these conditions is that if there are two or more parallel paths from one node to another, then in order to uniquely identify all modules there should be an excitation

followed by a measurement in every additional parallel path. This condition allows us to recover some modules within each path independently of the remaining paths. Thus, we must guarantee that some modules within each path are known or can be identified from the available input-output data.

In summary, we now have necessary and sufficient conditions for identifiability of edges in parallel paths from one source to one sink. We can use these results to determine what additional excitations and measurements are required for each path in more complicated structures. These results can be instrumental to the analysis to the case of more general acyclic dynamic networks, which will be investigated in the next section.

3.5.3 Directed acyclic graph

In this section, we investigate generic identifiability conditions for dynamic network whose associated digraph corresponds to a very general class of graphs: a *directed acyclic graph* (DAG). This topology is very general and allows many type of connections, except for feedbacks in the network. This means that this class of networks do not allow any cycle - see Section 2.3, i.e. there is no path $\mathcal{P}$ starting in node k and ending in node k. In all discussion that follows, we will assume that the nodes have been labeled sequentially. Recall that this can be done by a sorting algorithm (KAHN, 1962).

Before tackling the general identifiability problem for acyclic dynamic networks, we are going to consider special cases of these networks. Notice that the previously analyzed parallel and tree networks are just special cases of a DAG.

We will call a feedforward network an acyclic dynamic network with DAG topology whose every node is fully connected to all other nodes ahead of it. Recall that no cycles are allowed. The network matrix of a feedfoward network corresponds to a strictly lower triangular network matrix.

$$G^0(q) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ G^0_{21}(q) & 0 & 0 & \cdots & 0 \\ G^0_{31}(q) & G^0_{32}(q) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ G^0_{n1}(q) & G^0_{n2}(q) & \cdots & G^0_{n,n-1}(q) & 0 \end{bmatrix} \tag{57}$$

For a more general DAG, some $G^0_{ji}(q)$ in (57) for $j > i$ are known to be zero and the corresponding network matrix can be obtained from (57) by just using the knowledge of which G^0_{ji}'s are zero.

We formally define these two network structures in the following.

Definition 3.13. *A feedfoward network is one network whose network matrix can be written as (57) with $G^0_{ji}(q) \neq 0$ for $j > i$.*

The structure in (57) has one source and one sink and each node has a different number

of parallel interconnections, depending on its distance to the sink: node l has $n-l$ outgoing edges. Figure 10 illustrates an example of feedforward network with four nodes.

As for a directed acyclic graph (DAG), it is the more general class of acyclic digraphs, which we formally define as follows.

Definition 3.14. *A directed acyclic graph is a digraph in which there are no cycles.*

We are now going to obtain some relationships among the G^0_{ji}'s and T^0_{ji}'s for the network matrix in (57). These expressions will be useful for the input-output description and the analysis of identifiability. They are formally stated in the next lemma.

Lemma 3.3. *Consider a dynamic network with n nodes and network matrix (57). The following relationships hold for $l > j$:*

$$T^0_{ll}(q) = 1,\, T^0_{jl}(q) = 0, \tag{58}$$

$$T^0_{lj}(q) = \sum_{i=j}^{l-1} G^0_{li}(q) T^0_{ij}(q), \tag{59}$$

$$T^0_{lj}(q) = \sum_{i=j+1}^{l} T^0_{li}(q) G^0_{ij}(q), \tag{60}$$

$$G^0_{lj}(q) = T^0_{lj}(q) - \sum_{i=j+1}^{l-1} G^0_{li}(q) T^0_{ij}(q), \tag{61}$$

$$G^0_{lj}(q) = T^0_{lj}(q) - \sum_{i=j+1}^{l} T^0_{li}(q) G^0_{ij}(q). \tag{62}$$

Proof. Relationship (58) follows from straightforward calculations. Let $G^0_n(q)$ denote a network matrix in the form (57) with n nodes. Consider a network matrix with $n-1$ nodes. We can write $G^0_n(q)$ as a function of $G^0_{n-1}(q)$ from (57) as:

$$G^0_n(q) = \begin{bmatrix} G^0_{n-1}(q) & 0 \\ G^0_{n,r}(q) & 0 \end{bmatrix}, \tag{63}$$

where $G^0_{n,r}(q) = \begin{bmatrix} G^0_{n1}(q) & G^0_{n2}(q) & \cdots & G^0_{n,n-1}(q) \end{bmatrix}$. We can obtain the IO representation from $T^0_n(q) \triangleq (I - G^0_n(q))^{-1}$ and (63) as

$$T^0_n(q) = \begin{bmatrix} (I_{n-1} - G^0_{n-1}(q))^{-1} & 0 \\ G^0_{n,r}(q)(I_{n-1} - G^0_{n-1}(q))^{-1} & 1 \end{bmatrix} = \begin{bmatrix} T^0_{n-1}(q) & 0 \\ G^0_{n,r}(q) T^0_{n-1}(q) & 1 \end{bmatrix}. \tag{64}$$

It follows from (64) that the last row of $T^0_n(q)$ can be written as:

$$\begin{bmatrix} T^0_{n1}(q) \\ T^0_{n2}(q) \\ \vdots \\ T^0_{n,n-1}(q) \end{bmatrix}^T = \begin{bmatrix} G^0_{n1}(q) \\ G^0_{n2}(q) \\ \vdots \\ G^0_{n,n-1}(q) \end{bmatrix}^T T^0_{n-1}(q).$$

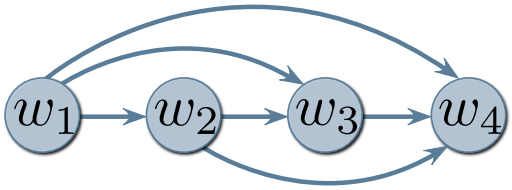
As $T^0_{n-1}(q)$ does not depend on $T^0_n(q)$, relationship (59) follows for $l = n$. Relationship (61) follows from (59) by isolating $G^0_{lj}(q)$ and using (58). Finally, relationships (60) and (62) follows from the fact that:

$$\begin{aligned} T^0(q) \triangleq (I - G^0(q))^{-1} &\iff T^0(q)(I - G^0(q)) = I = (I - G^0(q))T^0(q) \\ &\iff T^0(q)G^0(q) = G^0(q)T^0(q). \end{aligned} \tag{65}$$

and using (58) in (65) gives the desired result. □

An important remark is that (59) remains valid for any DAG network, provided we substitute the edges that do not exist by zero. With these expressions, we are now able to obtain conditions on the minimal EMPs. For a feedforward network a minimal EMP must excite and measure all internal nodes, as it is proven in the following theorem.

Figure 10 – A feedforward network with four nodes.



Source: The Author.

Theorem 3.8. *A feedfoward dynamic network is generically identifiable if and only if* $\mathcal{F} \subseteq \mathcal{B}$, $\mathcal{S} \subseteq \mathcal{C}$, *and* $\mathcal{I} \subseteq \mathcal{B} \cap \mathcal{C}$.

Proof. First, the IO model of the network obeys (58) and (59).

Necessity: From Theorem 3.1, every source must be excited and every sink must be measured. This network has a total of $n(n-1)/2$ unknowns. After removing the equations that have no information (58), it only remains $n(n-1)/2$ useful equations. Hence, if one node in $\mathcal{I}$ is not measured or excited we will have more unknowns than available equations.

Sufficiency: The conditions in the statement are known to be sufficient for any network – see Theorem 6.2 in GEVERS; BAZANELLA; PARRAGA (2017). □

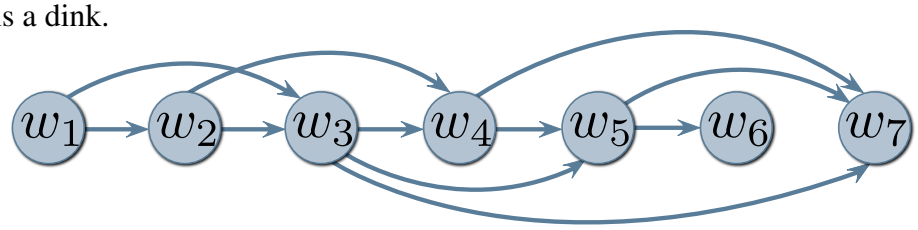
These are very strong conditions that this class of networks must satisfy, since they would provide generic identifiability even for a "full" network, that is, one which would also have feedback paths. On the other hand, it is not surprising, since most nodes need to be excited and measured in a simple parallel network – see Theorem 3.6. This strong requirement is due to the presence of some parallel connections among the nodes. In fact, certain internal nodes must be excited while other type of nodes must be measured. Before introducing the special kind of nodes, we say that node i is *connected* to node j if $G^0_{ji}(q) \neq 0$. We now introduce the two types of internal nodes that will be useful in the analysis.

Definition 3.15. *An internal node j is called a **dource** if it has at least one out-neighbor that is connected to all in-neighbors of j.*

Definition 3.16. *An internal node j is called a **dink** if it has at least one in-neighbor that connects to all out-neighbors of j.*

Notice that these two definitions depend solely on the neighborhood of the node, whether or not a node can be classified as a dource or dink depends on the interconnection of its neighborhood. Furthermore, a given internal node can be both dource and dink. Figure 11 depicts an example of acyclic network, where nodes 2 and 5 are classified as dources and node 4 is a dink. Notice that for a feedfoward network, as depicted in Figure 10, all internal nodes are both dources and dinks.

Figure 11 – An example of acyclic dynamic network where nodes 2 and 5 are dources and node 4 is a dink.



Source: The Author.

A more general class of feedforward networks is one in which not all of the forward paths exist, which corresponds to the class of directed acyclic graphs (DAGs) – see Definition 3.14. For this class of networks, the following theorem provides a necessary condition on the EMP requirements of dources and dinks.

Theorem 3.9. *Consider an acyclic dynamic network whose corresponding graph associated to the network matrix is a directed acyclic graph. All transfer functions of the network are generically identifiable only if $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$, all sources and dources are excited, and all sinks and dinks are measured.*

Proof. The first condition is known to be necessary for any network as presented in Theorem 3.2. That sources and sinks must be excited and measured, respectively, was presented in Theorem 3.1. Let us start by showing that dources must be excited. For this purpose, we are going to divide the network into two subnetworks, such that the network matrix (57) can be partitioned as (38). The first subnetwork has the set of nodes defined by $\mathcal{W}_1 = \{1, 2, \ldots, l\}$, while for the second part the set of nodes is $\mathcal{W}_2 = \{l+1, l+2, \ldots, n\}$, with l a dource. As l is a dource, there exists a $j \in \mathcal{N}_l^+$ such that $G_{jk}^0(q) \neq 0$ for all $k \in \mathcal{N}_l^-$. We have seen in (41) that all modules of each subnetwork can be identified if a valid EMP is applied to each subnetwork. It remains to know whether the edges of the submatrix $G_{\mathcal{W}_2,\mathcal{W}_1}^0(q) = G_{21}^0(q)$ are identifiable. Consider that node l is not excited. Now, to recover

$G_{21}^0(q)$ we can form the linear system $Ag_{21} = b$ as in (42), where $g_{21} = \text{vec}(G_{21}^0(q))$, $b = \text{vec}(C_2T_{21}^0(q)B_1)$ and

$$A = (T_1^0B_1)^T \otimes C_2T_2^0 = \begin{bmatrix} C_2T_2^0 & t_{21}C_2T_2^0 & t_{31}C_2T_2^0 & \cdots & \cdots & t_{l1}C_2T_2^0 \\ \mathbf{0} & C_2T_2^0 & t_{32}C_2T_2^0 & \cdots & \cdots & t_{l2}C_2T_2^0 \\ \mathbf{0} & \mathbf{0} & C_2T_2 & t_{43}C_2T_2^0 & \cdots & t_{l3}C_2T_2^0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & C_2T_2^0 & t_{l,l-1}C_2T_2^0 \end{bmatrix}, \tag{66}$$

where t_{ji} is the (j,i) element of $T_1^0B_1$. From now on we drop q as argument for notation convenience. Notice that as l is not excited, the corresponding T_{kl}'s are not present in (66) as they are not available for identification. The edges from g_{21} can be uniquely identified if and only if A is full column rank. Each block column of A is related to some edges in G_{21}^0, specifically, its columns. From (59), the last block column of (66) can be written as:

$$\begin{bmatrix} t_{l1}C_2T_2^0 \\ t_{l2}C_2T_2^0 \\ \vdots \\ t_{l,l-1}C_2T_2^0 \end{bmatrix} = G_{l1}\begin{bmatrix} C_2T_2^0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + G_{l2}\begin{bmatrix} t_{21}C_2T_2^0 \\ C_2T_2^0 \\ \vdots \\ \mathbf{0} \end{bmatrix} + \cdots + G_{l,l-1}\begin{bmatrix} t_{l-1,1}C_2T_2^0 \\ t_{l-1,2}C_2T_2^0 \\ \vdots \\ C_2T_2^0 \end{bmatrix}. \tag{67}$$

Notice that (67) implies that the last block column of (66) is a linear combination of the other block columns that depend on the incoming edges of l: G_{lk}^0 for $k \in \mathcal{N}_l^-$. Recall that for a DAG some G_{ji}^0 are zero. Now, each block column in (67) is associated with a group of edges in G_{21}^0 through g_{21}. If some edge from g_{21} is zero, then the corresponding column in A is multiplied by zero and we can remove it from A. Now, if the edges in g_{21} associated with the blocks columns corresponding to G_{lk}^0 are different from zero, then A will be rank deficient. If l is a dource, then there exists a nonzero edge (G_{jk}^0, with $k \in \mathcal{N}_l^-$) in g_{21} for each corresponding block column related to the incoming edges G_{lk}^0 in (67). Thus, A is rank deficient and the corresponding edges can not be identified.

Let us now consider the measurement of dinks. The proof follows from very similar arguments, except that this time the network will be divided in two subnetworks such that the first part has the nodes set as $\mathcal{W}_1 = \{1, 2, \ldots, l-1\}$ and the second part $\mathcal{W}_2 = \{l, l+1, \ldots, n\}$, with l a dink. Since l is a dink, there exists a $j \in \mathcal{N}_l^-$ such that $G_{kj}^0 \neq 0$ for all $k \in \mathcal{N}_l^+$. In a similar fashion, we can form a linear system ($Ag_{21} = b$) on the unknowns and get a similar matrix as (66):

$$A = [T_1^0B_1]^T \otimes C_2T_2 = \begin{bmatrix} C_2T_2 & t_{21}C_2T_2 & t_{31}C_2T_2 & \cdots & t_{l1}C_2T_2 \\ \mathbf{0} & C_2T_2 & t_{32}C_2T_2 & \cdots & t_{l2}C_2T2 \\ \mathbf{0} & \mathbf{0} & C_2T_2 & \cdots & t_{l3}C_2T2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & C_2T_2 \end{bmatrix}.$$

Notice that

$$\begin{aligned}&\operatorname{rank}(A) = \operatorname{rank}\left((T_1^0 B_1)^T \otimes C_2 T_2^0\right) = \\ &\operatorname{rank}\left((T_1^0 B_1)^T\right)\operatorname{rank}(C_2 T_2^0) = \operatorname{rank}\left(C_2 T_2^0 \otimes (T_1^0 B_1)^T\right).\end{aligned}$$

It will be more useful to analyze rank of $A' \triangleq C_2 T_2 \otimes (T_1^0 B_1)^T$ instead of the rank of A. For notation convenience, assume without loss of generality that $B_1 = I$, i.e all nodes from the first subnetwork are excited. The structure of A' is as follows

$$A' = \begin{bmatrix} t_{21}[T_1^0]^T & [T_1^0]^T & \mathbf{0} & \cdots & \mathbf{0} \\ t_{31}[T_1^0]^T & t_{32}[T_1^0]^T & [T_1^0]^T & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ t_{n-l+1,1}[T_1^0]^T & t_{n-l+1,2}[T_1^0]^T & \cdots & \cdots & [T_1^0]^T \end{bmatrix}, \tag{68}$$

where t_{ji} is the (j,i) element of $C_2 T_2^0$. Matrix A' has the same structure of (66). Similarly one can write the first block columns of A' as:

$$\begin{aligned}\begin{bmatrix} t_{21}[T_1^0]^T \\ t_{31}[T_1^0]^T \\ \vdots \\ t_{n-l+1,1}[T_1^0]^T \end{bmatrix} =& G_{nl}\begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ [T_1^0]^T \end{bmatrix} + G_{n-1,l}\begin{bmatrix} \mathbf{0} \\ \vdots \\ [T_1^0]^T \\ t_{n-l+1,n-l}[T_1^0]^T \end{bmatrix} + \cdots \\ &+ G_{l+1,l}\begin{bmatrix} [T_1^0]^T \\ t_{32}[T_1^0]^T \\ \vdots \\ t_{n-l+1,2}[T_1^0]^T \end{bmatrix}.\end{aligned} \tag{69}$$

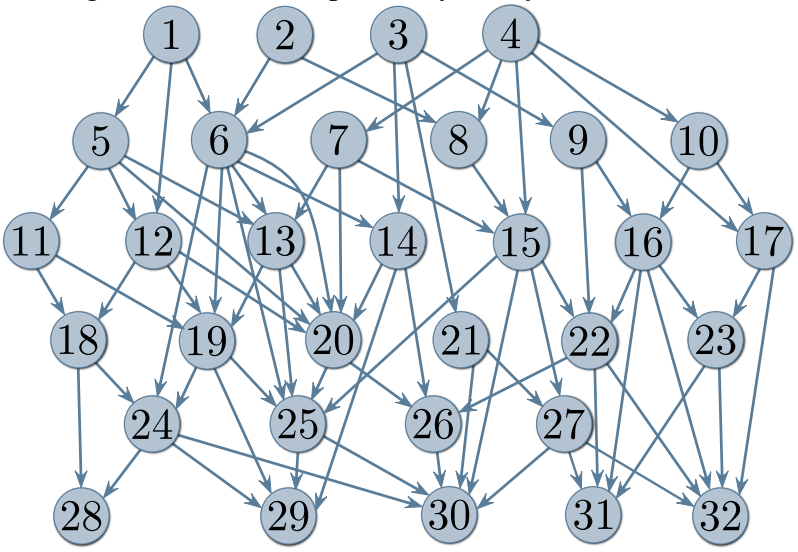
The dual conclusion can be made for this case if any of $\{G_{l+1,l}^0, \ldots, G_{nl}^0\}$ is zero and for the unknowns in g_{21} associated with each block column of A'. Since l is a dink, there exists nonzero edges (G_{kj}^0 for $k \in \mathcal{N}_l^+$) in g_{21} that cause a rank drop in A'. □

This result allows one to determine which nodes must be excited and which nodes must be measured in a quite general class of dynamic networks whose topologies correspond to a directed acyclic graph. The condition presented in this theorem is very useful from a local perspective. This means that a necessary condition for excitation or measurement of an internal node $i \in \mathcal{I}$ is based just on its in-neighbors ($\mathcal{N}_i^-$) and out-neighbors ($\mathcal{N}_j^+$) – see Definitions 3.15 and 3.16. The reason why a feedfoward network has such strong conditions is due to the fact that all internal nodes of this class of networks are dources and dinks. Using this result we can determine which nodes need to be excited and measured based only on the neighborhood of the nodes. This condition is, however necessary, not sufficient to determine whether some node should be excited (measured) or not. A requirement for a node to be excited (measured) could come from the applied EMP in the network. That is, even if a node $l \in \mathcal{I}$ does not obey the conditions in Theorem 3.9, it could need to be excited (measured) due to other conditions on the EMP.

In order to illustrate the usefulness of this result we provide the following example.

Example 3.2. *Consider the dynamic network depicted in Figure 12. From the topology of the network we determine the set of sources ($\mathcal{F} = \{1, 2, 3, 4\}$), and sinks ($\mathcal{S} = \{28, 29, 30, 31, 32\}$). The former need to be excited, while the latter need to be measured. We now determine which nodes are dources or dinks based on the neighborhood of each node. Consider node 23, for which we have $\mathcal{N}_{23}^{-} = \{16, 17\}$ and $\mathcal{N}_{23}^{+} = \{31, 32\}$. Node 23 is a dource because nodes 16 and 17 are connected to node 32. For this reason, it must be excited. Node 23 is also a dink since node 16 is connected to nodes 31 and 32. Thus, node 23 must also be measured. For the other nodes in the network we find that the following nodes are dources $\{5, 7, 10, 13, 23, 27\}$, while the following are dinks $\{8, 13, 23\}$.*

Figure 12 – An example of acyclic dynamic network.



Source: The Author.

Theorem 3.9 allows us to allocate excitation for nodes that are dources and measurements for dinks in a general class of acyclic dynamic networks. In this way, as shown in this example, we can determine by inspection the nodes that must be excited and the ones that must be measured. This is a necessary condition that is independent of the EMP applied in the network and only depends on the local topology of the node. However, it may be the case that for a given particular EMP, node i would need to be excited, even if it is not a dource, due to the EMP employed. A similar remark can be made for nodes that need to be measured in a DAG network. The fundamental result is that if a node l is a dource and a dink, then every valid EMP must contain node l as both excited and measured.

We are now going to present results related to paths embedded in a general acyclic network. These results can be used along with the necessary conditions for excitation and measurements presented in Theorem 3.9 and the results for general parallel networks

presented in Theorem 3.7. For all results that follow, we consider paths, say $\mathcal{P}$, with the property that no two nodes from a path $\mathcal{P}$ are connected through any other path in the network. We refer to paths with this property as a *channel* path. The following result gives a sufficient condition for recovering the modules within a channel path in an acyclic dynamic network.

Theorem 3.10. *Consider a path $\mathcal{P}(\mathcal{V}_\mathcal{P}, \mathcal{E}_\mathcal{P})$, with nodes $\mathcal{V}_\mathcal{P} = \{k_1, k_2, \dots k_{n_p}\}$ such that k_1 is connected to k_{n_p} only through $\mathcal{P}$ embedded in a dynamic network with topology of a directed acyclic graph. If there exists a node $s \in \mathcal{B}$ such that some path $\mathcal{P}_s$ connects s to k_1 but no other path connects s to $\{k_2, \dots, k_{n_p}\}$, then all edges from $\mathcal{E}_\mathcal{P}$ can be generically identified if $\mathcal{V}_\mathcal{P}, \subseteq \mathcal{B} \cup \mathcal{C}$ and $k_{n_p} \in \mathcal{C}$.*

Proof. If $k_1 \in \mathcal{B}$ then all edges from $\mathcal{P}$ can be uniquely identified following the Theorem 3.4. Now, consider that $k_1 \in \mathcal{C}$. If $k_2 \in \mathcal{C}$ then we can recover $G^0_{k_2,k_1}(q) = T^0_{k_2 s}(q)/T^0_{k_1 s}(q)$, otherwise $k_2 \in \mathcal{B}$ and we can recover $G^0_{k_2,k_1}(q) = T^0_{k_{n_p} s}(q)/(T^0_{k_1 s}(q) T^0_{k_{n_p} k_2}(q))$. We can proceed similarly for the next node. Suppose $k_3 \in \mathcal{C}$, then we can identify

$$G^0_{k_3 k_2}(q) = T^0_{k_3 s}(q)/(T^0_{k_1 s}(q) G^0_{k_2 k_1}(q)), \tag{70}$$

otherwise $k_3 \in \mathcal{B}$ and we can recover $G^0_{k_3 k_2}(q) = T^0_{k_{n_p} s}(q)/(T^0_{k_1 s}(q) G^0_{k_2 k_1}(q) T^0_{k_{np} k_2}(q))$. We can do this $\forall i \in \{2, 3, \dots, n_p\}$:

$$G^0_{k_i,k_{i-1}}(q) = \frac{T^0_{k_{np} s}(q)}{T^0_{k_{n_p} k_i}(q) G^0_{k_{i-1},k_{i-2}}(q) \cdots G^0_{k_2 k_1}(q) T^0_{k_1 s}(q)}, \text{ if } k_i \in \mathcal{B}, \tag{71}$$

$$G^0_{k_i,k_{i-1}}(q) = \frac{T^0_{k_i s}(q)}{T^0_{k_1 s}(q) G^0_{k_{i-1},k_{i-2}}(q) \cdots G^0_{k_2 k_1}(q)}, \text{ if } k_i \in \mathcal{C}. \tag{72}$$

□

This theorem is related to channel paths, those with the property that no two nodes within the path are connected through other paths in the network. For this kind of path, this theorem shows that we can identify all edges from a channel path provided there exists some excited node in the network connected to the first node of that path, but it is not connected to any other node from that path. The next theorem provides a dual result for the case where there is some measured node ahead of a channel path.

Theorem 3.11. *Consider a path $\mathcal{P}(\mathcal{V}_\mathcal{P}, \mathcal{E}_\mathcal{P})$, with nodes $\mathcal{V}_\mathcal{P} = \{k_1, k_2, \dots k_{n_p}\}$ such that k_1 is connected to k_{n_p} only through $\mathcal{P}$ embedded in a dynamic network with topology of a directed acyclic graph. If there exists a node $s \in \mathcal{C}$ such that some path $\mathcal{P}_s$ connects k_{n_p} to s, but no other path connects $\{k_1, k_2, \dots, k_{n_p - 1}\}$ to s, then all edges from $\mathcal{E}_\mathcal{P}$ can be identified if $\mathcal{V}_\mathcal{P} \subseteq \mathcal{B} \cup \mathcal{C}$ and $k_1 \in \mathcal{B}$.*

Proof. If k_{n_p} is measured, then all edges from $\mathcal{P}$ can be identified following Theorem 3.4. Now, consider that $k_{n_p} \in \mathcal{B}$. If $k_2 \in \mathcal{C}$, then we can recover $G^0_{k_2 k_1}(q) =$

$T^0_{k_2k_1}(q)$, otherwise if $k_2 \in \mathcal{B}$ we can recover $G^0_{k_2k_1}(q) = T^0_{sk_2}(q)/T^0_{sk_1}(q)$. Proceeding in a similar fashion for the next nodes. Suppose $k_3 \in \mathcal{B}$, then we can recover $G^0_{k_3k_2}(q) = T^0_{sk_1}(q)/(T^0_{sk_3}(q)G^0_{k_2k_1}(q))$, otherwise if $k_3 \in \mathcal{C}$ we can recover $G^0_{k_3k_2}(q) = T^0_{k_3k_1}(q)/(G^0_{k_2k_1}(q))$. Applying the same reasoning for $i \in \{2, 3, \dots n_p\}$:

$$G^0_{k_ik_{i-1}}(q) = \frac{T^0_{sk_1}(q)}{T^0_{sk_i}(q)G^0_{k_{i-1}k_{i-1}}(q)\cdots G^0_{k_2k_1}(q)}, \text{ if } k_i \in \mathcal{B}, \tag{73}$$

$$G^0_{k_ik_{i-1}}(q) = \frac{T^0_{k_ik_1}(q)}{G^0_{k_{i-1}k_{i-1}(q)}\cdots G^0_{k_2k_1}(q)}, \text{ if } k_i \in \mathcal{C}. \tag{74}$$

□

This theorem presents the dual result of Theorem 3.10. If the first node of the channel path $\mathcal{P}$ is excited and all its other nodes are either excited or measured, then we can identify all modules associated with $\mathcal{P}$ provided there exists some measured node in the network after $\mathcal{P}$, assuming that the nodes have been labeled sequentially, such that all paths connecting the nodes from $\mathcal{P}$ to that measured node are connected only through $\mathcal{P}$. The next result provides a sufficient condition that it is a combination of the previous results.

Theorem 3.12. *Consider a path $\mathcal{P}(\mathcal{V}_\mathcal{P}, \mathcal{E}_\mathcal{P})$, with nodes $\mathcal{V}_\mathcal{P} = \{k_1, k_2, \dots k_{n_p}\}$ such that k_1 is connected to k_{n_p} only through $\mathcal{P}$ embedded in a dynamic network with topology of a directed acyclic graph. If there exist nodes $f \in \mathcal{B}, s \in \mathcal{C}$, a path $\mathcal{P}_f$ that connects f to k_1, and a path $\mathcal{P}_s$ that connects k_{n_p} to s, and all paths from f to s pass through $\mathcal{P}$, then all edges from $\mathcal{E}_\mathcal{P}$ can be identified if $\mathcal{V}_\mathcal{P} \in \mathcal{B} \cup \mathcal{C}$.*

Proof. This result can be obtained by combining Theorems 3.10 and 3.11. If there is some node $k_j \in \mathcal{C}$, then all edges before k_j can be recovered using Theorem 3.10. On the other hand, if there is some node $k_i \in \mathcal{B}$, then all edges after it can be recovered using Theorem 3.11. The remaining edge that can not be identified from the results of these theorems is $G^0_{k_{l+1},k_l}(q)$, with $k_l \in \mathcal{C}, k_{l+1} \in \mathcal{B}$. This edge can be recovered as $G^0_{k_{l+1},k_l}(q) = T^0_{sf}(q)/(T^0_{k_lf}(q)T^0_{sk_{l+1}}(q))$. □

This theorem relaxes the necessity of exciting the first node of the channel path or measuring the last node of the channel path. It can be interpreted as a combination of the results of the previous Theorems 3.10 and 3.11. By combining these results, we can investigate how to deal with parallel paths embedded in a general cyclic dynamic network. The following result gives a sufficient condition to identify the modules of a path linking any two nodes when there is more than a single path connecting these two nodes.

Theorem 3.13. *Consider a dynamic network whose topology corresponds to a directed cyclic graph. Suppose that there are n_p parallel paths $\left(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{n_p}\right)$ from a node i to a node j in this network, such that $\mathcal{V}_{\mathcal{P}_1} \cap \mathcal{V}_{\mathcal{P}_2} \cap \cdots \cap \mathcal{V}_{\mathcal{P}_{n_p}} = \{i, j\}$. Consider the nodes from path $\mathcal{P}_1$: $\mathcal{V}_{\mathcal{P}_1} = \{i, k_1, \dots, k_{n_{p_1}}, j\}$. Suppose there exist nodes $f \in \mathcal{B}, s \in \mathcal{C}$, a*

path $\mathcal{P}_f$ that connects f to i, a path $\mathcal{P}_s$ that connects j to s, and that all paths from f to j and from i to s pass through only one of the paths $\mathcal{P}_k$, for $k = 1, \ldots, n_p$. All transfer functions related to the edges of $\mathcal{P}_1(\mathcal{V}_{\mathcal{P}_1}, \mathcal{E}_{\mathcal{P}_1})$ are generically identifiable if $\mathcal{V}_{\mathcal{P}_1} \subseteq \mathcal{B} \cup \mathcal{C}$, and in addition, all edges from the other $n_p - 1$ paths $(\mathcal{P}_2, \ldots, \mathcal{P}_{n_p})$ are known or can be identified from the excitation and measurement pattern.

Proof. If there exist two nodes i and j in the network that are linked through n_p parallel paths, we could collect all these nodes and form a subnetwork with parallel structure – see Definition 3.11. The last condition in Theorem 3.7 guarantees that all edges from the $n_p - 1$ can be generically identified. Since it is assumed that all edges from $\mathcal{E}_{\mathcal{P}_k}$ for $k = 2, \ldots, n_p$ are known, all edges from $\mathcal{E}_{\mathcal{P}_1}$ can be identified using Theorem 3.12. □

This theorem provides a sufficient condition on the excitation and measurement pattern of parallel paths embedded in a general dynamic network. It asserts that if all edges from the parallel paths are known, or can be identified from other relationships such as in Theorems 3.10 - 3.12, then we can recover all edges from the remaining path by simple exciting or measuring all its nodes.

All these results can be combined to identify the modules of a wide range of dynamic network whose topology corresponds to a DAG. A useful result for this class of networks is to identify all incoming edges or outgoing edges for a particular node in the network. The following lemma gives a sufficient condition to identify these edges related to a node in an acyclic network.

Lemma 3.4. *Consider a dynamic network whose topology corresponds to a directed acyclic graph. For any node l in this dynamic network, we can recover the outgoing edges of l from:*

$$T^0_{\{l+1,\ldots,n\},\mathcal{N}_l^+}(q) G^0_{\mathcal{N}_l^+,\{l\}}(q) = T^0_{\{l+1,\ldots,n\},\{l\}}(q). \tag{75}$$

The outgoing edges of l can be identified as:

$$[T^0_{\mathcal{N}_l^-,\{1,\ldots,l-1\}}(q)]^T [G^0_{\{l\},\mathcal{N}_l^-}(q)]^T = [T^0_{\{l\},\{1,\ldots,l-1\}}(q)]^T. \tag{76}$$

Proof. These relationships follow by forming a linear system with the incoming and outgoing edges of node l using expressions (61) and (62) in Lemma 3.3. □

This lemma provides a sufficient condition to identify the incoming and outgoing edges of a particular node in an acyclic dynamic network. Notice that a particular module $G^0_{ji}(q)$ can be identified from either the incoming edge of j in (75) or as outgoing edge of node i in (76). Now, we can combine all these results to determine a valid EMP for a general network with DAG topology. In order to demonstrate the usefulness of these results, let us now revisit Example 3.2.

Example 3.3. *Consider the acyclic dynamic network from Example 3.2 depicted in Figure 12. From the previous example we have shown that the following nodes must be excited:* $\mathcal{B} = \{1, 2, 3, 4, 5, 7, 10, 13, 23, 27\}$ *, while the following must be measured* $\mathcal{C} = \{8, 13, 23, 28, 29, 30, 31, 32\}$. *For every excited node we are going to verify the channel paths that exist in the network. Let us apply the Theorems 3.10 - 3.12 in order to determine whether to excite or measure the remaining nodes. The next table provides all channel paths from each excited node to the other nodes in the network. In this example we adopted the following convention* $v_1 \to v_2 \to \cdots \to v_p$ *to denote a path* $\mathcal{P} = (\mathcal{V}_\mathcal{P} = \{v_1, v_2, \ldots, v_p\}, \mathcal{E}_\mathcal{P} = \{(v_2, v_1), (v_3, v_2), \ldots, (v_p, v_{p-1})\})$.

Table 1 – All channel paths from the set of excited nodes $\mathcal{B} = \{1, 2, 3, 4, 5, 7, 10, 13, 23, 27\}$ to other nodes from network depicted in Figure 12.

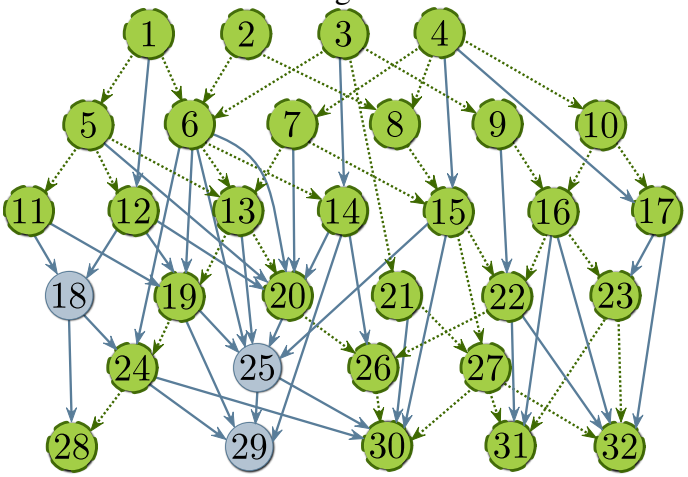
node	path	node	path	node	path
1	$1 \to 5 \to 11$ $1 \to 6 \to 14$	23	$23 \to 31$ $23 \to 32$	27	$27 \to 30$ $27 \to 31$
2	$2 \to 6 \to 13$	4	$4 \to 7 \to 13 \to 19 \to 24 \to 28$		$27 \to 32$
	$2 \to 8 \to 15 \to 22$ $2 \to 8 \to 15 \to 27$		$4 \to 8$ $4 \to 10 \to 16$	13	$13 \to 20 \to 26$ $13 \to 19 \to 24 \to 28$
3	$3 \to 6 \to 13$ $3 \to 21 \to 27$ $3 \to 9 \to 16 \to 22$	5	$5 \to 11$ $5 \to 12$ $5 \to 13$	7	$7 \to 13 \to 19 \to 24 \to 28$ $7 \to 15 \to 22$ $7 \to 15 \to 27$
	$3 \to 9 \to 16 \to 23$				
10	$10 \to 17$ $10 \to 16 \to 22 \to 26 \to 30$				

This table shows all channel paths from every node that needs to be excited to every other node in the network. That is, for every channel path there is no other path that links the two nodes for a particular excited node. This induces a subgraph with a topology of a multitree for every node that is excited in the network. Figure 13 depicts the union of all multitrees induced by the excited nodes and their respective edges.

We can now use Theorems 3.5, 3.10-3.12 to assign a valid EMP for each multitree and recover their respective edges. For instance, take node 1*, which is a source and it needs to be excited. If we excite* $\{1, 5\} \subset \mathcal{B}$*, measure* $\{11, 14\} \subset \mathcal{C}$*, and either excite or measure* 6*, then we can recover* $G^0_{51}(q)$*,* $G^0_{61}(q)$*,* $G^0_{11,5}(q)$*, and* $G^0_{14,6}(q)$*. Applying the same procedure for every source and dource, all dotted edges in Figure 13 can be recovered if* $\{1, 2, 3, 4, 5, 7, 10, 13, 23, 27\} \subset \mathcal{B}$*,* $\{8, 11, 12, 13, 14, 17, 22, 23, 26, 27, 28, 30, 31, 32\} \subset \mathcal{C}$*, and* $\{6, 9, 15, 16, 19, 20, 21, 24\} \subset \mathcal{B} \cup \mathcal{C}$.

As we can recover the dotted edges in Figure 13, the edges $G^0_{12,1}(q)$*,* $G^0_{14,3}(q)$*,* $G^0_{15,4}(q)$*,* $G^0_{17,4}(q)$*,* $G^0_{22,9}(q)$*.* $G^0_{23,17}(q)$*, and* $G^0_{32,17}(q)$ *can be recovered by using Theorem 3.13. Take the subnetwork formed by nodes* $\{1, 5, 12\}$ *and edges* $\{(5, 1), (12, 5), (12, 1)\}$ *as an example. Since we can recover* $G^0_{51}(q)$ *and* $G^0_{12,5}(q)$*, the remaining edge* $G^0_{12,1}(q)$ *can be*

Figure 13 – The collection of multitrees induced by the excited nodes and their edges. The nodes that are part of the collection of multitrees are depicted as dashed in green color, while their edges are dashed arrows also in green.



Source: The Author.

identified from $T^0_{12,1}(q)$ *as node* 1 *is excited and* 12 *is measured. Similar arguments can be made for the other edges.*

Let us do the same procedure of listing all unique paths as in Table 1, but this time for all measured nodes in the network. The next table shows all channel paths from some nodes of the network to all measured nodes from previous steps.

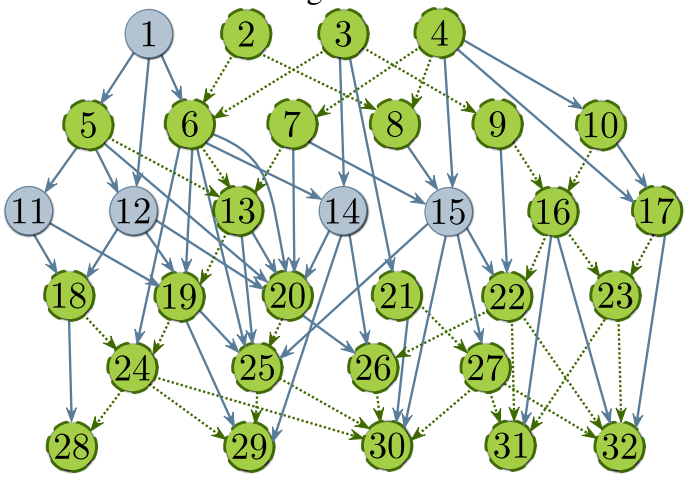
Table 2 – All channel paths from other nodes of the network to the measured nodes.

node	path	node	path	node	path
8	$2 \to 8$ $4 \to 8$	23	$3 \to 9 \to 16 \to 23$ $17 \to 23$	29	$18 \to 24 \to 29$ $20 \to 25 \to 29$
13	$2 \to 6 \to 13$ $3 \to 6 \to 13$ $4 \to 7 \to 13$	30	$18 \to 24 \to 30$ $25 \to 30$ $27 \to 30$	31	$17 \to 23 \to 31$ $22 \to 31$ $21 \to 27 \to 31$
	$5 \to 13$		$10 \to 16 \to 22 \to 26 \to 30$	28	$4 \to 7 \to 13 \to 19 \to 24 \to 28$
32	$21 \to 27 \to 32$ $22 \to 32$ $23 \to 32$				

Table 2 shows all channel paths from other nodes of the network to the set of sinks and dinks. Figure 14 depicts all multitrees generated by these channel paths.

Notice that most edges that could be recovered by this step were already identified in the previous steps. Therefore, we must allocate the excitations and measurements to identify only the unknown edges. We proceed by exciting the following nodes $\{18, 20, 22, 25\} \subset \mathcal{B}$*. Now, let us use Theorem 3.13 to identify some edges in the parallel paths as we did before, such as* $G^0_{28,18}(q)$*.*

Figure 14 – The collection of multitrees induced by the measured nodes and their edges. The nodes that are part of the collection of multitrees are depicted as dashed in green color, while its edges are dashed arrows also in green.



Source: The Author.

The remaining edges can be recovered from the expressions in Lemma 3.3 or 3.4. Take node 5 *as an example, to identify the outgoing edges of node* 5 *we can form the linear system from (76) (omitting* q*):*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T^0_{18,11} & T^0_{18,12} & 0 & 0 \\ T^0_{19,11} & T^0_{19,12} & T^0_{19,13} & 0 \\ 0 & T^0_{20,12} & T^0_{20,13} & 1 \\ T^0_{24,11} & T^0_{24,12} & T^0_{24,13} & 0 \\ T^0_{25,11} & T^0_{25,12} & T^0_{25,13} & T^0_{25,20} \\ 0 & T^0_{26,12} & T^0_{26,13} & T^0_{26,20} \\ T^0_{28,11} & T^0_{28,12} & T^0_{28,13} & 0 \\ T^0_{29,11} & T^0_{29,12} & T^0_{29,13} & T^0_{29,20} \\ T^0_{30,11} & T^0_{30,12} & T^0_{30,13} & T^0_{30,20} \end{bmatrix} \begin{bmatrix} G_{11,5} \\ G_{12,5} \\ G_{13,5} \\ G_{20,5} \end{bmatrix} = \begin{bmatrix} T^0_{11,5} \\ T^0_{12,5} \\ T^0_{13,5} \\ T^0_{18,5} \\ T^0_{19,5} \\ T^0_{20,5} \\ T^0_{24,5} \\ T^0_{25,5} \\ T^0_{26,5} \\ T^0_{28,5} \\ T^0_{29,5} \\ T^0_{30,5} \end{bmatrix}. \tag{77}$$

The only unknown in this system of equations is $G_{20,5}(q)$*. Thus, we can recover it by also exciting node* 12 *as*

$$G_{20,5}(q) = \frac{T^0_{26,5}(q) - T^0_{26,12}(q)G^0_{12,5}(q) - T^0_{26,13}(q)G^0_{13,5}(q)}{T^0_{26,20}(q)}. \tag{78}$$

Now, we can repeat the same procedure and verify if there are more parallel paths in which there is only a single path with unknowns edges and apply Theorem 3.13. Repeating

this procedure for the other nodes we obtain the following valid EMP:

$$\mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 20, 22, 23, 24, 25, 27\}, \quad (79)$$

$$\mathcal{C} = \{8, 11, 12, 13, 14, 17, 21, 22, 23, 26, 27, 28, 29, 30, 31, 32\}. \quad (80)$$

This EMP has cardinality equal to 37.

This example illustrates a procedure to obtain a valid EMP for a large acyclic dynamic network with topology corresponding to a DAG. It has been obtained by combining the results presented so far. Particularly, the role played by the channel paths forming subnetworks with multitree topology was responsible for allocating most excitations and measurements in the network.

In this section we have presented a number of conditions for exciting or measuring a particular node within a general acyclic dynamic network. These conditions can be combined to generate an algorithm for the synthesis of valid EMPs for any dynamic network whose topology corresponds to a directed acyclic graph (DAG). We have derived necessary and sufficient conditions for the identification of all modules for parallel interconnections between a source and a sink, here called parallel network. We have provided sufficient conditions to uniquely identify the edges from a path embedded in an acyclic network. Moreover, we have demonstrated a necessary condition for excitation/measurement of a node within a DAG that depends only on the neighborhood of the node of interest.

3.6 Isolated cycles

In this section we aim to provide necessary and sufficient conditions for the generic identifiability of dynamic networks whose topology corresponds to a cycle – see Section 2.3. A cycle graph (also called loop graph) is a graph that consists of a single loop and nothing more. Its network matrix is in the form

$$G^0(q) = \begin{bmatrix} 0 & 0 & \dots & 0 & G^0_{1n}(q) \\ G^0_{21}(q) & 0 & \dots & 0 & 0 \\ 0 & G^0_{32}(q) & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & G^0_{n,n-1}(q) & 0 \end{bmatrix}. \quad (81)$$

The cycle we are interested in can be a graph by itself, as in (81), or part of a larger graph. When the cycle of interest is part of a larger graph, some of its nodes may belong to other cycles. When this is not the case – that is, no other cycle in the graph contains any of the nodes of the loop of interest – we will say that it is an *isolated cycle*. All results in this section pertain to the identifiability of isolated cycles.

The necessary condition $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ of Theorem 3.2, which applies to all dynamic networks, means that all nodes must be involved in the identification process: they must be

either measured or excited. When it comes to cycles, we would like to determine whether there is any EMP satisfying that necessary condition that would make $G^0(q)$ identifiable (it would then be a minimal EMP with smallest cardinality), or whether additional conditions apply.

In BAZANELLA; GEVERS; HENDRICKS (2019) a sufficient condition for identifiability of cycles was given. The authors have shown that adding a single measurement or a single excitation to such EMP is sufficient to achieve generic identifiability of an isolated cycle. This result is formally stated in the next theorem.

Theorem 3.14. *(BAZANELLA; GEVERS; HENDRICKS, 2019). All transfer functions in an isolated cycle are generically identifiable if $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ and $\mathcal{B} \cap \mathcal{C} \neq \emptyset$.*

In order to provide necessary and sufficient conditions for the identifiability of an isolated cycle, we now derive some properties of cycles. We first recall some expressions and properties derived in the proof of Theorem V.2 of BAZANELLA; GEVERS; HENDRICKS (2019).

Assume, without loss of generality, that the node indices in the cycle go from 1 to n and that the arrows go from i to $i+1$ in the cycle, as in (81). Define the product of all transfer functions in the cycle as follows:

$$R \triangleq G^0_{1n}(q)G^0_{n,n-1}(q)\dots G^0_{32}(q)G^0_{21}(q). \tag{82}$$

Observe that the closed loop transfer function from one node to itself is

$$T^0_{ii}(q) = (1-R)^{-1}. \tag{83}$$

For distinct i, k, we also define

$$R_{ik} \triangleq G^0_{i,i-1}(q)G^0_{i-1,i-2}(q)\dots G^0_{k+1,k}(q) \text{ if } k < i, \tag{84}$$

$$R_{ik} \triangleq G^0_{i,i-1}(q)G^0_{i-1,i-2}(q)\dots G^0_{1n}(q)G^0_{n,n-1}(q)\dots G^0_{k+1,k}(q) \text{ if } k > i. \tag{85}$$

The next lemma provides relations between the quantities $R, R_{ik}, T^0_{ik}(q)$ and $G^0_{ik}(q)$ that will be useful for proving our main result of this section.

Lemma 3.5. *The transfer functions R, R_{ik}, T_{ik} and $G^0_{ik}(q)$ are related by the following expressions for any i, k, j:*

$$T^0_{ik}(q) = R_{ik}(1-R)^{-1}, \tag{86}$$

$$R = R_{ki}R_{ik}, \tag{87}$$

$$G^0_{i+1,i}(q) = \frac{R_{ji}}{R_{j,i+1}} = \frac{R_{i+1,j}}{R_{ij}} = \frac{T^0_{ji}(q)}{T^0_{j,i+1}(q)} = \frac{T^0_{i+1,j}(q)}{T^0_{ij}(q)}. \tag{88}$$

Proof. Expressions (86) and (87) follow immediately from the input-output relationship and definition of R_{ik}. The expressions in (88) can be verified by direct computation. For $j > i+1$, we have

$$\frac{R_{ji}}{R_{j,i+1}} = \frac{G^0_{j,j-1}(q)G^0_{j-1,j-2}(q)\cdots G^0_{i+2,i+1}(q)G^0_{i+1,i}(q)}{G^0_{j,j-1}(q)G^0_{j-1,j-2}(q)\cdots G^0_{i+2,i+1}(q)} = G^0_{i+1,i}(q). \tag{89}$$

When $i > j$, the same relationship can be obtained using (87). □

With these expressions under our belt, we are now ready to prove our main result. We first consider the special case where the cycle has either 2 nodes or 3 nodes, i.e. $n = 2$ or $n = 3$.

Theorem 3.15. *All transfer functions in an isolated cycle with $n \leq 3$ are generically identifiable if and only if $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ and $\mathcal{B} \cap \mathcal{C} \neq \emptyset$.*

Proof. For $n = 2$ we have:

$$T^0(q) = \begin{bmatrix} \frac{1}{1-G^0_{12}(q)G^0_{21}(q)} & \frac{G^0_{12}(q)}{1-G^0_{12}(q)G^0_{21}(q)} \\ \frac{G^0_{21}(q)}{1-G^0_{12}(q)G^0_{21}(q)} & \frac{1}{1-G^0_{12}(q)G^0_{21}(q)} \end{bmatrix}. \tag{90}$$

For $n = 3$ we have:

$$T^0(q) = \begin{bmatrix} \frac{1}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} & \frac{G^0_{13}(q)G^0_{32}(q)}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} & \frac{G^0_{13}(q)}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} \\ \frac{G^0_{21}(q)}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} & \frac{1}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} & \frac{G^0_{13}(q)G^0_{21}(q)}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} \\ \frac{G^0_{21}(q)G^0_{32}(q)}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} & \frac{G^0_{32}(q)}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} & \frac{1}{1-G^0_{13}(q)G^0_{21}(q)G^0_{32}(q)} \end{bmatrix}. \tag{91}$$

Sufficiency follows directly from Theorem 3.14 above, which has been proven in BAZANELLA; GEVERS; HENDRICKS (2019). For necessity, inspection of the $T^0(q)$ matrix shows that if no node is excited and measured, then for $n = 2$, $T^0(q)$ contains only a single known element (impossible to recover 2 $G^0_{ji}(q)$). For $n = 3$, there are only 2 independent elements of $T^0(q)$ (impossible to recover 3 $G^0_{ji}(q)$). □

What this theorem shows is that for an EMP to guarantee generic identifiability of a cycle with less than four nodes it must have at least one node excited and measured, while the others must be either excited or measured. A minimal EMP is therefore characterized by one node being excited and measured, which results in a total of four minimal EMPs for $n = 2$ and twelve minimal EMPs for $n = 3$.

We now consider isolated cycles that have at least 4 nodes. In BAZANELLA; GEVERS; HENDRICKS (2019) it was shown that the conditions $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ and $\mathcal{B} \cap \mathcal{C} \neq \emptyset$, which are necessary and sufficient conditions for isolated cycles with 2 or 3 nodes, are actually sufficient conditions for isolated loops of any size. An alternative sufficient condition for identifiability, also derived in BAZANELLA; GEVERS; HENDRICKS (2019), applied to cycles that have an even number of nodes, larger than 3. It was shown that identifiability is achieved if the EMP obeys the following interleaving condition between excited and measured nodes.

Theorem 3.16. *(BAZANELLA; GEVERS; HENDRICKS, 2019). Let n be even and larger than 3. All transfer functions in an isolated loop can be identified if its nodes are alternately measured and excited.*

This interleaving condition on the EMP is clearly not necessary for identifiability of an isolated cycle, as follows from Theorem 3.14, but it has inspired the development of the necessary and sufficient conditions for identifiability of an isolated cycle formally stated in the next theorem.

Theorem 3.17. *All transfer functions in an isolated cycle are generically identifiable if and only if $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ and, in addition: (i) either $\mathcal{B} \cap \mathcal{C} \neq \emptyset$, or (ii) there exist at least two measured nodes in the cycle, each of which is immediately followed by an excited node.*

Proof. That each node must be excited or measured follows from Theorem 3.2.

Sufficiency:

If (i) holds, the cycle is identifiable by Theorem V.2 of BAZANELLA; GEVERS; HENDRICKS (2019). Consider now that (ii) holds. Without loss of generality let nodes k and n be the two measured nodes that are followed immediately by nodes $k+1$ and 1, which are excited. From these measurements and excitations we obtain: $T^0_{k1}(q), T^0_{n1}(q), T^0_{n,k+1}(q)$, and $T^0_{k,k+1}(q)$, and we can form the product:

$$\frac{T^0_{n1}(q)T^0_{k,k+1}(q)}{T^0_{k1}(q)T^0_{n,k+1}(q)} = \frac{R_{n1}R_{k,k+1}}{R_{k1}R_{n,k+1}} = R_{nk}R_{kn} = R,$$

where the equalities follow from Lemma 3.5. Once R is known, the transfer functions $G^0_{ji}(q)$ can be calculated step by step from the available $T^0_{ji}(q)$, remembering that each node in the cycle is either measured or excited.

Suppose node 2 is excited, we can recover $G^0_{21}(q) = T^0_{k1}(q)/T^0_{k2}(q)$. If 2 is measured, we can identify $G^0_{21}(q) = T^0_{21}(q)(1 - R)$. Consider that the next node 3 is excited, then we can recover $G^0_{32}(q) = T^0_{k1}(q)/(G^0_{21}(q)T^0_{k3}(q))$. If 3 is measured, we identify $G^0_{32}(q) = T^0_{31}(q)(1 - R)/G^0_{21}(q)$. Proceeding in a similar fashion we can recover $G^0_{i,i-1}(q)$ from knowledge of the previous identified modules as:

$$G^0_{i,i-1}(q) = \frac{T^0_{n1}(q)}{R_{i-2,1}T^0_{ni}(q)}, \text{ if } i \in \mathcal{B}, \tag{92}$$

$$G^0_{i,i-1}(q) = \frac{T^0_{i1}(q)(1 - R)}{R_{i-1,1}}, \text{ if } i \in \mathcal{C}, \tag{93}$$

for $i = 3, \ldots, n$. The last transfer function can be recovered from R since $G^0_{1n}(q) = R/R_{n1}$.

Necessity:

Consider that no node is both excited and measured. We show that condition (ii) must then hold to guarantee identifiability. If only one node is excited, then we can identify only

$n-1$ transfer functions $T^0_{ji}(q)$; hence we cannot identify the n elements $G^0_{ji}(q)$. The same holds if only one node is measured. This shows that we need at least two excited and two measured nodes in the loop. Suppose now that the loop contains at least two measured nodes and two excited nodes, but that condition (ii) does not hold. This implies that all the measured nodes are consecutive, and so are all the excited nodes. Without loss of generality, let nodes 1 to k be the excited nodes and let $k+1$ to n be the measured nodes.

Then, with R_{ik} defined in (84) for $i > k$, the corresponding matrix input-output $CT^0(q)B$ has the following form.

$$CT^0(q)B = (1-R)^{-1} \begin{pmatrix} R_{k+1,1} & R_{k+1,2} & \dots & R_{k+1,k-1} & R_{k+1,k} \\ R_{k+2,1} & R_{k+2,2} & \dots & R_{k+2,k-1} & R_{k+2,k} \\ \vdots & \vdots & & \vdots & \vdots \\ R_{n1} & R_{n,2} & \dots & R_{n,k-1} & R_{nk} \end{pmatrix}.$$

It now follows from (88) that the first row of $CT^0(q)B$ allows one to successively compute $G^0_{21}(q), G^0_{32}(q), \dots, G^0_{k,k-1}(q)$. It follows from (88) that all elements of the second row of $CT^0(q)B$ are equal to the corresponding elements of the first row multiplied by $G^0_{k+2,k+1}(q)$. Thus, knowledge of the second row allows one to compute one additional element of $G^0(q)$, namely $G^0_{k+2,k+1}(q)$. Pursuing row by row downwards up to the last row shows that we can compute $G^0_{k+2,k+1}(q), \dots, G^0_{n,n-1}(q)$. Collecting these results shows that, with this EMP (i.e. the cycle consists of k consecutive excited nodes followed by $n-k$ measured nodes), the corresponding $CT^0(q)B$ allows one to identify the edges $G^0_{21}(q), G^0_{32}(q), \dots, G^0_{k,k-1}(q)$ as well as the edges $G^0_{k+2,k+1}(q), \dots, G^0_{n,n-1}(q)$ can be identified, but not the edges $G^0_{k+1,k}(q)$ and $G^0_{1n}(q)$. □

We assume from now on that the cycles we consider have at least 3 nodes, noting that a cycle with two nodes is just a simple feedback system for which the identifiability conditions are well established. The following corollary provides an alternative formulation for the results of Theorem 3.15 and Theorem 3.17 which yields an even simpler way of checking the identifiability of an isolated cycle.

Corollary 3.4. *All transfer functions in an isolated cycle are generically identifiable if and only if $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ and, in addition: (i) either $\mathcal{B} \cap \mathcal{C} \neq \emptyset$, or (ii) the excited nodes (and hence also the measured nodes) are not all consecutive along the cycle.*

Proof. The result is included in the proof of Theorem 3.17. For the special case of $n = 3$, it is easy to see that if $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$, then necessarily condition (ii) is violated. □

Not only are the conditions of Corollary 3.4 necessary and sufficient, but in addition their verification on a given cycle graph is trivial: it can be done by visual inspection. Conversely, if the objective is to establish minimal EMPs for the identification of a cycle

graph, condition (ii) in this Corollary provides all minimal EMPs. There is a total of $2^n - n(n-1) - 2$ minimal EMPs from which the user can choose from. Once the minimal EMPs are characterized, all other valid EMPs can be obtained by just picking at least one node to be both excited and measured. These EMPs correspond to a number of $\sum_{k=1}^{n-1} 2^{n-k} n!/(k!(n-k)!) + 1$ out of the total EMPs.

It is easy to spot invalid EMPs. Consider, for instance, the two EMPs for a 5-node cycle depicted in Figure 15, where the one on the left is minimal and the one on the right is not valid. An EMP satisfying the necessary conditions of Theorem 3.2 is *invalid* only if no node in the cycle is both excited and measured, and all excitations and measurements are arranged in an uninterrupted sequence as in Figure 15-b); in other words, the excited nodes (and therefore the measured nodes) are all contiguous.

When we consider all possible EMPs, the number of valid EMPs outnumber those that are not generically identifiable for any number of nodes in the cycle, since it is sufficient to have at least one node excited and measured with any combination of the remaining nodes. This means that if the user were to choose randomly an EMP, it is likely that the chosen EMP is identifiable. Table I is a clear illustration of this. It provides, for cycles having 2 to 10 nodes, the number of minimal EMPs, of valid EMPs, and of invalid ones, assuming that in all cases the necessary condition is satisfied, i.e. each node is either excited or measured.

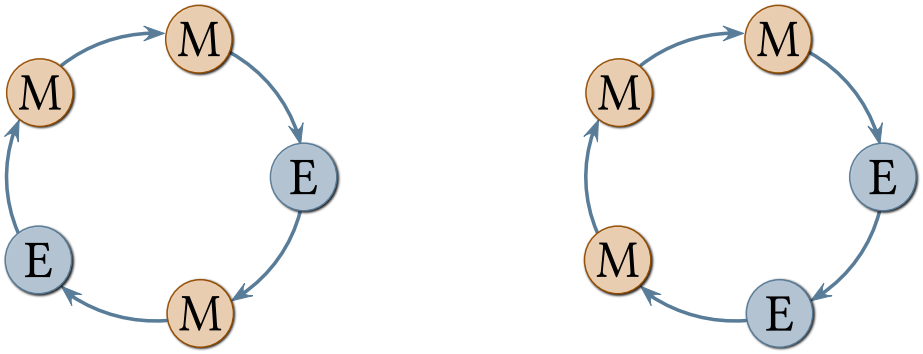
Table 3 – Number of minimal/valid/invalid EMPs for a cycle where each node is at least excited or measured.

nodes	minimal EMPs	valid EMPs	invalid EMPs
2	4	5	2
3	12	19	8
4	2	67	14
5	10	221	22
6	32	697	32
7	84	2143	44
8	198	6503	58
9	438	19609	74
10	932	58957	92

In conclusion, we have derived necessary and sufficient conditions for the generic identifiability of a cycle network. The conditions are very simple to check by visual inspection of the corresponding graph. Unlike the other results on identifiability of networks, no recourse to rank conditions or to vertex disjoint paths are required, as in the results of the previous sections. Besides the necessary requirement that each node must be either excited or measured, as is the case for all network structures, the requirement for cycles with 4 nodes or more is that either one node is both excited and measured,

Figure 15 – Two possible EMPs for a 5-node loop. **E** stands for an excited node and **M** for a measured node.

a) A minimal EMP. b) Not valid EMP.



Source: MAPURUNGA; GEVERS; BAZANELLA (2022).

or that not all measured (and hence not all excited) nodes are contiguous. Our results make it extremely easy to choose an EMP that makes the network identifiable and that is convenient for the user from a practical point of view.

3.7 Examples with mixed structures

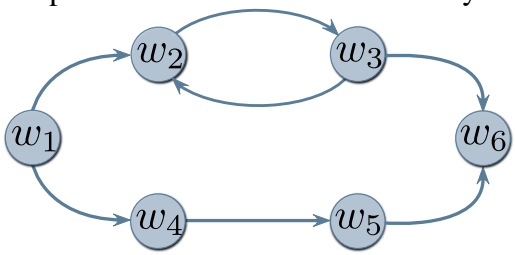
In this section we consider examples to illustrate the theorems presented in the previous sections. We deal with dynamic networks with more complex structures, which may contain many sub-graphs and feedback loops. The results presented so far were valid for particular network structures and they should be valid even when they are embedded in more complex structures. The identifiability conditions obtained for these structures can then be combined in order to obtain valid EMPs for the whole network by following a strategy of divide and conquer. Let us analyze the following example to see whether we can obtain valid EMPs by combining the identifiability results for the structures presented so far.

Example 3.4. *Consider the dynamic network depicted in Figure 16. The network matrix has the following form:*

$$G^0(q) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ G^0_{21}(q) & 0 & G^0_{23}(q) & 0 & 0 & 0 \\ 0 & G^0_{32}(q) & 0 & 0 & 0 & 0 \\ G^0_{41}(q) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G^0_{54}(q) & 0 & 0 \\ 0 & 0 & G^0_{63}(q) & 0 & G^0_{65}(q) & 0 \end{bmatrix}. \tag{94}$$

This network is a combination of a parallel network with a two-node cycles in one path. We have presented necessary and sufficient identifiability conditions for parallel networks in Theorem 3.7 and for cycles in Theorem 3.17. First, for any dynamic network we must

Figure 16 – A parallel network with a 2-node cycle embedded.



Source: The author.

have that each node is either excited or measured: $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$. *Additionally, from the conditions presented for these particular structures, it is expected that at least one node in the cycle must be both excited and measured, that is, either node 2 or 3 in Figure 16. As for the parallel network, from Theorem 3.7 we must obey one of the following conditions: either node* 2 *or* 3 *must be excited and measured (same for the cycle), node 2 is excited and 3 is measured, either node 4 or 5 is both excited and measured, and node 4 is excited and 5 is measured.*

Taking all the conditions together we must have: each node must be either excited or measured, node 1 *must be excited because it is a source, node* 6 *must be measured as it is a sink, either node 2 or node 3 must be excited and measured. This leads to the valid EMPs:* $1 \in \mathcal{B}, 6 \in \mathcal{C}, \{3,4,5\} \subset \mathcal{B} \cup \mathcal{C}, \{2\} \subset \mathcal{B} \cap \mathcal{C}$ *with cardinality 7. For this network there is a single minimal EMP with cardinality equal to 6:* $\text{EMP}_m = (\mathcal{B} = \{1,3,4\}; \mathcal{C} = \{2,5,6\})$. *One can recover the modules from the input-output relationship as follows (omitting* q*)*

$$
\begin{aligned}
G_{21}^0 &= T_{21}^0 - \frac{T_{23}^0}{T_{63}^0}\left(T_{61}^0 - \frac{T_{64}^0 T_{51}^0}{T_{54}^0}\right), \\
G_{23}^0 &= T_{23}^0 - \frac{[T_{23}^0]^2}{T_{63}^0 T_{21}^0}\left(T_{61}^0 - \frac{T_{64} T_{51}^0}{T_{54}^0}\right), \\
G_{32}^0 &= \frac{T_{54}^0 T_{61}^0 - T_{51}^0 T_{64}^0}{T_{21}^0 T_{54}^0 T_{63}^0 + T_{23}^0 \left(T_{51}^0 T_{64}^0 - T_{54}^0 T_{61}^0\right)}, \\
G_{41}^0 &= \frac{T_{51}^0}{T_{54}^0}, \\
G_{54}^0 &= T_{54}^0, \\
G_{63}^0 &= T_{63}^0 - \frac{T_{23}^0}{T_{21}^0}\left(T_{61}^0 - \frac{T_{64}^0 T_{51}^0}{T_{54}^0}\right), \\
G_{65}^0 &= \frac{T_{64}^0}{T_{54}^0}.
\end{aligned}
$$

Thus, the conditions for each subnetwork are sufficient for generic identifiability of the composed network, but not necessary. This implies that it is not necessary to comply with the excitation and measurement requirements of the cycle to make the network identifiable. Notice that the conditions on isolated cycles are for the identification of the cycle only. If we are willing to identify all modules from the network, it may be the case that we can

identify the modules corresponding to the cycle with other excitation/measurement from outside the cycle, as in the case here. Nevertheless, we can apply these conditions and obtain valid EMPs with cardinality $n+1=7$, which need only an additional measurement when compared to the minimal EMP. All valid EMPs with cardinality equal to seven are displayed in Table 4.

Table 4 – Valid EMPs with cardinality equal to seven for dynamic network from Figure 16.

EMP	$(\mathcal{B};\,\mathcal{C})$	EMP	$(\mathcal{B};\,\mathcal{C})$
EMP_1	$(\{1,3\};\{2,3,4,5,6\})$	EMP_2	$(\{1,3,5\};\{2,4,5,6\})$
EMP_3	$(\{1,3,5\};\{2,3,4,6\})$	EMP_4	$(\{1,3,4\};\{2,4,5,6\})$
EMP_5	$(\{1,3,4\};\{2,3,5,6\})$	EMP_6	$(\{1,3,4\};\{1,2,5,6\})$
EMP_7	$(\{1,3,4,6\};\{2,5,6\})$	EMP_8	$(\{1,3,4,5\};\{2,5,6\})$
EMP_9	$(\{1,3,4,5\};\{2,4,6\})$	EMP_{10}	$(\{1,3,4,5\};\{2,3,6\})$
EMP_{11}	$(\{1,2\};\{2,3,4,5,6\})$	EMP_{12}	$(\{1,2,5\};\{2,3,4,6\})$
EMP_{13}	$(\{1,2,4\};\{2,3,5,6\})$	EMP_{14}	$(\{1,2,4,5\};\{2,3,6\})$
EMP_{15}	$(\{1,2,3\};\{3,4,5,6\})$	EMP_{16}	$(\{1,2,3\};\{2,4,5,6\})$
EMP_{17}	$(\{1,2,3,5\};\{3,4,6\})$	EMP_{18}	$(\{1,2,3,5\};\{2,4,6\})$
EMP_{19}	$(\{1,2,3,4\};\{3,5,6\})$	EMP_{20}	$(\{1,2,3,4\};\{2,5,6\})$
EMP_{21}	$(\{1,2,3,4,5\};\{3,6\})$	EMP_{22}	$(\{1,2,3,4,5\};\{2,6\})$

From this table we can see that $\text{EMPs}_{2,4,8,9}$ allow us to relax the excitation and measurement for the cycle at the cost of exciting and measuring either node 4 or 5. With exception of $\text{EMPs}_{6,7}$, all other valid EMPs from Table 4 can be obtained by using the results of Theorems 3.17 and 3.7.

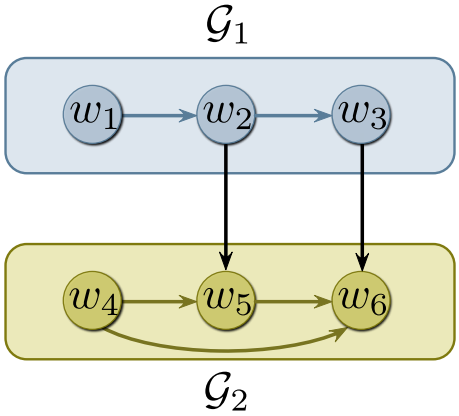
What this example shows is that we can combine the known identifiability conditions for the structures forming the network and still obtain valid EMPs. Although applying the identifiability conditions in the subnetworks render the whole network identifiable, one can not simply infer a minimal EMP from these conditions. Let us now analyze another example in which we can use the identifiability conditions for the structures and their interconnection.

Example 3.5. *Consider the dynamic network depicted in Figure 17.*

The network matrix is given by:

$$G^0(q)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ G^0_{21}(q) & 0 & 0 & 0 & 0 & 0 \\ 0 & G^0_{32}(q) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G^0_{52}(q) & 0 & G^0_{54}(q) & 0 & 0 \\ 0 & 0 & G^0_{63}(q) & G^0_{64}(q) & G^0_{65}(q) & 0 \end{bmatrix}. \tag{95}$$

Figure 17 – An example of network divided into two subnetworks: $\mathcal{G}_1$ is a branch, while $\mathcal{G}_2$ is a feedforward network.



Source: The author.

For this particular network there are two structures for which we can exploit the identifiability results, namely a branch subnetwork indicated as $\mathcal{G}_1$ and a feedforward subnetwork denoted as $\mathcal{G}_2$ in Figure 17. For the branch subnetwork we can use the results from Corollary 3.3, while for the feedforward network we can the results of Theorem 3.8. Combining these results we see that all edges within $\mathcal{G}_1$ and $\mathcal{G}_2$ can be uniquely identified if we excite nodes $1, 4, 5$, measure nodes $3, 5, 6$, and in addition to that we need to either excite or measure node 2. The question is now whether we can uniquely recover the modules $G^0_{52}(q)$ and $G^0_{63}(q)$. In order to answer this question we can use item (e) from Theorem 3.3. Using the partition in Figure 17 we have that $T^0_i(q) = (I - G^0_i(q))^{-1}$:

$$T^0_1(q) = \begin{bmatrix} 1 & 0 & 0 \\ G^0_{21}(q) & 1 & 0 \\ G^0_{32}(q)G_{21}(q) & G^0_{21}(1) & 1 \end{bmatrix}, \tag{96}$$

$$T^0_2(q) = \begin{bmatrix} 1 & 0 & 0 \\ G^0_{54}(q) & 1 & 0 \\ G^0_{65}(q)G^0_{54}(q) + G^0_{64}(q) & G^0_{65}(q) & 1 \end{bmatrix}. \tag{97}$$

The k-rank (See Definition 3.6) of $(T^0_1(q)B_1)^T$ is always equal to 1 if $B_1 \neq I$. Notice that from the perspective of $\mathcal{G}_1$, there is no requirement for exciting node 3 since this node is a sink for $\mathcal{G}_1$. As for $\mathcal{G}_2$ we need to measure at least nodes 5 and 6. This implies that the k-rank of $C_2T^0_2(q)$ is equal to 2, and the sufficient condition in item (e) from Theorem 3.3 holds. This results in the EMP $\mathcal{B} = \{1, 4, 5\}, \mathcal{C} = \{3, 5, 6\}$. Therefore, we can apply valid EMPs for the two subnetworks and this is sufficient for the whole network to be identifiable.

Even though for the subnetwork $\mathcal{G}_2$ node 5 needs to be both excited and measured, the case for the whole network is that node 5 does not need to be excited, but it needs to be measured. This follows from Theorem 3.9 by examining the neighborhood of node 5. Since there is one in-neighbor ($2 \in \mathcal{N}^-_5$) of node 5 that is not connected to the single

out-neighbor ($\mathcal{N}_5^+ = \{6\}$) of node 5, then node 5 is not a dource – see Definition 3.16 – and might not be excited. However, node 5 is a dink and must be measured. One valid EMP that does not have this requirement is $(\mathcal{B} = \{1, 3, 4\}, \mathcal{C} = \{2, 3, 5, 6\})$.

These examples show that identifiability conditions for particular structures are useful to determine valid EMPs for more complex dynamic networks. This can be the case even when the known subnetworks are embedded in a larger network. However, there is no guarantee that the combination of minimal EMPs for each subnetwork will lead to a minimal EMP for the whole network. Example 3.4 have illustrated one occasion where this does not happen.

3.8 Conclusions

In this chapter we presented the concept of the identifiability in dynamic networks. Conditions on the identifiability of dynamic networks usually depend on the rank of certain submatrices. An algebraic formulation for the identifiability of dynamic networks was presented which led to the concept of generic identifiability. This definition allowed to provide topological conditions for the identifiability problem using tools from graph theory.

Identifiability in dynamic networks can be equivalently expressed in terms of the experimental setting applied at the network. The concept of minimum excitation and measurement pattern (EMP) was introduced, which is based on the experimental setting to achieve network identifiability. This new approach to look at the identifiability problem is to determine which nodes need to be excited and which nodes need to be measured. Based on this new approach, we have thus characterized the minimal EMPs for some classes of networks. Necessary and sufficient conditions for parallel, feedforward, and cycle networks were derived. Furthermore, we have derived necessary conditions for excitation/measurement of certain nodes in an acyclic dynamic network that are based on the local topology of the nodes.

We have also explored how these EMPs can be applied to subnetworks and whether additional conditions were necessary to achieve identifiability of the whole network. These conditions depend on the excitation and measurement of some nodes of the subnetworks. We have provided examples that illustrated how these developed conditions can be used in more complex networks.

The results of this chapter generated two publications: MAPURUNGA; BAZANELLA (2021b); MAPURUNGA; GEVERS; BAZANELLA (2022). A third paper is currently under preparation (MAPURUNGA; BAZANELLA; GEVERS, 2022).

4 SELECTION OF EMPS FOR DYNAMIC NETWORKS

This chapter deals with the problem of selecting, among all possibles excitation and measurement patterns, the one which yields the most accurate parameter estimates. For identifying the modules of the network we use the prediction error method, which under some mild conditions achieves the Cramer-Rao lower bound. The analysis is focused on two classes of networks: branches and cycles. This chapter begins by presenting the problem under analysis and providing the framework under which it will be solved. Afterwards, we focus in solving the problem for branch networks and then for cycles. We provide a number of key principles that can be used as guidelines to choose the best experimental setup according to certain objectives.

4.1 Introduction

Experiment design is one of the fields of research in the system identification community. This field of research deals with the design of the variables involved in the identification experiment. Examples of such variables are the number of samples N to collect from the experiment, the sampling time employed in the experiment, the decision of which variables will be used as inputs and which variables will be used as outputs.

One of the main problems in the experiment design for system identification is the selection of an appropriate input signal to adequately excite the system at hand. Many methods for designing an appropriate input signal have been proposed in the literature, many of them can be cast into an convex optimization problem. This is particularly advantageous because there are plenty of optimization algorithms available for this task. For the prediction error method to provide consistent estimates it is necessary that the model structure is identifiable and also that the input-output data used are informative enough with respect to the model structure. Therefore, an appropriate design of the input signal is of paramount importance for the identification method performance.

A crucial aspect when dealing with multivariable systems is the selection of which variables should be used as inputs and which variables should be used as outputs. In some scenarios, the user may not have access to the measurement of certain signals or there may

be some constraints that make the manipulation of certain signals unfeasible. Furthermore, the associated costs of exciting may be different across the possible inputs of the system. This issue can be drastically aggravated when we deal with large interconnected systems.

A relevant problem of the experiment design is how to appropriately choose the inputs and outputs of the system at hand. This problem becomes of paramount importance when dealing with large-scale networks. How to obtain relevant information content from the network dynamics depends on the experimental setting applied to the network. The focus of this chapter is to provide an answer to the problem of how to select the inputs and outputs in a dynamic network, that is, its Excitation and Measurement Pattern (EMP). Is it better to excite or measure a certain node? Are there some EMPs that provide most accurate estimates than others? We provide answers and principles for these questions and we analyze the structural property that an appropriate input-output selection can make on the accuracy of the parameter estimates.

We will see that there are some key principles that can be used to determine which EMP yields the most accurate estimates. They will serve as guidelines for the user to appropriately select which nodes to excite and which nodes to measure in a dynamic network. We will provide these answers for two particular classes of networks: branches and cycles. The principles presented here for these two classes of networks can, under some conditions, serve as guiding principles for more general topologies.

4.2 Network setup and assumptions

We have seen in Chapter 3 that there are many combinations of inputs and outputs that render a particular class of network identifiable. In fact, the conditions for identifiability of the networks took the form as conditions on which nodes to excite and which nodes to measure. In other words, the conditions are based on the excitation and measurement pattern (EMP) of the network. For a given network there is typically a number of EMPs available for the user to choose. Here, we are interested in providing a framework for the user to choose an appropriate experimental setup with the best possible accuracy.

Let us recall how we have defined a dynamic network model. For the analysis, we are going to consider that only some nodes are available for direct measurement and only some nodes can be excited by a known excitation signal. The data is generated by the following network given the standard module representation without process noise (VAN DEN HOF *et al.*, 2013):

$$\begin{aligned} w(t) &= G^0(q)w(t) + Br(t), \\ y(t) &= Cw(t) + e(t), \end{aligned} \tag{98}$$

where $w(t) \in \mathbb{R}^n$ represents the internal signals of the network. $r(t) \in \mathbb{R}^m$ is the vector of external excitation signals, $y(t) \in \mathbb{R}^p$ is the vector of available measurements of the network corrupted by sensor noise $e(t) \in \mathbb{R}^p$. We assume that the topology of the network

is known. Recall that matrices B and C are selection matrices responsible for indicating where the inputs are applied in the network and which measurements are taken. Associated with these matrices are the set of excited nodes $\mathcal{B}$ and the set of measured nodes $\mathcal{C}$, together they define an EMP – see Definition 3.5. The data generating system (98) can be written into an input-output representation:

$$y(t) = CT^0(q)Br(t) + e(t),\ T^0(q) \triangleq (I - G^0(q))^{-1}. \tag{99}$$

The modules associated with the network matrix can be identified from input-output data $\{r(t), y(t)\}$, $t = 1, \ldots, N$. In order to identify the modules from $G^0(q)$, we are going to use the prediction error method (PEM) and the parametrized model:

$$y(t, \theta) = C\left(I - G(q, \theta)\right)^{-1} Br(t) + e(t). \tag{100}$$

The corresponding optimal one-step ahead predictor of $y(t)$ is given by:

$$\hat{y}(t|t-1, \theta) = C\left(I - G(q, \theta)\right)^{-1} Br(t) = CT(q, \theta)Br(t). \tag{101}$$

For the identification of the modules of the network the following assumption will be valid throughout this chapter.

Assumption 4.1.

(a) the transfer functions $G^0_{ji}(q)$ are proper and $T^0(q) \triangleq (I - G^0(q))^{-1}$ is stable;

(b) There exists a unique parameter vector θ^0 such that $G(q, \theta^0) \equiv G^0(q)$;

(c) Each module in $G(q, \theta)$ is independently parametrized: $G_{ji}(q, \theta_{ji})$;

(d) The input-output data is sufficiently rich so that for any pair C, B, the transfer matrix $CT^0(q)B$ can be consistently estimated from network data.

(e) the external signals $\{r_i(t)\}$ are independent zero mean white noise processes with variance σ_i^2 and uncorrelated with all noise processes $\{e_j(t)\}$;

(f) the corrupting noise sequences $\{e_j(t)\}$ are independent stationary Gaussian white noise processes with zero mean and variance λ_j.

Notice that assuming that the inputs are white noise process is not a practical restriction, since one can approximate the spectrum of a white noise process by using a pseudo-binary random signal (PRBS) (LJUNG, 1999). We further assume that the nodes have been labeled sequentially in the network.

Whether the modules can be uniquely recovered from input-output data will depend on the choice of the matrices B and C. There may be many combinations of these matrices such that the whole network is identifiable. For this purpose, we use *valid* EMPs, see

Definition 3.5 from Chapter 3. Recall that a valid EMP is a tuple of the set of excited and measured nodes that renders a network generically identifiable.

Our interest is to determine which minimal EMP – one with the minimum number of excitations and measurements combined – yields the most accurate parameter estimates. The search for minimal EMPs can be justified by the minimum requirements that the user must use to identify a network. Furthermore, these EMPs provide a framework in which comparison between exciting and measuring certain nodes is possible. It should be noted that valid EMPs can be obtained from a minimal EMP by just adding more excitations and/or measurements.

In order to determine which minimal EMP yields the most most accurate results, we must assess the accuracy of the parameter estimates $\hat{\theta}$ from input-output data $\{r(t), y(t)\}$. It is well-known that PEM achieves asymptotically the Cramer-Rao lower bound under the Gaussian assumption (LJUNG, 1999; SÖDERSTRÖM; STOICA, 1989; SÖDERSTRÖM, 2006). The asymptotic covariance matrix of PEM can be evaluated as:

$$P = \frac{1}{N}[\mathbb{E}\psi(t,\theta)\Lambda^{-1}\psi^T(t,\theta)]^{-1}|_{\theta=\theta^0} \triangleq M^{-1}, \tag{102}$$

where $\mathbb{E}$ denotes mathematical expectation, $\psi(t,\theta)$ is the gradient of the prediction error $(y(t) - \hat{y}(t|t-1,\theta))$ with respect to the parameter vector θ, Λ is the noise covariance matrix, and M is the information matrix. We will be concerned with the choice of the minimal EMP that achieves the most accurate parameter estimates according to a measure of the asymptotic covariance matrix P. We point out that all the analysis in the following is based on the Cramer-Rao lower bound, and therefore it is valid for any efficient estimator. The problem we tackle in this chapter can be formulated as follows. Given a dynamic network satisfying Assumption 4.1, determine which minimal EMP provides the smallest trace of the asymptotic covariance matrix P.

Here we adopted the trace of P as a criterion, which in the literature of optimal experiment design is known as A-optimally criterion (PUKELSHEIM, 2006). This criterion will allow us to choose a minimal EMP according to the precision it yields for the parameter estimates. We remark that similar conclusions can be derived if one considers the determinant of P, which is known as D-optimality criterion.

There are various factors that compete against each other to determine which EMP leads to the most accurate estimates: the parametrization of the modules, the location of poles and zeros of each transfer function, signal-to-noise ratio at some nodes. In order to isolate these factors and provide meaningful insights with respect to the choice of the minimal EMPs, the following assumptions are instrumental in the theoretical analysis provided in this chapter.

Assumption 4.2. *All transfer functions of the dynamic network are* identical*:* $G_k^0(e^{j\omega}) \equiv G^0(e^{j\omega})$.

Assumption 4.3. *The external excitation signals* $\{r_i(t)\}$ *have the same variance* $\sigma_i^2 = \sigma^2$ *for* $i = 1, 2, \ldots m$*. The covariance matrix* Λ *associated with the noise* $e(t)$ *can be written as* λI_p*, where* I_p *is the identify matrix of size* p*.*

These assumptions are not going to be employed in all results that follow. They may seem restrictive, but they are necessary to isolate particularities of the modules from the structural property of the experimental setting. Furthermore, a fair comparison should consider an equally exciting scenario for all EMPs. This is precisely what Assumption 4.3 does. Notice that Assumption 4.3 implies that all excitation signals have the same second-order statistical properties, the same is valid for the measurement noise. When this assumption holds, the comparison among different EMPs does not depend on the magnitude of the variances of the input and noise signals. We refer to an EMP where Assumption 4.3 holds as an *equally excited* EMP. Regarding this assumption, it will be not used in all results that follow. For this reason, we will introduce the following definition that will be useful later on.

Definition 4.1. *The signal-to-noise ratio from excitation at node* i *to measurement taken from node* j*, denoted as SNR*$_{ji}$*, is defined as* σ_i^2/λ_j*.*

This definition is related to the ratio of input energy from excitation $\{r_i(t)\}$ in node j with respect to the measurement noise $\{e_j(t)\}$. The larger the SNR$_{ji}$ is, the better is the information content from node i to node j. The next definition concerns the excitation and measurement of a particular module.

Definition 4.2. *A module* $G_{ji}^0(q)$ *is called a* direct module *if* $i \in \mathcal{B}$ *and* $j \in \mathcal{C}$*.*

Direct modules will be one of the key factors for determining which minimal EMP provides the most accurate results for dynamic networks. For the remainder of this chapter, we are going to adopt the following index for the modules: $G_{ji}^0(q) \triangleq G_i^0(q)$, that is, we use only the in-neighbor i to refer to module $G_{ji}^0(q)$.

4.3 Computation of the information matrix

Before we start the analysis of which EMP provides the most accurate outcomes, we are going to show how the information matrix can be computed for a given EMP. In order to compute the asymptotic covariance matrix we only need to compute the gradient $\psi(t, \theta)$ of the optimal predictor and we need to know the noise covariance matrix. If the noise covariance matrix is not known, it can be estimated from data and used to get an approximation of the asymptotic covariance matrix. For the purpose of our analysis we will assume that both the "true" parameter system θ^0 and the noise covariance matrix are known. This is, of course, no limitation of the prediction error method as when the number of data grows the PEM estimate will asymptotically converge to θ^0, assuming that a model

structure capable of representing the "true system" has been selected. Let us define the gradient with respect to output j and input i as:

$$\psi_{ji}(t,\theta) \triangleq \frac{\partial T_{ji}(q,\theta) r_i(t)}{\partial \theta}. \tag{103}$$

With this definition we can decompose the gradient of the optimal predictor as follows:

$$\psi(t,\theta) = \begin{bmatrix} \sum_{i\in\mathcal{B}} \psi_{c_1 i}(t,\theta) & \sum_{i\in\mathcal{B}} \psi_{c_2 i}(t,\theta) & \cdots & \sum_{i\in\mathcal{B}} \psi_{c_p i}(t,\theta) \end{bmatrix}, \tag{104}$$

for $c_k \in \mathcal{C}$. We can use this decomposition to simplify the calculation of the information matrix under Assumption 4.1. For this purpose, let us define the partial information matrix.

Definition 4.3. *The* partial *information matrix from input $r_i(t), i \in \mathcal{B}$ to output $y_j, j \in \mathcal{C}$ is defined as*

$$M_{ji} \triangleq \frac{N}{\lambda_j} \mathbb{E}\psi_{ji}(t,\theta)\psi_{ji}^T(t,\theta). \tag{105}$$

With this definition at hand, we can further simplify the computation of the information matrix as formally stated in the following lemma.

Lemma 4.1. *Consider a dynamic network defined in (98). Under Assumption 4.1, the information matrix can be written as*

$$M = \sum_{i\in\mathcal{B}, j\in\mathcal{C}} M_{ji}. \tag{106}$$

Proof. Under the stated assumption the information matrix can be computed as $M = N\mathbb{E}\psi(t,\theta)\Lambda^{-1}\psi(t,\theta)$. Using (104) and the fact that $\Lambda = diag(\lambda_{c_1}, \lambda_{c_2}, \dots \lambda_{c_p})$, we can decompose M as:

$$M = \frac{N}{\lambda_{c_1}} \mathbb{E} \sum_{i\in\mathcal{B}} \psi_{c_1,i}(t,\theta) \sum_{i\in\mathcal{B}} \psi_{c_1,i}^T(t,\theta) + \frac{N}{\lambda_{c_2}} \mathbb{E} \sum_{i\in\mathcal{B}} \psi_{c_2,i}(t,\theta) \sum_{i\in\mathcal{B}} \psi_{c_2,i}^T(t,\theta) + \\ \cdots + \frac{N}{\lambda_{c_p}} \mathbb{E} \sum_{i\in\mathcal{B}} \psi_{c_p,i}(t,\theta) \sum_{i\in\mathcal{B}} \psi_{c_p,i}^T(t,\theta).$$

Since all inputs are mutually independent, it holds that the terms $\mathbb{E}\psi_{k,i}(t,\theta)\psi_{l,j}^T(t,\theta) = 0$, for $k, l \in \mathcal{C}$ and $i \neq j \in \mathcal{B}$. As all these terms are zero, the remaining terms are the ones in (106). □

This lemma states that, under Assumption 4.1, we can compute the information matrix of a given EMP by considering the contribution of the partial information matrices from each input to each output. This result will be particularly useful in the analysis that follow, specially for cycles. We are now ready to start our analysis for branches and cycles.

4.4 Branch networks

Before tackling the problem of which minimal EMP provides the most accurate parameter estimates for a branch network, let us recall what characterizes a branch network. Recall that a branch network is composed of n nodes sequentially linked, where each node is only connected to the next node until the last node is reached, assuming that the nodes have been labeled sequentially. The first node is the source, while the last node is the sink of the branch. A branch network can be characterized by the structure of its network matrix, which has the following form

$$G(q,\theta^0) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ G_1(q,\theta_1^0) & 0 & 0 & \cdots & 0 \\ 0 & G_2(q,\theta_2^0) & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & G_{n-1}(q,\theta_{n-1}^0) & 0 \end{bmatrix}. \tag{107}$$

Recall that we have labeled the modules according to its in-neighbor node: $G_{ji}^0(q) \triangleq G_i^0(q)$. A branch network is also called a cascade network. The accuracy of such networks has been investigated in WAHLBERG; HJALMARSSON; MÅRTENSSON (2009). More recent works include EVERITT; ROJAS; HJALMARSSON (2013, 2014). In all these works, the interest was either in a particular module within the network or in analyzing the effects of common dynamics of the modules. The analysis provided in them is valid for just one EMP: where the source is excited and all other nodes are measured.

Recall that the identifiability conditions and the EMPs for this class of network were presented in Chapter 3. In summary, the source node must be excited, the sink node must be measured, and all other nodes must be either excited or measured. In this way, depending on the number of nodes, there will be a considerable number of EMPs to analyze, more precisely 2^{n-2}. The set of candidate minimal EMPs is characterized by the conditions formally stated in the next corollary.

Corollary 4.1. *(BAZANELLA; GEVERS; HENDRICKS, 2019). In a branch network (98) with network matrix (107) an EMP is minimal if and only if* $1 \in \mathcal{B}$, $n \in \mathcal{C}$, $\mathcal{B} \cup \mathcal{C} = \mathcal{W}$, $\mathcal{B} \cap \mathcal{C} \neq \emptyset$.

This corollary is a restatement of the conditions presented for generic identifiability for branches in Corollary 3.3 of Chapter 3. Here, we are interested in the different experimental settings that give the best accuracy of the parameter estimates, that is, which minimal EMPs will have the most accurate results.

Let us first see how to compute the information matrix associated with branch networks. We are going to proceed by using the concept of partial information matrix – see Definition 4.3. In order to compute the information matrix we first need the expression of

the input-output relationship for branch networks. For a branch network we have that:

$$T_{ii}^0(q) = 1, \text{ for } i \in \mathcal{W}, \tag{108}$$

$$T_{ij}^0(q) = 0, \text{ for } i < j \in \mathcal{W}, \tag{109}$$

$$T_{ji}^0(q) \triangleq \rho_{ji}^0(q) = \prod_{i=1}^{j-1} G_i^0(q) \text{ for } i < j \in \mathcal{W}. \tag{110}$$

Using these expressions we are able to obtain the general form of the gradient of the optimal predictor $\psi(t,\theta)$. By decomposing the gradient $\psi(t,\theta)$ according to (104) we obtain for $i < j < n$:

$$\begin{aligned}\psi_{ji}^T(t,\theta) = \frac{\partial T_{ji}(q,\theta) r_i(t)}{\partial \theta}^T &= \begin{bmatrix} \mathbf{0}_{i-1} & \frac{\partial \rho_{ji}}{\partial \theta_i}^T & \cdots & \frac{\partial \rho_{ji}}{\partial \theta_{j-1}}^T & \mathbf{0}_{n-j} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{i-1} & G_i'^T \frac{\rho_{ji}}{G_i(q,\theta_i)} & \cdots & G_{j-1}'^T \frac{\rho_{ji}}{G_{j-1}(q,\theta_{j-1})} & \mathbf{0}_{n-j} \end{bmatrix}, \end{aligned} \tag{111}$$

where $\rho_{ji} \triangleq \prod_{k=i}^{j-1} G_k(q,\theta_k) r_i(t)$, and $\mathbf{0}_k$ is a zero matrix with dimension corresponding to either the first $G_k(q,\theta_k)$ modules or the last $G_k(q,\theta_k)$ modules. The prime denotes differentiation with respect to the parameter vectors: $G_i' \triangleq \frac{\partial G_i(q,\theta_i)}{\partial \theta_i}$. We will start the analysis using small branch networks with just a few nodes to provide the key insights for the problem at hand. We further extend these results by increasing the number of nodes in the network.

4.4.1 3-node branches

We start our analysis by looking into branch networks with three nodes. A branch network with three nodes has the following network matrix.

$$G(q,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ G_1(q,\theta_1) & 0 & 0 \\ 0 & G_2(q,\theta_2) & 0 \end{bmatrix}. \tag{112}$$

This dynamic network has two modules $G_1(q,\theta_1)$ and $G_2(q,\theta_2)$ to be identified. The set of minimal EMPs for this network is given in Corollary 4.1. They are characterized as follows. The first node needs to be excited and the last node needs to be measured. The second node could be either excited or measured, and this defines the two minimal EMPs for this network:

I. $\text{EMP}_1 = (\mathcal{B} = \{1\}, \mathcal{C} = \{2,3\})$;

II. $\text{EMP}_2 = (\mathcal{B} = \{1,2\}, \mathcal{C} = \{3\})$.

The decision between these two minimal EMPs can be equivalently stated as the decision on whether to excite or measure the second node. In WAHLBERG; HJALMARSSON; MÅRTENSSON (2009) minimal EMP_1 was analyzed and some insights were given assuming *identical* transfer functions ($G_1^0(q) \equiv G_2^0(q)$), i.e. under Assumption 4.2. Under this

premise, the covariance matrix for EMP_1 was derived in WAHLBERG; HJALMARSSON; MÅRTENSSON (2009) as:

$$P_1 = \begin{bmatrix} A^{-1} & -A^{-1} \\ -A^{-1} & A^{-1} + B^{-1} \end{bmatrix}, \tag{113}$$

where

$$A \triangleq \frac{N}{\lambda_2}\mathbb{E}[G_1' r_1 \times G_1'^T r_1], \tag{114}$$

$$B \triangleq \frac{N}{\lambda_3}\mathbb{E}[G_2' G_1 r_1 \times G_2'^T G_1 r_1]. \tag{115}$$

From now on, we drop the arguments q and t in order to stress the dependence on the modules $G_k(q)$ for these matrices expressions. The subscript in P_1 is used to refer to minimal EMP_1, we will adopt this convention for the rest of this chapter. Under Assumption 4.2, the main conclusions drawn in WAHLBERG; HJALMARSSON; MÅRTENSSON (2009) were

- The quality of the estimate $\hat{\theta}_1$ is not improved by measurement of $\{y_3(t)\}$.
- The covariance of estimate $\hat{\theta}_2$ is larger than or equal to covariance of $\hat{\theta}_1$.

The first observation is intriguing, since one would expect that by adding more information with two measurements would improve the quality of the estimates. The second observation is related to the influence of the direct modules, as pointed out in MAPURUNGA; BAZANELLA (2021a), direct modules – in this case $G_1(q, \theta_1)$, see Definition 4.2 – are estimated more accurately.

Before tackling the problem of whether to excite or measure node 2 in this network, let us provide a dual analysis for minimal EMP_2 under Assumption 4.2. The gradient of the minimal EMP_2 is as follows:

$$\psi_2(t) = \begin{bmatrix} G_1' G_2(q, \theta_2) r_1(t) \\ G_2' G_1(q, \theta_1) r_1(t) + G_2' r_2(t) \end{bmatrix}. \tag{116}$$

The asymptotic covariance matrix can be found as the inverse of the information matrix:

$$\text{cov}\left(\begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix}\right) \sim M_2^{-1}, \tag{117}$$

where

$$M_2 = \begin{bmatrix} F & H \\ H^T & B + L \end{bmatrix}, \tag{118}$$

$$F \triangleq \frac{N}{\lambda_3}\mathbb{E}[G_1' G_2 r_1 \times G_1'^T G_2 r_1], \tag{119}$$

$$H \triangleq \frac{N}{\lambda_3}\mathbb{E}[G_2' G_1 r_1 \times G_1'^T G_2 r_1], \tag{120}$$

$$L \triangleq \frac{N}{\lambda_3}\mathbb{E}[G_2' r_2 \times G_2'^T r_2]. \tag{121}$$

If the two modules are identical then $F = H = B$. In this case, the asymptotic covariance matrix is:

$$P_2 = \begin{bmatrix} L^{-1} + B^{-1} & -L^{-1} \\ -L^{-1} & L^{-1} \end{bmatrix}. \tag{122}$$

Hence, under Assumption 4.2, we can make the following observations with respect to minimal EMP_2:

- The covariance of estimate $\hat{\theta}_1$ is larger than or equal to the covariance of $\hat{\theta}_2$.
- The quality of $\hat{\theta}_2$ is not improved by excitation $\{r_1(t)\}$, since $\text{cov}(\hat{\theta}_2)$ does not depend on $\{r_1(t)\}$ – see (122).

These conclusions are dual to those for the EMP_1, in this case we are analyzing the effect of the addition of an excitation source instead of a new measurement. The measurement $\{y_3(t)\}$ in EMP_1 is to $\text{cov}(\hat{\theta}_1)$ as the excitation signal $\{r_1(t)\}$ in EMP_2 is to $\text{cov}(\hat{\theta}_2)$. Furthermore, we observe again that the direct module, in this case $G_2(q, \theta_2)$, is estimated more accurately. These conclusions will also be observed for branch networks with more nodes, by maintaining the structures of these two EMPs. In EMP_1 we only excite the source and measure the remaining nodes, and for EMP_2 we only measure the sink and excite all other nodes. In fact, Assumption 4.2 is not necessary for this phenomenon to happen as we will see later.

The next theorem allows to decide whether to measure or excite node two based on the trace of the covariance matrix.

Theorem 4.1. *Consider a 3-node branch network with network matrix given in (112). Under Assumption 4.2, EMP_2 yields a smaller trace of the covariance matrix if and only if SNR_{32} in EMP II is larger than SNR_{21} in EMP I:*

$$\frac{\sigma_2^2}{\lambda_3} > \frac{\sigma_1^2}{\lambda_2}, \tag{123}$$

otherwise EMP_1 is more accurate. Under Assumptions 4.2 and 4.3, it holds that both EMPs result in the same trace of the covariance matrix.

Proof. According to (113) and (122), the trace of the covariance matrices of the minimal EMPs are:

$$\mathrm{tr}(P_1) = \mathrm{tr}(A^{-1}) + \mathrm{tr}(A^{-1} + B^{-1}). \tag{124}$$

$$\mathrm{tr}(P_2) = \mathrm{tr}(L^{-1}) + \mathrm{tr}(L^{-1} + B^{-1}), \tag{125}$$

From these expressions, we see that the difference in the trace of the covariance matrix relies on $\mathrm{tr}(A^{-1})$ and $\mathrm{tr}(L^{-1})$. We can extend the expressions for A and L as:

$$A = \frac{N}{\lambda_2}\mathbb{E}[G_1' r_1 \times G_1'^T r_1] = \Gamma_A N \frac{\sigma_1^2}{\lambda_2},$$
$$L = \frac{N}{\lambda_3}\mathbb{E}[G_2' r_2 \times G_2'^T r_2] = \Gamma_L N \frac{\sigma_2^2}{\lambda_3},$$

where Γ_A, $\Gamma_L \succ 0$ are associated with the covariance function of the vector signal $G_i'(q) r_i(t)$ for $i = 1, 2$. Since $G_1^0(q) \equiv G_2^0(q)$ we have that $\Gamma_A = \Gamma_L$. In this way, if $\lambda_2/\sigma_1^2 > \lambda_3/\sigma_2^2$ then $\mathrm{tr}(A^{-1}) > \mathrm{tr}(L^{-1})$, implying that EMP_2 is more accurate. Otherwise, EMP_1 is the more accurate. Under Assumption 4.3, SNR_{21} is equal to SNR_{32}, then we have that $\mathrm{tr}(P_1) = \mathrm{tr}(P_2)$. □

The choice of EMP_2 over EMP_1 is equivalent to the decision of either exciting or measuring node 2. As stated in this theorem, this choice depends on whether SNR_{32} of EMP_2 is larger than SNR_{21} of EMP_1 – see Definition 4.1. Notice that this implies that the choice is related to the power applied in the direct modules – $G_1^0(q)$ for EMP_1 and G_2^0 for EMP_2. If the users have control over the input energy, they can use (123) as a tool to choose a value $\overline{\sigma}_2^2 > \sigma_1^2 \frac{\lambda_3}{\lambda_2}$, for which a better precision will be achieved by using EMP_2.

In order to understand the role that direct modules play in the accuracy of the EMPs we will consider specific parametrizations for the modules of the network. We start by analysing FIR modules:

$$G_{ji}(q, \theta_k^0) = \sum_{i=0}^{n_{ji}} \theta_{ki}^0 q^{-i}. \tag{126}$$

Recall that a transfer function is completely characterized by its impulse response. The next result shows how the numeric values of the parameters influence the accuracy obtained by the minimal EMPs.

Theorem 4.2. *Consider a 3-node branch network with first order FIR modules: $G_k(q, \theta_k^0) = \theta_k^0 q^{-1}$ for $k = 1, 2$. Under Assumption 4.3, minimal EMP_1 yields a least trace of the covariance matrix when compared to EMP_2 if and only if*

$$[\theta_1^0]^2 > [\theta_2^0]^2. \tag{127}$$

Proof. For notation convenience, we drop the superscript from θ_k^0 in all expressions that follow. With modules $G_k(q, \theta_k) = \theta_k q^{-1}$ for $k = 1, 2$, the asymptotic covariance matrices for EMPs$_{1,2}$ are:

$$P_1 = \begin{bmatrix} \frac{\lambda_2}{\sigma_1^2} & -\frac{\lambda_2\theta_2}{\sigma_1^2\theta_1} \\ -\frac{\lambda_2\theta_2}{\sigma_1^2\theta_1} & \frac{\lambda_2\theta_2^2+\lambda_3}{\sigma_1^2\theta_1^2} \end{bmatrix}, \tag{128}$$

$$P_2 = \begin{bmatrix} \frac{\lambda_3\theta_1^2}{\sigma_2^2\theta_2^2} + \frac{\lambda_3}{\sigma_1^2\theta_2^2} & -\frac{\lambda_3\theta_1}{\sigma_2^2\theta_2} \\ -\frac{\lambda_3\theta_1}{\sigma_2^2\theta_2} & \frac{\lambda_3}{\sigma_2^2} \end{bmatrix}. \tag{129}$$

Now, by applying the trace for each covariance matrix we have the following expression:

$$\frac{\mathrm{tr}(P_2)}{\mathrm{tr}(P_1)} = \frac{\lambda_3\theta_1^2\left(\sigma_1^2\theta_1^2 + \sigma_1^2\theta_2^2 + \sigma_2^2\right)}{\sigma_2^2\theta_2^2\left(\lambda_2\theta_1^2 + \lambda_2\theta_2^2 + \lambda_3\right)}. \tag{130}$$

Under Assumption 4.3, we have that $\sigma_1^2 = \sigma_2^2$ and $\lambda_2 = \lambda_3$. Now, we can rewrite (130) as:

$$\frac{\mathrm{tr}(P_2)}{\mathrm{tr}(P_1)} = \frac{\theta_1^2}{\theta_2^2}, \tag{131}$$

from which follows that $\mathrm{tr}(P_2) > \mathrm{tr}(P_1)$ if and only if (131) is larger than one. That is, minimal EMP$_1$ is most accurate than EMP$_2$ if and only if $\theta_1^2 > \theta_2^2$. □

This theorem illustrates the role the direct module plays in the decision of the most accurate EMP. The most accurate EMP will be one where the module with largest magnitude of the network is a direct module. For this simple network we have demonstrated that the selection of the most accurate EMP also depends on the numeric values of the parameters.

In conclusion, if the network is fully symmetrical then both EMPs give the same overall accuracy, which is due to the fact that the EMPs are "mirrored" (see Definition 4.4) – one is obtained from the other by changing each excitation for a measurement and vice-versa, except the sink and the source. Moreover, each module is identified more accurately when it is a direct module. On the other hand, when the SNRs are not uniform, best overall accuracy is obtained by the EMP in which the direct module has a larger SNR. These are general principles that, as will be seen further ahead in this chapter, also apply to more general networks.

4.4.2 4-node branches

In this section we consider branch networks composed of four nodes. For this class of networks, there are three modules to be identified: $G_1^0(q), G_2^0(q)$, and $G_3^0(q)$. From Corollary 4.1, there are four different minimal EMPs from one to choose, listed below:

I EMP$_1 = (\mathcal{B} = \{1\}, \mathcal{C} = \{2, 3, 4\})$;

II EMP$_2 = (\mathcal{B} = \{1, 2, 3\}; \mathcal{C} = \{4\})$;

III $\text{EMP}_3 = (\mathcal{B} = \{1,2\}; \mathcal{C} = \{3,4\})$;

IV $\text{EMP}_4 = (\mathcal{B} = \{1,3\}; \mathcal{C} = \{2,4\})$.

Minimal $\text{EMPs}_{1,2}$ in the four-node case can be seen as an extension of the minimal $\text{EMPs}_{1,2}$ from the three-node case, since they preserve a similar structure. For these two EMPs, we will show that the results from the previous section also apply for the four-node case and more generally for any number of nodes.

Now, under Assumption 4.2, i.e. *identical* modules $G_1^0(q) = G_2^0(q) = G_3^0(q)$, with the following parametrization $G_k(q, \theta_k^0) \triangleq G_*$ for $k = 1, 2, 3$. The information matrix for the minimal EMP_1 becomes:

$$M_1 = \begin{bmatrix} A_1 + B_1 + C_1 & B_1 + C_1 & C_1 \\ B_1 + C_1 & B_1 + C_1 & C_1 \\ C_1 & C_1 & C_1 \end{bmatrix}, \tag{132}$$

where

$$A_i \triangleq \frac{N}{\lambda_{i+1}} \mathbb{E}[G_*' r_i \times G_*'^T r_i], \tag{133}$$

$$B_i \triangleq \frac{N}{\lambda_{i+2}} \mathbb{E}[G_*' G_* r_i \times G_*'^T G_* r_i], \tag{134}$$

$$C_i \triangleq \frac{N}{\lambda_{i+3}} \mathbb{E}[G_*' G_* G_* r_i \times G_*'^T G_* G_* r_i]. \tag{135}$$

For minimal EMP_2 one can find in a similar way:

$$M_2 = \begin{bmatrix} C_1 & C_1 & C_1 \\ C_1 & B_2 + C_1 & B_2 + C_1 \\ C_1 & B_2 + C_1 & A_3 + B_2 + C_1 \end{bmatrix}. \tag{136}$$

Information matrix of minimal EMP_3 is as follows:

$$M_3 = \begin{bmatrix} B_1 + C_1 & B_1 + C_1 & C_1 \\ B_1 + C_1 & A_2 + B_1 + B_2 + C_1 & B_2 + C_1 \\ C_1 & B_2 + C_1 & B_2 + C_1 \end{bmatrix}. \tag{137}$$

Finally, minimal EMP_4 yields the information matrix:

$$M_4 = \begin{bmatrix} A_1 + C_1 & C_1 & C_1 \\ C_1 & C_1 & C_1 \\ C_1 & C_1 & A_3 + C_1 \end{bmatrix}. \tag{138}$$

Table 5 – Covariance of the parameter estimates for each EMP under Assumptions 4.2 and 4.3.

EMP/θ	$\mathrm{cov}(\hat{\theta}_1)$	$\mathrm{cov}(\hat{\theta}_2)$	$\mathrm{cov}(\hat{\theta}_3)$
EMP_1	A_1^{-1}	$A_1^{-1}+B_1^{-1}$	$B_1^{-1}+C_1^{-1}$
EMP_2	$B_2^{-1}+C_1^{-1}$	$A_3^{-1}+B_2^{-1}$	A_3^{-1}
EMP_3	$[A_2 \star B_1 + B_2 \star C_1]^{-1}$[1]	$[A_2 + B_1 \star B_2 \star C_1]^{-1}$	$[A_2 \star B_1 + B_2 \star C_1]^{-1}$
EMP_4	A_1^{-1}	$A_1^{-1}+A_3^{-1}+C_1^{-1}$	A_3^{-1}

By inverting the information matrix of each EMP we obtain the covariance matrices for each parameter estimates, which are displayed in Table 5. If we additionally adopt the conditions stated in Assumption 4.3, we have that $A_i = A$, $B_i = B$ and $C_i = C$ for $i = 1, 2, 3$ in (132)-(138) and in Table 5.

Notice that $\mathrm{EMPs}_{1,2}$ are symmetrical, that is, they yield the same results but in the reverse order. The reason for that is that these EMPs are related by a symmetry related to the distribution of excitations and measurements, which we will further explore later on. For these two EMPs the key observations made in the previous section are also valid as can be seen from Table 5. The covariance matrix of $\hat{\theta}_1$ does not depend on additional measurements in EMP_1, while $\mathrm{cov}(\hat{\theta}_3)$ in EMP_2 does not depend on the first and second input signals. We will show later that this result also holds for an arbitrary number of nodes. If we consider the covariances of $\hat{\theta}_1$ and $\hat{\theta}_2$ from EMP_1 in the three-node case, we can see from Table 5 that EMP_1 in the four-node case does not improve the corresponding accuracy of $\hat{\theta}_1$ and $\hat{\theta}_2$ with respect to the three-node counterpart. This implies that adding a new measurement (from three nodes to four nodes) for EMP_1 does not improve the precision of the parameter estimates under Assumptions 4.2 and 4.3. A dual argument holds for EMP_2. The covariances of $\hat{\theta}_1$ and $\hat{\theta}_3$ are the same in EMP_4, which corresponds to $\hat{\theta}_1$ in EMP_1 and $\hat{\theta}_3$ in EMP_2 . This phenomenon is linked to how the location of excitations and measurements is distributed in the EMPs. We will show later in Section 4.4.3 when this effect happens for general branch networks.

Conversely to the three-node branch network case, where all minimal EMPs lead to equivalent overall accuracy, in the four-node case there is a minimal EMP that provides better precision. The next result shows that minimal EMP_3 yields a smaller trace of the covariance matrix than $\mathrm{EMPs}_{1,2}$.

Theorem 4.3. *Consider a 4-node branch network with dynamic matrix (107), for $n = 4$. Under Assumptions 4.2 and 4.3, minimal EMP_3 yields a smaller trace of covariance matrix than minimal $\mathrm{EMPs}_{1,2}$:*

$$\mathrm{tr}(P_3) < \mathrm{tr}(P_1) = \mathrm{tr}(P_2). \tag{139}$$

[1] $A \star B \triangleq [A^{-1} + B^{-1}]^{-1}$, $A \star B \star C \triangleq [A^{-1} + B^{-1} + C^{-1}]^{-1}$.

Proof. To prove (139) we are going to compare the covariance of each module of both minimal EMPs in Table 5 under Assumptions 4.2 and 4.3. If the following conditions hold:

$$A^{-1} \succ (A + (2B^{-1} + C^{-1})^{-1})^{-1}, \tag{140}$$

$$A^{-1} + B^{-1} \succ [[A^{-1} + B^{-1}]^{-1} + [B^{-1} + C^{-1}]^{-1}]^{-1}, \tag{141}$$

$$B^{-1} + C^{-1} \succ [[A^{-1} + B^{-1}]^{-1} + [B^{-1} + C^{-1}]^{-1}]^{-1}, \tag{142}$$

then $\mathrm{tr}(P_1) > \mathrm{tr}(P_3)$. This follows from the implication: $A \succ B \implies \mathrm{tr}(A) > \mathrm{tr}(B)$, with $A \succ B$ in the sense that $A - B \succ 0$ is positive definite. The above conditions can be made equivalent to:

$$A^{-1}(A^{-1} + B^{-1} + B^{-1} + C^{-1})^{-1}A^{-1} \succ 0, \tag{143}$$

$$A^{-1} + B^{-1} \succ 0, \tag{144}$$

$$B^{-1} + C^{-1} \succ 0, \tag{145}$$

after some manipulation and using the fact that $A \succcurlyeq B \iff A^{-1} \preccurlyeq B^{-1}$. The last two conditions hold true since they define the covariance matrices of $\hat{\theta}_2$, $\hat{\theta}_3$ in Table 5 for EMP_1. Condition (143) holds by definition and it was obtained using the matrix inversion lemma. □

This result implies that under Assumptions 4.2 and 4.3 using EMP_3 is more advantageous than $\mathrm{EMPs}_{1,2}$. As for EMP_4 there are values of the modules for which EMP_4 can be made better than the others. EMP_3 will be more accurate than any other minimal EMP if: $\mathrm{tr}(P_1) \leq \mathrm{tr}(P_4)$. From Table 5, this could be achieved as:

$$3A^{-1} + C^{-1} \succcurlyeq A^{-1} + 2B^{-1} + C^{-1}$$
$$A^{-1} \succcurlyeq B^{-1} \iff B \succcurlyeq A.$$

From (133)-(134) we notice that for "large" values of G the above expression will hold true. The rationale is that A is filtered by G to produce B, which implies that for "large" values of G the expression $B - A$ will tend to be positive definite. In this scenario, EMP_3 will be the experimental setting that yields the smallest trace of the covariance matrix.

In order to analyze the influence of the numeric values of the modules we will analyze whether EMP_3 is better for state-space branch networks, i.e. a network with first order FIR modules as in (126). The following result shows that for a first order FIR network EMP_3 yields the most accurate results.

Theorem 4.4. *Consider a 4-node branch networks with network matrix in (107). Assume that the modules have the following structure $G_k(q, \theta_k^0) = \theta_k^0 q^{-1}$. Under Assumptions 4.2 and 4.3, EMP_3 yields the smallest trace of the covariance matrix among the minimal EMPs.*

Proof. That EMP_3 yields a smaller trace of covariance matrix than $\text{EMPs}_{1,2}$ has been proved in Theorem 4.3. Now, it remains to show that under the stated assumptions EMP_3 is most accurate than EMP_4. Under Assumption 4.2 we have that $G_k(q,\theta_k^0) = \theta^0 q^{-1}$ for $k = 1, 2, 3$. The covariance matrices of EMP_3 and EMP_4 under the stated assumptions are:

$$P_3 = \begin{bmatrix} \frac{\lambda_4}{\sigma_1^2[\theta^0]^2} & -\frac{\lambda_4}{\sigma_1^2([\theta^0]^2+1)} & 0 \\ -\frac{\lambda_4}{\sigma_1^2([\theta^0]^2+1)} & \frac{\lambda_4\left(2[\theta^0]^2+1\right)}{\sigma_1^2([\theta^0]^4+2[\theta^0]^2+1)} & -\frac{\lambda_4}{\sigma_1^2([\theta^0]^2+1)} \\ 0 & -\frac{\lambda_4}{\sigma_1^2([\theta^0]^2+1)} & \frac{\lambda_4}{\sigma_1^2[\theta^0]^2} \end{bmatrix}, \tag{146}$$

$$P_4 = \begin{bmatrix} \frac{\lambda_4}{\sigma_1^2} & -\frac{\lambda_4}{\sigma_1^2} & 0 \\ -\frac{\lambda_4}{\sigma_1^2} & \frac{\lambda_4\left(2[\theta^0]^4+1\right)}{\sigma_1^2[\theta^0]^4} & -\frac{\lambda_4}{\sigma_1^2} \\ 0 & -\frac{\lambda_4}{\sigma_1^2} & \frac{\lambda_4}{\sigma_1^2} \end{bmatrix}, \tag{147}$$

from which we can form

$$\frac{\text{tr}(P_4)}{\text{tr}(P_3)} = \frac{4[\theta^0]^8 + 8[\theta^0]^6 + 5[\theta^0]^4 + 2[\theta^0]^2 + 1}{4[\theta^0]^6 + 5[\theta^0]^4 + 2[\theta^0]^2}. \tag{148}$$

We have that EMP_3 yields a smaller trace of the covariance matrix if and only if (148) is larger than one, which is equivalent to:

$$4[\theta^0]^8 + 4[\theta^0]^6 + 1 > 0. \tag{149}$$

This expression is positive for all real values of θ^0. □

This theorem shows that for a first order FIR module structure, EMP_3 is the most accurate among the minimal EMPs. As we have seen in Theorem 4.2 by relaxing Assumption 4.2, the choice of the best EMP depends on the "size" of the numeric parameters of the modules. The next result provides a result regarding which minimal EMP is more accurate.

Theorem 4.5. *Consider a 4-node branch network with modules with a FIR structure* $G_k(q,\theta_k^0) = \theta_k^0 q^{-1}$. *Under Assumption 4.3,* EMP_3 *is the most accurate minimal EMP if* $|\theta_2^0| > |\theta_1^0|$ *and* $|\theta_2^0| > |\theta_3^0|$.

Proof. For notation convenience, we drop the superscript θ_k^0 in all expressions that follow. Under Assumption 4.3, assume without loss of generality that $\sigma_i^2 = \lambda_j = 1$ for $i \in \mathcal{B}, j \in \mathcal{C}$. We can write the asymptotic covariance matrices of the minimal EMPs as:

$$P_1 = \begin{bmatrix} 1 & -\frac{\theta_2}{\theta_1} & 0 \\ -\frac{\theta_2}{\theta_1} & \frac{\theta_2^2+1}{\theta_1^2} & -\frac{\theta_3}{\theta_1^2\theta_2} \\ 0 & -\frac{\theta_3}{\theta_1^2\theta_2} & \frac{\theta_3^2+1}{\theta_1^2\theta_2^2} \end{bmatrix}, P_2 = \begin{bmatrix} \frac{\theta_1^2+1}{\theta_2^2\theta_3^2} & -\frac{\theta_1}{\theta_2\theta_3^2} & 0 \\ -\frac{\theta_1}{\theta_2\theta_3^2} & \frac{\theta_2^2+1}{\theta_3^2} & -\frac{\theta_2}{\theta_3} \\ 0 & -\frac{\theta_2}{\theta_3} & 1 \end{bmatrix} \tag{150}$$

$$P_3 = \begin{bmatrix} \frac{\theta_1^2+1}{\theta_2^2\left(\theta_3^2+1\right)} & -\frac{\theta_1}{\theta_2\left(\theta_3^2+1\right)} & 0 \\ -\frac{\theta_1}{\theta_2\left(\theta_3^2+1\right)} & \frac{\theta_1^2+\theta_3^2+1}{\theta_1^2\theta_3^2+\theta_1^2+\theta_3^2+1} & -\frac{\theta_3}{\theta_2\left(\theta_1^2+1\right)} \\ 0 & -\frac{\theta_3}{\theta_2\left(\theta_1^2+1\right)} & \frac{\theta_3^2+1}{\theta_2^2\left(\theta_1^2+1\right)} \end{bmatrix}, P_4 = \begin{bmatrix} 1 & -\frac{\theta_2}{\theta_1} & 0 \\ -\frac{\theta_2}{\theta_1} & \frac{\theta_2^2}{\theta_3^2} + \frac{\theta_2^2}{\theta_1^2} + \frac{1}{\theta_1^2\theta_3^2} & -\frac{\theta_2}{\theta_3} \\ 0 & -\frac{\theta_2}{\theta_3} & 1 \end{bmatrix}. \tag{151}$$

After lengthy manipulations, we can compare EMP_1 and EMP_3 using the covariance expressions:

$$\text{tr}(P_1) - \text{tr}(P_3) = \frac{\theta_1^2(\theta_1^2 + \theta_2^4\theta_3^2 + \theta_2^4 + \theta_2^2\theta_3^4 + \theta_2^2\theta_3^2 + \theta_2^2 + 2) + \theta_2^4\theta_3^2 + \theta_2^4 + \theta_2^2\theta_3^2 + \theta_2^2 - \theta_3^6 - 2\theta_3^4 - \theta_3^2 + 1}{\theta_2^2\theta_3^2\left(\theta_1^2\theta_3^2 + \theta_1^2 + \theta_3^2 + 1\right)}.$$

Since we assume that $|\theta_2| > |\theta_1|$ and $|\theta_2| > |\theta_3|$, we have that

$$\begin{aligned} & & \theta_2^2 - \theta_3^2 > 0; \\ \theta_2^4\theta_3^2 - \theta_3^6 > 0 & \iff & \theta_2^4 - \theta_3^4 > 0; \\ \theta_2^2\theta_3^2 - \theta_3^4 > 0 & \iff & \theta_2^2 - \theta_3^2 > 0; \\ & & \theta_2^4 - \theta_3^4 > 0, \end{aligned}$$

which implies that $\text{tr}(P_1) - \text{tr}(P_3) > 0$. A similar argument holds for $\text{tr}(P_2) - \text{tr}(P_3)$, provided the necessary changes are made. Proceeding in a similar way for comparison between EMP_3 and EMP_4:

$$\begin{aligned} &\text{tr}(P_4) - \text{tr}(P_3) = \\ &\frac{\theta_1^4\theta_2^4\theta_3^2 + \theta_1^4\theta_2^4 + 2\theta_1^4\theta_2^2\theta_3^4 + \theta_1^4 + \theta_1^2\theta_2^4\theta_3^4 + 2\theta_1^2\theta_2^4\theta_3^2 + \theta_1^2\theta_2^4 + \theta_1^2\theta_2^2\theta_3^4 + 2\theta_1^2\theta_2^2\theta_3^2 + \theta_1^2\theta_2^2}{\theta_1^2\theta_2^2\theta_3^2\left(\theta_1^2+1\right)\left(\theta_3^2+1\right)} + \\ &\frac{-\theta_1^6\theta_2^2\theta_3^2 - 2\theta_1^4\theta_3^2 - \theta_1^2\theta_3^6 - 2\theta_1^2\theta_3^4 - 2\theta_1^2\theta_3^2 + \theta_2^4\theta_3^4 + \theta_2^4\theta_3^2 + \theta_2^2\theta_3^2 + \theta_2^2 + \theta_3^2}{\theta_1^2\theta_2^2\theta_3^2\left(\theta_1^2+1\right)\left(\theta_3^2+1\right)}. \end{aligned}$$

Now, we can verify that if $\theta_2^2 > \theta_1^2$ and $\theta_2^2 > \theta_3^2$ then:

$$\begin{aligned} \theta_1^4\theta_2^4\theta_3^2 - \theta_1^6\theta_2^2\theta_3^2 > 0 & \iff \theta_2^2 - \theta_1^2 > 0; \\ \theta_1^2\theta_2^2\theta_3^2 - \theta_1^4\theta_3^2 > 0 & \iff \theta_2^2 - \theta_1^2 > 0; \\ \theta_1^2\theta_2^2\theta_3^2 - \theta_1^2\theta_3^4 > 0 & \iff \theta_2^2 - \theta_3^2 > 0; \\ \theta_2^4\theta_3^2 - \theta_1^4\theta_3^2 > 0 & \iff \theta_2^4 - \theta_1^4 > 0; \\ \theta_1^2\theta_2^4 - \theta_1^2\theta_3^4 > 0 & \iff \theta_2^4 - \theta_3^4 > 0; \\ \theta_1^2\theta_2^2 - \theta_1^2\theta_3^2 > 0 & \iff \theta_2^2 - \theta_3^2 > 0; \\ \theta_2^2\theta_3^2 - \theta_1^2\theta_3^2 > 0 & \iff \theta_2^2 - \theta_1^2 > 0; \\ \theta_1^2\theta_2^2\theta_3^4 - \theta_1^2\theta_3^6 > 0 & \iff \theta_2^2 - \theta_3^2 > 0, \end{aligned}$$

from which follows that EMP_3 yields the least trace of the covariance matrix. □

This theorem shows the influence of the numeric values on the accuracy obtained by the EMPs. We have shown that if the corresponding numeric value of $G_2(q, \theta_2^0)$ is large than the other modules, then EMP_3 is the most accurate EMP under Assumption 4.3. This result is in line with the one in Theorem 4.2 for a 3-node branch network, in which the selection of the best EMP depends also on the numeric values.

The dominance of EMP_3 with respect to the others is related to its structure. In this EMP, nodes near the source are excited, while nodes near the sink are measured. This structure represents a more balanced pattern since EMP_3 has equal shares of excitations and measurements and they are distributed such that half of the network is excited and the other half is measured. This kind of EMP will also have an advantage over others in a branch network with n number of nodes.

Now if we relax Assumption 4.3 and consider Assumption 4.2 only, one can get a similar result as Theorem 4.1 by inverting (132), (136), and (137).

Theorem 4.6. *Consider a branch network with four nodes and dynamic matrix as in (107). Under Assumption 4.2 we have the following.*

1. *If $SNR_{43} > SNR_{21}$ and $SNR_{42} > SNR_{31}$, then EMP_2 yields more accurate results than EMP_1.*

2. *If $SNR_{32} > SNR_{21}$, then EMP_3 results in better accuracy than EMP_1.*

3. *If $SNR_{32} > SNR_{43}$, then EMP_3 is more accurate than EMP_2.*

Proof. Let us start with item 1. We first notice that under Assumption 4.2 the following holds: $A_i = \frac{\sigma_i^2}{\lambda_{i+1}}\Gamma_A$, and $B_i = \frac{\sigma_i^2}{\lambda_{i+2}}\Gamma_B$. From (132)-(138) and Table 5 we have that EMP_2 is more accurate than EMP_1 if:

$$A_3 - A_1 \succ 0 \iff \left(\frac{\sigma_3^2}{\lambda_4} - \frac{\sigma_1^2}{\lambda_2}\right)\Gamma_A \succ 0,$$
$$B_2 - B_1 \succ 0 \iff \left(\frac{\sigma_2^2}{\lambda_4} - \frac{\sigma_1^2}{\lambda_3} -\right)\Gamma_B \succ 0,$$

from which follows the expressions for item 1. Now, items 2 and 3 can be proved in a similar fashion, for this reason we prove only the former. From the information matrix (137) and the expressions (140) and (141) in Theorem 4.3 we have that EMP_3 is more accurate than EMP_1 if:

$$A_1^{-1} \succ [A_2 + [B_1^{-1} + B_2^{-1} + C_1^{-1}]^{-1}]^{-1} \iff$$
$$[B_1^{-1} + B_2^{-1} + C_1^{-1}]^{-1} + \left(\frac{\sigma_2^2}{\lambda_3} - \frac{\sigma_1^2}{\lambda_2}\right)\Gamma_A \succ 0$$
$$A_1^{-1} + B_1^{-1} \succ [[A_2^{-1} + B_1^{-1}]^{-1} + [B_2^{-1} + C_1^{-1}]^{-1}]^{-1} \iff$$
$$[[A_2^{-1} + B_1^{-1}]^{-1} - [A_1^{-1} + B_1^{-1}]^{-1}] + [B_2^{-1} + C_1^{-1}]^{-1} \succ 0$$
$$\implies \left(\frac{\sigma_2^2}{\lambda_3} - \frac{\sigma_1^2}{\lambda_2}\right)\Gamma_A \succ 0.$$

From the above expression one can infer the conditions stated in this theorem, since expression (142) holds generally also in this case. □

The first result is a direct extension of the result in Theorem 4.1. These results reveal that the selection of the best EMP will also depend on the SNR_{ji} relationship among some nodes. It is worth noticing the comparison between EMP_3 and EMP_1 and also between $\text{EMPs}_{2,3}$. In both cases, the key factor is the difference among the SNR_{ji}'s of the direct modules of EMP_3 (G_2), EMP_1 (G_1), and EMP_2 (G_3).

We have shown that the principles from the previous section are valid for the four nodes branch networks. As demonstrated in the last section, $\text{EMPs}_{1,2}$ provide equal overall accuracy, since they are mirrored versions of each other. However, in contrast with the case of networks with three nodes, EMP_3 yields better precision when compared to $\text{EMPs}_{1.2}$. For FIR branch networks EMP_3 dominates over all other minimal EMPs. A principle that emerges from this result is that EMP_3 yields better accuracy due to its uniform pattern with equal shares of excitations and measurements. This principle is also valid for branch networks with more nodes as we will show in Section 4.4.3. When the network is not uniformly excited, we have extended the result from the previous section and we have shown that comparison among the EMPs_{1-3} can be made based on the SNR_{ji} of the direct modules. The influence of the direct modules is clear in the case of FIR networks, when other factors are equal, i.e. in a uniformly excited network, the EMPs with "large" direct modules have an advantage over the others.

4.4.3 Branches with n nodes

In the previous sections we considered branch dynamic networks with just a few number of nodes. Under Assumptions 4.2 and 4.3, we have observed the following. Minimal $\text{EMPs}_{1,2}$ yield the same accuracy for branch networks with three and four nodes. However, in the four-node case, EMP_3 outperforms $\text{EMPs}_{1,2}$ with respect to the trace of the covariance matrix. In this section, we will show that the phenomenon of yielding similar covariance matrices is not unique to $\text{EMPs}_{1,2}$. We provide a result that characterizes all minimal EMPs with same overall accuracy. Furthermore, we will show that, as in the 4-node case, there is a minimal EMP that results in better estimates when compared to other minimal EMPs.

The reason that minimal $\text{EMPs}_{1,2}$ yield, under Assumptions 4.2 and 4.3, same overall accuracy is due to the symmetry of excitation and measurements. This happens because these EMPs are mirrored versions of each other. For EMP_1, we excite only the source and measure the other nodes, while for EMP_2 it holds the converse, only the sink is measured and the other nodes are excited. We now introduce the concept of mirrored EMP as follows.

Definition 4.4. *Consider an n-node branch network, for which minimal EMP$_1$ $(\mathcal{B}_1, \mathcal{C}_1)$ and minimal EMP$_2$ $(\mathcal{B}_2, \mathcal{C}_2)$ apply. Minimal EMP$_1$ is a* mirrored *version of minimal EMP$_2$ if the set of excited and measured nodes are formed as $\mathcal{B}_1 = \{n - j + 1 \,|\, j \in \mathcal{C}_2\}$ and $\mathcal{C}_1 = \{n - j + 1 \,|\, j \in \mathcal{B}_2\}$.*

This definition implies that there is a symmetry with respect to the source and sink nodes for any minimal EMP and its mirrored version. If there is an excited (measured) node that is k nodes ahead from the source, then in the *mirrored* EMP the node that is k nodes behind the sink must be measured (excited). Using this definition we see that EMP_2 is a mirrored version of EMP_1 and vice-versa. With this definition at hand we are in position to state the next result, which relates the accuracy of a given EMP and its mirrored version.

Theorem 4.7. *Consider an n-node branch network for which there are minimal EMPs that apply. Under Assumptions 4.2 and 4.3, a minimal EMP and its mirrored version yield the same trace of the asymptotic covariance matrix.*

Proof. We are going to show that for a minimal EMP and its mirrored version, the information matrix can be written as:

$$M = Q\overline{M}Q^T \tag{152}$$

where $\overline{M}$ is the information matrix associated with the mirrored version of an EMP and

$$Q = \begin{bmatrix} & & & I \\ & & \iddots & \\ & I & & \\ I & & & \end{bmatrix}.$$

Notice that $Q = Q^T = Q^{-1}$ is a permutation matrix, thus both trace and determinant of M and $\overline{M}$ are equal. The effect of pre and pos multiplying Q is equivalent to reversing the order of the rows and columns of M. Therefore, we just need to show that a mirrored version of an EMP has information matrix with reversed rows and columns. Now, recall that we can decompose the gradient of the optimal predictor according to the transfer function from input i to output j as (111) for $i < j < n$:

$$\psi_{ji}(t,\theta) = \begin{bmatrix} \mathbf{0}_{i-1} \\ \frac{\partial \rho_{ji}}{\partial \theta_i} \\ \vdots \\ \frac{\partial \rho_{ji}}{\partial \theta_{j-1}} \\ \mathbf{0}_{n-j} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{i-1} \\ G_i' \frac{\rho_{ji}}{G_i(q,\theta_i)} \\ \vdots \\ G_{j-1}' \frac{\rho_{ji}}{G_{j-1}(q,\theta_{j-1})} \\ \mathbf{0}_{n-j} \end{bmatrix}, \tag{153}$$

where $\rho_{ji} \triangleq \prod_{k=i}^{j-1} G_k r_i$, and $\mathbf{0}_k$ is a zero matrix with dimension corresponding to either the first $G_k(q,\theta_k)$ modules or the last $G_k(q,\theta_k)$ modules. The partial information matrix – see Definition 4.3 – associated with input i and output j is $M_{ji} \triangleq \frac{N}{\lambda_j}\mathbb{E}\psi_{ji}\psi_{ji}^T$. We can compute the information matrix from the partial information matrices using Lemma 4.1 as $M = \sum_{j\in\mathcal{C},i\in\mathcal{B}} M_{ji}$. From the structure of $\psi_{ji}(t,\theta)$, the first $i-1$ block rows and columns of M_{ji} are zero. Similarly, the last $n-j$ block rows of M_{ji} are also zero. Under

Assumptions 4.2 and 4.3, it holds that all nonzero elements of $\psi_{ji}(t,\theta)$ are equal, which implies that the elements of M_{ji} are the same. Consider an arbitrary EMP defined by its set of excited nodes $\mathcal{B}$ and measured nodes $\mathcal{C}$. For every pair $j \in \mathcal{C}$ and $i \in \mathcal{B}$, there is a reflected version of ψ_{ji} in the mirrored EMP, such that:

$$\overline{\psi}_{n-i+1,n-j+1}(t,\theta) = \begin{bmatrix} \mathbf{0}_{n-j} \\ G'_{n-j+1}\frac{\rho_{n-i+1,n-j+1}}{G_{n-j+1}(q,\theta_{n-j+1}} \\ \vdots \\ G'_{n-i}\frac{\rho_{n-i+1,n-j+1}}{G_{j-1}(q,\theta_{j-1}} \\ \mathbf{0}_{i-1} \end{bmatrix}. \tag{154}$$

This means that $\overline{\psi}_{n-i+1,n-j+1}(t,\theta)$ is a reversed, from top to bottom, version of $\psi_{ji}(t,\theta)$. From this relationship, we have that $\overline{M}_{n-i+1,n-j+1}$ can be written as $QM_{ji}Q^T$. Therefore, the information matrix of the mirrored EMP is as follows

$$\overline{M} = \sum_{k\in\overline{\mathcal{C}},l\in\overline{\mathcal{B}}} \overline{M}_{kl} = \sum_{j\in\mathcal{C},i\in\mathcal{B}} QM_{ji}Q^T = QMQ^T.$$

□

This theorem gives a framework for which one can exchange an excitation for a measurement (or the converse) without affecting the overall accuracy of the estimates. This property reveals a duality between excitation and measurement for which the key property is the symmetry of the EMPs for the branch network. Among the EMPs there is always a mirrored EMP that yields the same accuracy under Assumptions 4.2 and 4.3. However, for some minimal EMPs, the mirrored equivalent is not a different EMP but the EMP itself. This will be the case when the EMP is symmetrical with respect to the excitations and measurements. For branch networks with an odd number of nodes, there are no minimal EMPs identical to their mirrored versions. Therefore, the number of minimal EMPs to be analyzed is halved with respect to the total number of minimal EMPs (2^{n-2}), since a mirrored EMP produces the same accuracy. When the number of nodes in the network is even, there will be a total of $3 \cdot 2^{n-4}$ minimal EMPs yielding different covariance matrices. The phenomenon of equal trace of covariance matrices for $\text{EMPs}_{1,2}$ in the case of three and four nodes is therefore a general result and valid for any number of nodes in a branch network.

In a similar way, the observations made in Section 4.4.1 for minimal $\text{EMPs}_{1,2}$ can be further generalized. As we have seen in Section 4.4.1 and 4.4.2, accuracy of the first module in EMP_1 is not improved by other measurements and its quality is related to the equality between the first two modules of the network. In WAHLBERG; HJALMARSSON; MÅRTENSSON (2009), the authors have conjectured that a similar phenomenon would apply instead to the last module in EMP_1, but they have shown that this does not happen.

However, an extension of this reasoning holds when we consider EMP_2. This conclusion is dependent on the EMP employed. The next result formalizes the above statements.

Theorem 4.8. *Consider an n-node branch network with dynamic matrix as in (107), for which minimal EMPs apply. The following holds:*

1. *The accuracy of $\hat{\theta}_1$ in any minimal EMP such that $2 \in \mathcal{C}$ and $G_1 \equiv G_2$ is not improved by any additional excitation signals or measurements;*

2. *The accuracy of $\hat{\theta}_{n-1}$ in any minimal EMP such that $n-1 \in \mathcal{B}$ and $G_{n-2} \equiv G_{n-1}$ is not improved by any additional excitation signals or measurements.*

Proof. First, let us prove item 1. We decompose the gradient of the optimal predictor as in (111). Thus, the information matrix can be written as $M = \sum_{j\in\mathcal{C}, i\in\mathcal{B}} \frac{N}{\lambda_j}\mathbb{E}\psi_{ji}\psi_{ji}^T$. Notice, however, that for any $\psi_{ji}(t,\theta)$ for $j,i \geq 3$ there is no dependence on $G_1(q,\theta_1)$ and $G_2(q,\theta_2)$. The only terms that have the influence of $G_1(q,\theta_1)$ and $G_2(q,\theta_2)$ are $\psi_{k1}(t,\theta)$ for $k = 2,\ldots,n$. Since we assume that $G_1^0(q)^0 \equiv G_2^0(q)$, we have that the first and second block rows of any $\psi_{ji}(t,\theta^0)$ are exactly the same. Therefore, the same holds for M with exception of the first block element. Since $2 \in \mathcal{C}$ the term $\psi_{21}(t,\theta^0)$ appears in M and is the term which adds only to the first block element. Thus, we can write M as:

$$M = \begin{bmatrix} A + X_1 & X_1 & \cdots & X_n \\ X_1 & X_1 & \ldots & X_n \\ \vdots & \vdots & \tilde{M}_{11} & \tilde{M}_{12} \\ X_n & X_n & \tilde{M}_{12}^T & \tilde{M}_{22} \end{bmatrix},$$

where $A \triangleq N/\lambda_2 \mathbb{E} G_1' r_1 G_1' r_1$. Now, define:

$$Q \triangleq \begin{bmatrix} I & -I & 0\cdots & \cdots 0 \\ 0 & I & 0\cdots & \cdots 0 \\ \vdots & \vdots & \tilde{Q}_{11} & \tilde{Q}_{12} \\ 0 & 0 & \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix}.$$

The covariance matrix can thus be obtained as $P_1 = Q^T \overline{M}^{-1} Q$, where

$$\overline{M}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & \tilde{S}_{22} \end{bmatrix}.$$

Thus, the accuracy of the first module is independent of the rest of the network signals under the conditions stated. We can proceed similarly in the proof of item 2. However, for this case, we have that the last two block rows are exactly the same, since $G_{n-2}^0(q) \equiv G_{n-1}^0(q)$ and $n-2 \in \mathcal{B}$. The term $\psi_{n,n-1}(t,\theta^0)$ adds only to the last block element in last block row. The corresponding covariance matrix can be obtained as:

$$P_2 = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12}^T & A^{-1} \end{bmatrix},$$

where $A \triangleq N/\lambda_n \mathbb{E} G'_{n-1} r_{n-1} G'^T_{n-1} r_{n-1}$. Therefore, the accuracy of the last module is independent of the network signals under the conditions stated. □

The results presented in this theorem come as no surprise. They can be interpreted as follows. The additional information about $G_1^0(q)$ that the network signals could provide is only through the measurements. Thus, not even in the case when all signals are measured the accuracy of $G_1^0(q)$ is improved when $G_1^0(q) \equiv G_2^0(q)$ and $2 \in \mathcal{C}$. The dual situation happens with G_{n-1} and the additional excitation signals.

We have shown in Theorem 4.3 that for a four-node branch network, EMP_3 achieves better overall accuracy when compared to $\text{EMPs}_{1,2}$. A principle that emerged from this result is that it would be better to excite the nodes near the source and measure the remaining nodes close to the sink. This observation will also be true for networks with an arbitrary number of nodes as shown in Section 4.4.4. Before analyzing whether this principle is valid for branches with an arbitrary number of nodes, The next result will be useful in the analysis o the next theorem.

Lemma 4.2. *Consider an n-node branch network with network matrix as in (107). Under Assumptions 4.2 and 4.3, minimal EMP$_1$ $= (\mathcal{B} = \{1\}, \mathcal{C} = \{2, 3, \ldots, n\})$ yields the following asymptotic covariance of the parameter estimates:*

$$\text{cov}(\hat{\theta}_1) = X_1^{-1}, \tag{155}$$

$$\text{cov}(\hat{\theta}_i) = X_i^{-1} + X_{i+1}^{-1}, \text{ for } i = 2 \ldots n-1, \tag{156}$$

with $X_i = N/\lambda \mathbb{E} G' \prod_{k=1}^{i-1} Gr \times G'^T \prod_{k=1}^{i-1} Gr$.

Proof. Let $G \triangleq G_i^0(q)$ and $\lambda \triangleq \lambda_i$ for $i = 1, 2, \ldots, n-1$. The gradient of the optimal predictor for minimal EMP_1 is the following.

$$\psi_1(t, \theta) = \begin{bmatrix} G'r_1 & G'Gr_1 & \cdots & G'G^{n-2}r_1 \\ 0 & G'Gr_1 & \cdots & G'G^{n-2}r_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & G'G^{n-2}r_1 \end{bmatrix}.$$

It can be shown that the information matrix of the minimal EMP_1 has the following structure:

$$M_1 = \begin{bmatrix} \sum_{i=1}^{n-1} X_i & \sum_{i=2}^{n-2} X_i & \sum_{i=3}^{n-1} X_i & \cdots & X_{n-1} \\ \sum_{i=2}^{n-1} X_i & \sum_{i=2}^{n-2} X_i & \sum_{i=3}^{n_1-} X_i & \cdots & X_{n-1} \\ \sum_{i=3}^{n-1} X_i & \sum_{i=3}^{n-1} X_i & \sum_{i=3}^{n-1} X_i & \cdots & X_{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ X_{n-1} & X_{n-1} & \cdots & \cdots & X_{n-1} \end{bmatrix},$$

where $X_i \triangleq N/\lambda \mathbb{E} G' G^{i-1} r_1 G' G^{i-1} r_1$. Now, define:

$$Q_1 \triangleq \begin{bmatrix} I & -I & & & \\ & I & -I & & \\ & & \ddots & \ddots & \\ & & & \ddots & -I \\ & & & & I \end{bmatrix}.$$

Let $\overline{M}_1 \triangleq \mathrm{bdiag}(X_1, X_2, \ldots, X_{n-1})$, with $\mathrm{bdiag}(\cdot)$ referring to a block diagonal matrix. We have that $\overline{M}_1 = Q_1 M_1 Q_1^T$, from which we can recover the covariance matrix as $P_1 = Q_1^T \overline{M}_1^{-1} Q_1$, yielding:

$$P_1 = \begin{bmatrix} X_1^{-1} & -X_1^{-1} & 0 & \ddots \\ -X_1^{-1} & X_1^{-1} + X_2^{-1} & -X_2^{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & X_{n-2}^{-1} & X_{n-1}^{-1} + X_{n-2}^{-1} \end{bmatrix}.$$

□

This lemma provides the asymptotic covariance expressions for EMP_1. The next theorem states that for EMP_1 in Lemma 4.2 it is better to exchange the measurement of the second node by an excitation.

Theorem 4.9. *Consider a branch network with network matrix in (107). Under Assumptions 4.2 and 4.3, a smaller trace of the asymptotic covariance matrix is obtained by* EMP_2*:* $(\{1,2\}, \{3, \ldots, n\})$ *when compared to* EMP_1*:* $(\{1\}, \{2, \ldots, n\})$.

Proof. Under Assumptions 2.1 and 2.2 we have that $G_k(q, \theta_k) = G$ for $k = 1, \ldots, n-1$, $\sigma_i^2 = \sigma^2$ and $\lambda_j = \lambda$ for $i \in \mathcal{B}$, $j \in \mathcal{C}$. Let $X_1 \triangleq \frac{N}{\lambda} \mathbb{E} G' r \times G'^T r$ and $X_i \triangleq \frac{N}{\lambda} \mathbb{E} G' \prod_{k=1}^{i-1} Gr \times \mathbb{E} G' \prod_{k=1}^{i-1} Gr$. The covariance of the parameter estimates for EMP_1 follows from Lemma 4.2:

$$\begin{aligned} \mathrm{cov}(\hat{\theta}_1^1) &= X_1^{-1} \\ \mathrm{cov}(\hat{\theta}_2^1) &= X_1^{-1} + X_2^{-1} \\ \mathrm{cov}(\hat{\theta}_3^1) &= X_2^{-1} + X_3^{-1} \\ &\vdots \\ \mathrm{cov}(\hat{\theta}_{n-1}^1) &= X_{n-2}^{-1} + X_{n-1}^{-1}. \end{aligned}$$

As for EMP_2, the information matrix can be obtained by using the partial information approach in Lemma 4.1 and (111). The desired result can be obtained by making the

following comparison among the covariances of the parameter estimates for both EMPs:

$$\begin{array}{c} \mathrm{cov}(\hat{\theta}_1^1) \succ \mathrm{cov}(\hat{\theta}_2^2) \\ \mathrm{cov}(\hat{\theta}_2^1) \succ \mathrm{cov}(\hat{\theta}_1^2) \\ \mathrm{cov}(\hat{\theta}_3^1) \succ \mathrm{cov}(\hat{\theta}_3^2) \\ \mathrm{cov}(\hat{\theta}_4^1) \succ \mathrm{cov}(\hat{\theta}_4^2) \\ \mathrm{cov}(\hat{\theta}_5^1) \succ \mathrm{cov}(\hat{\theta}_5^2) \\ \vdots \\ \mathrm{cov}(\hat{\theta}_{n-1}^1) \succ \mathrm{cov}(\hat{\theta}_{n-1}^2) \end{array} , \tag{157}$$

where the superscript i refers to EMP$_i$. Now, Recall that $X_1 \star X_2 \triangleq (X_1^{-1} + X_2^{-1})^{-1}$. After lengthy calculations using the Schur's complement, (157) is equivalent to

$$\begin{aligned} &X_1^{-1} \succ [X_1 + [X_2^{-1} + [X_2 \star X_3 + X_3 \star X_4 + X_4 \star X_5 + \cdots + X_{n-2} \star X_{n-1}]^{-1}]^{-1}]^{-1} \\ &X_1^{-1} + X_2^{-1} \succ [X_1 \star X_2 + X_2 \star X_3 + X_3 \star X_4 + \cdots + X_{n-2} \star X_{n-1}]^{-1} \\ &X_2^{-1} + X_3^{-1} \succ [X_2 + [Z_1^{-1} + [X_2 + W_1]^{-1}]^{-1}]^{-1} \\ &X_3^{-1} + X_4^{-1} \succ [X_3 + [Z_2^{-1} + [X_3 + W_2]^{-1}]^{-1}]^{-1} \\ &X_4^{-1} + X_5^{-1} \succ [X_4 + [Z_3^{-1} + [X_4 + W_3]^{-1}]^{-1}]^{-1} \\ &\qquad\vdots \\ &X_{n-2}^{-1} + X_{n-1}^{-1} \succ [X_{n-2} + [Z_{n-3}^{-1} + [X_{n-2} + W_{n-3}]^{-1}]^{-1}]^{-1}, \end{aligned} \tag{158}$$

where $W_i = \sum_{k=1}^{n-2} X_k \star X_{k+1} - X_i \star X_{i+1}$ and

$$\begin{aligned} &0 \prec Z_1 \triangleq X_3 - [X_2 + X_3][X_1 + 2X_2 + X_3]^{-1}[X_2 + X_3] \\ &0 \prec Z_2 \triangleq X_4 - [X_3 + X_4][X_2 + 2X_3 + X_4]^{-1}[X_3 + X_4] \\ &0 \prec Z_3 \triangleq X_5 - [X_4 + X_5][X_3 + 2X_4 + X_5]^{-1}[X_4 + X_5] \\ &\qquad\vdots \\ &0 \prec Z_i \triangleq X_{i+2} - [X_{i+1} + X_{i+2}][X_i + 2X_{i+1} + X_{i+2}]^{-1}[X_{i+1} + X_{i+2}]. \end{aligned}$$

To verify that (158) is indeed positive definite, we just need to apply the matrix inversion lemma to the right side. With an appropriate relabel of variables we have:

$$\begin{aligned} X_1^{-1} \succ [X_1 + H_1^{-1}]^{-1} &\iff X_1 \prec X_1 + H_1^{-1} \\ X_1^{-1} + X_2^{-1} \succ [X_1 \star X_2 + W_1]^{-1} &\iff \\ X_1^{-1} + X_2^{-1} \succ X_1^{-1} + X_2^{-1} - [X_1 \star X_2]^{-1}[[X_1 \star X_2]^{-1} + W_1^{-1}]^{-1}[X_1 \star X_2]^{-1} &\iff \\ [X_1 \star X_2]^{-1}[[X_1 \star X_2]^{-1} + W_1^{-1}]^{-1}[X_1 \star X_2]^{-1} \succ 0 & \\ X_i^{-1} + X_{i+1}^{-1} \succ [X_i + H_i^{-1}]^{-1} \iff X_i^{-1} + X_{i+1}^{-1} \succ X_i^{-1} - X_i^{-1}[X_i^{-1} + H_i]^{-1}X_i^{-1} &\iff \\ X_{i+1}^{-1} + X_i^{-1}[X_i^{-1} + H_i]^{-1}X_i^{-1} \succ 0. & \end{aligned} \tag{159}$$

□

Now we have established that there is a minimal EMP which provides better accuracy than at least another minimal EMP. Since mirrored EMPs yield equal overall accuracy, the same conclusion is valid for the mirrored versions of EMPs stated in this theorem. This is a key observation. To fully appreciate it, consider the special case of a network with five nodes ($n = 5$). The difference between EMP_1 and EMP_2 is that we exchange the measurement of the second node by an excitation signal. Since we are adding one more excitation, one should wonder whether adding more excitations would be beneficial to the accuracy. This is clearly not the case, since $\text{EMP}_{\overline{1}} = (\{1, 2, 3, 4\}, \{5\})$ is a mirrored version of EMP_1. Therefore, there is a trade-off between the number of excitations and measurements in a branch network.

It is better to excite the first half of the network and measure the last half. This was formally shown for 4-node branches, and by applying Theorem 4.9 to the 5-node case we reach the same conclusion. Thus, it is expected that this result also extends to an arbitrary number of nodes. For a more general case where Assumptions 4.2 and 4.3 do not hold, we expect that the EMP_2 from Theorem 4.9 will tend to be more accurate. Indeed, this will be observed in the numerical results presented in the next section. We also point to the fact that EMP_1 applied to an n-node branch network does not improve any estimates with respect to a network with $n - 1$ nodes. This does not happen with other minimal EMPs, like those where the first nodes are excited and the last ones are measured.

In summary, we have established three principles that influence the accuracy of the module estimates in a branch network. Firstly, we demonstrated that "mirrored" minimal EMPs provide the same overall accuracy for a fully symmetric network. Therefore, we expect that when all quantities involved are arbitrary, there is no preferred choice between a particular EMP and its mirrored version. This choice will depend upon the magnitude of certain modules within the network and the SNR_{ji} at some nodes. Moreover, we demonstrated a topological principle for branch networks: minimal EMPs where the nodes near the source are excited and the nodes close to the sink are measured yield most accurate results under Assumptions 4.2 and 4.3. This principle was previously observed for 4-node branch networks and it is once more confirmed for general branch networks. Finally, the key observations in WAHLBERG; HJALMARSSON; MÅRTENSSON (2009) – see Section 4.4.1, were shown to be dependent not only on common dynamics between some modules but also on the experimental setting employed. We have seen that direct modules play an important role in the selection of the best EMP for branches with three nodes. While this is not necessarily the case for branches with 4 nodes, where a specific EMP is often better than the others, they serve as an strong indicator for which EMPs should be considered.

4.4.4 Numerical analysis

In this section we analyze how the different factors presented so far work together and how they compare with each other. This is done through numerical experiments that demonstrate that the guiding principles developed until now also apply to the general case of n-node networks.

For the numerical experiments we consider that the following will be valid in all experiments. A total of 10,000 network simulations will be performed. In each run we consider branch networks with cardinality from four to eight nodes. All signals involved are realizations of Gaussian white noise processes. The input $\{r_i(t)\}$ is zero-mean Gaussian with variance σ_i^2 for $i \in \mathcal{B}$, while the corrupting noise $\{e_j(t)\}$ is also a zero mean process, but with variance λ_j for $j \in \mathcal{C}$. A new realization of the random signals involved is performed at each run of the simulation. With respect to the EMPs, we will consider two scenarios:

(i) Assumption 4.3 holds. For this scenario we have chosen $\sigma_i^2 = 1,\ \forall i \in \mathcal{B}$ and $\lambda_j = 0.01,\ \forall j \in \mathcal{C}$.

(ii) For the second analyzed scenario the variances σ_i^2 and λ_j will be drawn from a uniform distribution $\mathcal{U}\left(0.001, 50\right)$.

We remark that the numerical values of the SNR in the first scenario do not influence the decision of the best minimal EMP, since in this case the choice depends only on the numerical parameters and the EMP itself. For each cardinality of the network we will choose at every run a best EMP, the one with smallest trace of the covariance matrix. We are going to list the two best minimal EMPs for each cardinality. With respect to the structure of each model we will consider two structures: first and second order transfer functions. The first order module is parametrized as:

$$G_i(q, \theta_i) = \frac{b_i}{q + a_i}, \tag{160}$$

where $\theta_i = [a_i\ b_i]^T$. In each run of the numerical simulation each parameter is randomly selected. Each module parameter a_i is sampled from $\mathcal{U}\left(0.1, 0.9\right)$, while b_i's are sampled from $\mathcal{U}(0.5, 2)$. The results obtained for the first order transfer function (160) under scenarios (i) and (ii) are displayed in Table 6. This table shows the frequency (in percentage with respect to the total number of runs) in which the two most selected EMPs were chosen as the best for branch networks with number of nodes from 4 up to 8.

The results from this table show that for all network cadinalities the best minimal EMP was the one where the nodes near the source were excited and the nodes near the sink were measured. Thus, the results obtained for branch networks with four nodes are also valid for larger networks even when Assumptions 4.2 and 4.3 do not hold.

Table 6 – How often the two best ranked minimal EMPs$(\mathcal{B}, \mathcal{C})$ were selected considering first order modules for scenario (i) - under Assumption 4.3 - and (ii) where all quantities were randomly selected.

n	Scen.	best EMP	%	runner-up EMP	%
4	(i)	$(\{1,2\},\{3,4\})$	54.12	$(\{1,3\},\{2,4\})$	21.25
	(ii)	$(\{1,2\},\{3,4\})$	50.22	$(\{1,3\},\{2,4\})$	18.55
5	(i)	$(\{1,2\},\{3,4,5\})$	24.79	$(\{1,2,3\},\{4,5\})$	24.72
	(ii)	$(\{1,2,3\},\{4,5\})$	23.58	$(\{1,2\},\{2,4,5\})$	23.1
6	(i)	$(\{1,2,3\},\{4,5,6\})$	18.88	$(\{1,2,4\},\{3,5,6\})$	16.72
	(ii)	$(\{1,2,3\},\{4,5,6\})$	17.08	$(\{1,2,4\},\{3,5,6\})$	13.55
7	(i)	$(\{1,2,3,4\},\{5,6,7\})$	12.47	$(\{1,2,3\},\{4,5,6,7\})$	11.78
	(ii)	$(\{1,2,3,4\},\{5,6,7\})$	10.56	$(\{1,2,3\},\{4,5,6,7\})$	10.36
8	(i)	$(\{1,2,3,4\},\{5,6,7,8\})$	10.81	$(\{1,2,3\},\{4,5,6,7,8\})$	8.3
	(ii)	$(\{1,2,3,4\},\{5,6,7,8\})$	9.15	$(\{1,2,3\},\{4,5,6,7,8\})$	7.64

For small cardinalities, such as four nodes, the difference in frequency was more than two times with respect to the runner-up EMP, while for larger cardinalities a slightly increase in the frequency was observed. Notice that the best EMP and the runner up for five and seven nodes are mirrored EMPs and they provide similar accuracy. This means that together they account for almost half of the selections for five nodes and a fifth for seven nodes. The decrease in percentage of the best EMP when we increase the cardinality of the network is also due to the large number of available minimal EMPs, for instance, for a network with cardinality eight there are a total of 64 minimal EMPs to choose.

Once we have observed that exciting the first nodes and measuring the last ones is the best approach, we wonder by how much the best EMP yields better precision than the others. This is a crucial aspect since one could benefit simply from choosing the structure of excitations and measurements in the network. To answer this question we have compared the ratio of the trace of covariance matrices obtained by the best EMP and the runner-up. In addition, we have also compared the ratio between the best EMP and the worst EMP for the results. We depict in Table 7 the median of the ratio best/runner up and best/worst for the two scenarios analyzed.

We see in this table that for small network cardinalities (four and five nodes) we can have from 30% up to almost double precision improvement compared to the runner-up minimal EMP. Whereas in the case for larger cardinalities we have improvements of at least 8% (for the eight nodes case). The situation dramatically changes when the best EMP is compared to the EMP that yielded worst accuracy. For a network with few nodes

Table 7 – Median of the trace of covariance matrix ratio between the best EMP and the runner up and between best EMP and worst EMP for first order modules.

	(i) Under Assumption 4.3		(ii) All random	
n	runner-up	worst EMP	runner-up	worst EMP
4	1.59	10.23	1.96	15.45
5	1.31	26.14	1.54	46.40
6	1.19	56.64	1.36	111.69
7	1.12	111.42	1.24	240.86
8	1.08	227.18	1.19	555.80

we have at least ten times better precision, and for larger networks this number grows even bigger to 555 (for eight nodes). The large difference observed for cardinalities with more than six nodes is partly due to well-known fact that when we increase the number of parameters the variance also increases. Therefore, for larger networks it is even more important to not choose an EMP arbitrarily.

Now, let us consider a second order transfer function with the following structure:

$$G_i(q, \theta_i) = \frac{\theta_{i1} q + \theta_{i2}}{q^2 + \theta_{i3} q + \theta_{i4}}. \tag{161}$$

The poles of $G_i(q, \theta_i)$ are randomly selected from the right side of the unitary disk, while the zeros are drawn from a disk with radius three. We have performed the same experiments that we have done for first order modules. Similar results to Table 6 are presented in Table 8.

Table 8 – How often the best minimal EMPs$(\mathcal{B}, \mathcal{C})$ were selected for second order modules under scenarios (i) (Assumption 4.3) and (ii).

n	Scen.	best EMP	%	runner-up EMP	%
4	(i)	$(\{1,2\},\{3,4\})$	57.38	$(\{1,3\},\{2,4\})$	17.7
	(ii)	$(\{1,2\},\{3,4\})$	49.7	$(\{1\},\{2,3,4\})$	18.09
5	(i)	$(\{1,2,3\},\{4,5\})$	29.06	$(\{1,2\},\{3,4,5\})$	28.64
	(ii)	$(\{1,2\},\{3,4,5\})$	27.55	$(\{1,2,3\},\{4,5\})$	23.86
6	(i)	$(\{1,2,3\},\{4,5,6\})$	24.36	$(\{1,2,4\},\{3,5,6\})$	17.76
	(ii)	$(\{1,2,3\},\{4,5,6\})$	18.78	$(\{1,2,4\},\{3,5,6\})$	12.83
7	(i)	$(\{1,2,3\},\{4,5,6,7\})$	13.56	$(\{1,2,3,4\},\{5,6,7\})$	13.17
	(ii)	$(\{1,2,3\},\{4,5,6,7\})$	11.05	$(\{1,2,3,4\},\{5,6,7\})$	10.2
8	(i)	$(\{1,2,3,4\},\{5,6,7,8\})$	10.6	$(\{1,2,3,5\},\{4,6,7,8\})$	6.85
	(ii)	$(\{1,2,3,4\},\{5,6,7,8\})$	7.22	$(\{1,2,3\},\{4,5,6,7,8\})$	5.33

As can be observed from this table, the results observed for first order modules are also valid for second order modules. Once more, there is a minimal EMP that is selected more often than others, the one where nodes near the source are excited and the nodes near the sink are measured. These results suggest that the principles derived from the analysis under Assumptions 4.2 and 4.3 can be applied as guidelines for the selection of the best EMPs.

We also have analyzed the gains in accuracy of the best EMP compared to the runner up EMP and the worst EMP, which are displayed in Table 9.

Table 9 – Median of the trace of covariance matrix ratio between the best EMP and the runner up and between the best EMP and worst EMP for second order modules.

	(i) Under Assumption 2.2		(ii) All random	
n	runner-up	worst EMP	runner-up	worst EMP
4	1.72	7.46	2.09	11.72
5	1.52	17.57	1.75	31.35
6	1.44	37.10	1.61	71.29
7	1.39	72.41	1.51	153.80
8	1.37	136.27	1.47	288.92

Once again, we have observed that the selection of the minimal EMP is crucial in the precision of the parameter estimates. In the case of second order modules, the difference from the runner up is even larger than in the first order module, ranging from at least 37% better to more than two times the precision.

From the thousands of numerical experiments performed we have seen that the principles derived for the analytical results are also observed in the more general case where the network is arbitrarily excited and with different modules. First, mirrored EMPs tend to have a similar performance on the accuracy of the estimates. Second, we have observed that minimal EMPs where the first half of the nodes are excited while the remaining nodes are measured tend to give the most accurate results. For any minimal EMP there are competing factors that will influence the decision of which EMP provides the most accurate estimates. On the one hand, there is the influence of the SNR at some nodes on the precision of the parameter estimates. On the other hand, the magnitude of the module parameters may be a key decider for accuracy of the estimates. In any case, the structure of the excitations and measurements plays a major role in the selection of the best EMP as evidenced by the numeric examples.

4.4.5 Summary

We have established a number of key factors that influence the accuracy in branch networks based either on analytical results or on an extensive numerical analysis, or both.

These factors together form fundamental principles that an experiment design should account for when the objective is to decide which EMP yields the most accurate results. The first factor is a topological principle that states that EMPs where the first half nodes are excited and the remaining nodes are measured yield the most accurate estimates. Second, a large signal-to-noise ratio should be applied in the direct modules of the EMPs. If, in addition to that, some prior knowledge is available, then the user should choose EMPs for which direct modules have a large magnitude. Third, we have shown that some EMPs result in the same overall accuracy, which allows the user to exchange excitations for measurements and vice-versa without losing precision of the estimates.

Last, but not least, a very important finding is the large difference observed in the precision of the estimates when the best excitation and measurement pattern is compared to other candidates.

4.5 Cyclic networks

In this section, we address the accuracy analysis for cyclic dynamic networks in a similar fashion to what was done for branch networks. Our approach will be the same used for branch networks, we start with small cyclic networks with just 2 nodes, and from that we proceed by increasing the number of nodes. Recall that an isolated cycle network can be described by its network matrix, which has the following form:

$$G^0(q) = \begin{bmatrix} 0 & 0 & 0 & \cdots & G_n^0(q) \\ G_1^0(q) & 0 & 0 & \cdots & 0 \\ 0 & G_2^0(q) & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & G_{n-1}^0(q) & 0 \end{bmatrix}. \tag{162}$$

Recall that we have relabeled the modules as $G_{ji}^0(q) \triangleq G_i^0(q)$. Notice that this structure is very similar to the branch case, the main difference is in the additional transfer function $G_n^0(q)$ that closes the loop. There are n modules to be identified for a cycle network with n nodes. In the analysis that follows, it will be useful to analyze the features of this network by analyzing a particular case of (162)

$$G(q, \theta^0) = \begin{bmatrix} \mathbf{0} & \theta_n^0 \\ \mathrm{diag}(\theta_1^0, \theta_2^0, \ldots, \theta_{n-1}^0) & 0 \end{bmatrix} q^{-1}. \tag{163}$$

Recall that $\mathrm{diag}(\cdot)$ refers to a diagonal matrix. We refer to (163) as a state-space network matrix. We are interested in providing key insights on how to determine which EMPs are equivalent, which produce the same overall accuracy, and which ones yield the most accurate estimates. For this purpose, we need to establish a general formula for computing the information matrix of a cycle network. This is what we do in the following.

4.5.1 The information matrix for cycles

In order to compute the information matrix we need an expression for $\psi(t,\theta)$. Recall that we can compute the information matrix from Lemma 4.1 as the sum of partial information matrices associated to a given EMP. Therefore, we just need to provide a general formula for $\psi_{ji}(t,\theta)$ as defined in (103).

Let us briefly recall the expressions for the input-output description (82)-(88) of a cycle:

$$R \triangleq G_1(q,\theta_1)G_2(q,\theta_2)\cdots G_n(q,\theta_n), \tag{164}$$

$$R_{ik} \triangleq G_{i-1}(q,\theta_{i-1})G_{i-2}(q,\theta_{i-2})\cdots G_k(q,\theta_k) \text{ for } k<i, \tag{165}$$

$$R_{ik} \triangleq G_{i-1}(q,\theta_{i-1})G_{i-2}(q,\theta_{i-2})\cdots G_n(q,\theta_n)G_{n-1}(q,\theta_{n-1})\cdots G_k(q,\theta_k) \text{ for } k>i, \tag{166}$$

$$R = R_{ik}R_{ki}, \tag{167}$$

$$R_{ii} \triangleq 1. \tag{168}$$

From the input-output relationship given in Lemma 3.5 one can compute the expressions for the gradient as follows:

$$\frac{\partial T_{ii}(q,\theta)}{\partial \theta_k} = \frac{R_{ik}R_{k-1,i}G_k'}{(1-R)^2} \tag{169}$$

$$\frac{\partial T_{ji}(q,\theta)}{\partial \theta_k} = \frac{R_{jk}R_{k-1,i}G_k'}{(1-R)^2}, \text{ if } G_k(q,\theta_k) \in R_{ji} \tag{170}$$

$$\frac{\partial T_{ji}(q,\theta)}{\partial \theta_k} = \frac{R_{ji}R_{ik}R_{k-1,i}G_k'}{(1-R)^2}, \text{ if } G_k(q,\theta_k) \notin R_{ji}, \tag{171}$$

where $G_k \in R_{ji}$ means that G_k is a term of R_{ji}. From these expressions, we can form the gradient for $j>i$ as follows:

$$\psi_{ji}(t,\theta) = \frac{1}{(1-R_{ii})^2}\begin{bmatrix} R_{ji}R_{i1}R_{0,i}G_1' \\ \vdots \\ R_{ji}R_{ii}R_{i-1,i}G_i' \\ R_{ji+1}R_{i1}G_{i+1}' \\ R_{ji+2}R_{i+1,1}G_{i+2}' \\ \vdots \\ R_{jj}R_{j-1,1}G_j' \\ R_{ji}R_{ij+1}R_{j,i}G_{j+1}' \\ \vdots \\ R_{ji}R_{i,n-1}R_{n-2,i}G_{n-1}' \end{bmatrix}. \tag{172}$$

As for $j < i$ we have the following gradient:

$$\psi_{ji}(t,\theta) = \frac{1}{(1-R_{ii})^2} \begin{bmatrix} R_{j1}R_{0,i}G'_1 \\ R_{j2}R_{1,i}G'_2 \\ \vdots \\ R_{jj-1}R_{i,j-1}G'_{j-1} \\ R_{ji}R_{ij}R_{j-1,i}G'_j \\ R_{ji}R_{ij+1}R_{ji}G'_{j+1} \\ \vdots \\ R_{ji}R_{ii}R_{i-1i}G'_i \\ R_{ji+1}R_{i,i}G'_{i+1} \\ \vdots \\ R_{j,n-1}R_{n-2,i}G'_{n-1} \end{bmatrix}. \tag{173}$$

The information matrix of a cycle for a given EMP can be constructed from these gradients using Lemma 4.1. More generally, the partial information matrices are related under the conditions of the following lemma.

Lemma 4.3. *The partial information matrix $M_{i+l,i}$ from node i to $i+l$ is related to the partial information matrix $M_{i+l+k,i+k}$ from node $i+k$ to $i+l+k$:*

$$M_{i+l,i} = QM_{i+l+k,i+k}Q^T, \tag{174}$$

if $\lambda_{i+l}/\sigma_i^2 = \lambda_{i+l+k}/\sigma_{i+k}^2$ and $R_{i+l,i} = R_{i+l+k,i+k}$ with Q an appropriate permutation matrix.

Proof. If $\psi_{i+l,i}(t,\theta) = Q\psi_{i+l+k,i+k}(t,\theta)$ and $\lambda_{i+l}/\sigma_i^2 = \lambda_{i+l+k}/\sigma_{i+k}^2$, then (174) holds. These gradients will have the same elements if $T_{i+l,i} = T_{i+l+k,i+k}$, which is equivalent to $R_{i+l,i} = R_{i+l+k,i+k}$. Under this condition, they have the same elements but with a different order. Therefore, there exists a transformation matrix Q for which the partial information matrix are similar up to a transformation Q. □

This result will allow us to determine the conditions under which two minimal EMPs yield the same, or a similar, covariance matrix of the parameter estimates. These conditions will depend on whether the partial information matrices from a given minimal EMP are similar up to a similarity transformation to those of another minimal EMP. It is immediate from the definitions that

$$\text{SNR}_{ii} = \text{SNR}_{jj} = \frac{\sigma_i^2}{\lambda_i} = \frac{\sigma_j^2}{\lambda_j} \iff M_{ii} = M_{jj}.$$

This means that the partial information matrix from a node to itself will differ from another node based solely on the SNR_{ii} applied at the nodes.

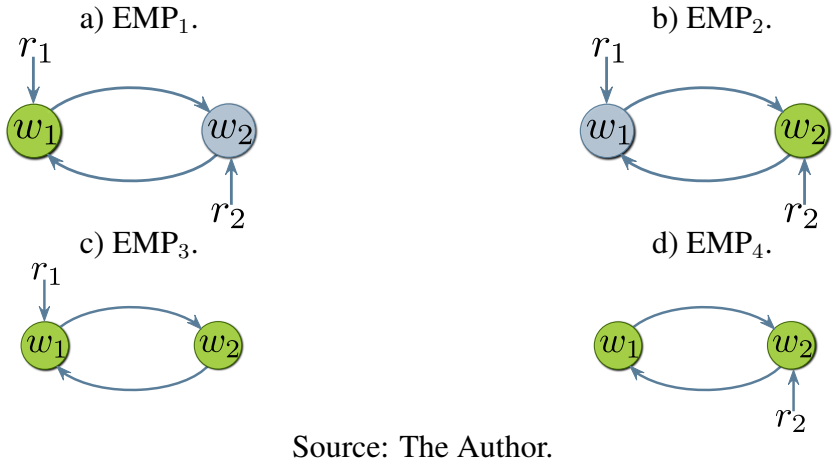
4.5.2 2-node cycles

The simplest cycle network is the one composed of only two nodes. There are only two modules to be identified $G_1^0(q)$ and $G_2^0(q)$. From the necessary and sufficient conditions in Theorem 3.17, a 2-node cycle network has only four minimal EMPs, namely:

1. $\text{EMP}_1 = (\mathcal{B} = \{1, 2\}; \mathcal{C} = \{1\})$,
2. $\text{EMP}_2 = (\mathcal{B} = \{1, 2\}; \mathcal{C} = \{2\})$,
3. $\text{EMP}_3 = (\mathcal{B} = \{1\}; \mathcal{C} = \{1, 2\})$,
4. $\text{EMP}_4 = (\mathcal{B} = \{2\}; \mathcal{C} = \{1, 2\})$.

Figure 18 depicts each of these minimal EMPs for the two nodes cycle network.

Figure 18 – Minimal EMPs for the 2-node cycle network. Excited nodes are indicated by incoming edges with label r_j. Measured nodes are depicted in green color.



Source: The Author.

The following theorem provides the relationship among the minimal EMPs for a 2-node cycle network.

Theorem 4.10. *Consider a 2-node cycle dynamic network with network matrix in (162), for which minimal EMPs$_{1-4}$ apply. Under Assumptions 4.2 and 4.3, it holds*

1. *EMP$_1$ and EMP$_4$ yield the same asymptotic covariance matrix;*
2. *EMP$_2$ and EMP$_3$ yield the same asymptotic covariance matrix;*
3. *All minimals EMPs yield the same trace of the asymptotic covariance matrix.*

Proof. We can compute the information matrices associated to each minimal EMP using the partial information matrices:

$$M_1 = M_{11} + M_{12}, M_2 = M_{21} + M_{22}, \tag{175}$$

$$M_3 = M_{11} + M_{21}, M_4 = M_{22} + M_{12}. \tag{176}$$

Let us start with items 1 and 2. Under Assumptions 4.2 and 4.3, we have $M_{11} = M_{22}$ and therefore $M_1 = QM_4Q^T$ and $M_2 = QM_3Q^T$. To prove item 3 we only need to show that $M_1 = QM_3Q^T$, which is equivalent to $M_{12} = QM_{21}Q^T$. Under the stated assumptions, we have from Lemma 4.3 that:

$$M_{12} = QM_{21}Q^T.$$

□

This result shows that under Assumptions 4.2 and 4.3 all minimal EMPs yield the same overall accuracy. In fact, these assumptions are not necessary for these results to hold. The following theorem gives conditions when the minimal EMPs provide similar accuracy.

Theorem 4.11. *Consider a 2-node cycle dynamic network with network matrix in (162), for which minimal EMPs*$_{1-4}$ *apply. Minimal EMP*$_1$ *and EMP*$_4$ *yield the same covariance matrix and minimal EMP*$_2$ *and EMP*$_2$ *yield the same covariance matrix if and only if SNR*$_{11}$ = *SNR*$_{22}$.

Proof. The proof follows from the observation that $M_1 = M_4$ and $M_2 = M_3$ if and only if $M_{11} = M_{22}$. Since $\psi_{11}(t, \theta^0)$ and $\psi_{22}(t, \theta^0)$ have the same elements, they will be equal if and only if $\frac{\sigma_1^2}{\lambda_1} = \frac{\sigma_2^2}{\lambda_2}$. □

This theorem shows that equality among some minimal EMPs for a cycle network with 2 nodes depends only on the SNR$_{ji}$ at the nodes. In the next sections we are going to show that this principle is more general to cycle networks.

Let us now investigate how the magnitude of the modules influence the accuracy of the minimal EMPs. In order to perform this analysis, we will consider a state-space cycle with network matrix (163) with $n = 2$ as follows:

$$G^0(q) = \begin{bmatrix} 0 & \theta_2^0 \\ \theta_1^0 & 0 \end{bmatrix} q^{-1}. \tag{177}$$

A similar analysis was provided for state-space branch networks in Theorems 4.2, 4.4, and 4.5, where we have seen that the direct modules play a key role in the decision of which minimal EMP yields most accurate results. For EMP$_1$ one can obtain the following variance of the parameter estimates:

$$\mathrm{var}(\hat{\theta}_2^1) = \frac{\lambda_1\gamma_0}{Nd_1}\left(\sigma_1^2 + \sigma_2^2\theta_2^{0\,2}\right), \tag{178}$$

$$\mathrm{var}(\hat{\theta}_1^1) = \frac{\lambda_1\gamma_0}{N\theta_2^{0\,2}d_1}\left(\theta_1^{0\,2}\sigma_1^2 + \sigma_2^2\right), \tag{179}$$

where $\gamma_k \triangleq \mathbb{E}r^F(t)r^F(t-k)$, $r_i^F(t) = \frac{r_i(t)}{\left(1-\theta_2^0\theta_1^0\right)^2}$, and

$$d_1 = \left(\gamma_0^2\left(\sigma_1^2 + \sigma_2^2\theta_2^{0\,2}\right)\left(\sigma_1^2 + \sigma_2^2\theta_2^{0\,2}\right) - \left(\gamma_0^2\sigma_1^2\theta_1^0 + \gamma_2\sigma_2^2\theta_2^0\right)^2\right).$$

These expressions follow from the information matrix, which is fully developed in Appendix A.1. Now, for EMP_2 we have the following results:

$$\text{var}(\hat{\theta}_2^2) = \frac{1}{N}\frac{\lambda_2\gamma_0}{\theta_1^{0^2}d_2}\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right), \tag{180}$$

$$\text{var}(\hat{\theta}_1^2) = \frac{1}{N}\frac{\lambda_2\gamma_0}{d_2}\left(\theta_1^{0^2}\sigma_1^2 + \sigma_2^2\right), \tag{181}$$

where $d_2 = \left(\gamma_0^2\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right)\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right) - \left(\gamma_0^2\sigma_2^2\theta_1^0 + \gamma_2\sigma_1^2\theta_2^0\right)^2\right)$. As for EMP_3, we have the following results for the variances:

$$\text{var}(\hat{\theta}_2^3) = \frac{1}{N}\frac{\lambda_1\lambda_2\gamma_0}{\theta_1^{0^2}d_3}\left(\lambda_1 + \lambda_2\theta_2^{0^2}\right) \tag{182}$$

$$\text{var}(\hat{\theta}_1^3) = \frac{1}{N}\frac{\lambda_1\lambda_2\gamma_0}{d_3}\left(\theta_1^{0^2}\lambda_1 + \lambda_2\right), \tag{183}$$

with $d_3 = \sigma_1^2\left(\gamma_0^2\left(\lambda_1 + \lambda_2\theta_2^{0^2}\right)\left(\theta_1^{0^2}\lambda_1 + \lambda_2\right) - \left(\gamma_0^2\lambda_2\theta_2^0 + \gamma_2\lambda_1\theta_1^0\right)^2\right)$. Finally, EMP_4 yields the following variance of the estimates:

$$\text{var}(\hat{\theta}_2^4) = \frac{1}{N}\frac{\lambda_1\lambda_2\gamma_0}{d_4}\left(\lambda_1 + \lambda_2\theta_2^{0^2}\right), \tag{184}$$

$$\text{var}(\hat{\theta}_1^4) = \frac{1}{N}\frac{\lambda_1\lambda_2\gamma_0}{\theta_2^{0^2}d_4}\left(\theta_1^{0^2}\lambda_1 + \lambda_2\right), \tag{185}$$

with $d_4 = \sigma_2^2\left(\gamma_0^2\left(\lambda_1 + \lambda_2\theta_2^{0^2}\right)\left(\theta_1^{0^2}\lambda_1 + \lambda_2\right) - \left(\gamma_0^2\lambda_2\theta_2^0 + \gamma_2\lambda_1\theta_1^0\right)^2\right)$.

A number of conclusions can be drawn from these equations. The next theorem gives conditions under which we can compare the minimal EMPs for state-space cycles.

Theorem 4.12. *Consider a 2-node cycle network with network matrix in (177). Under Assumptions 4.2 and 4.3, the direct modules of the minimal EMPs are estimated more accurately. Under Assumption 4.3, the following holds*

1. *if* $[\theta_1^0]^2 > [\theta_2^0]^2$ *then* $\text{tr}(P_1) = \text{tr}(P_4) > \text{tr}(P_2) = \text{tr}(P_3)$.

2. *if* $[\theta_1^0]^2 < [\theta_2^0]^2$ *then* $\text{tr}(P_1) = \text{tr}(P_4) < \text{tr}(P_2) = \text{tr}(P_3)$.

Proof. First notice that under Assumption 4.3 we have $P_1 = P_4$ and $P_2 = P_3$ from Theorem 4.10. We are going to show now that direct modules are estimated more accurately. Recall that $\theta_2^0 q^{-1}$ is a direct module of $\text{EMPs}_{1,4}$ and $\theta_1^0 q^{-1}$ is a direct module in $\text{EMPs}_{2,3}$. Now we just need to compare (178)-(180) and (179)-(181):

$$\frac{\text{var}(\hat{\theta}_2^1)}{\text{var}(\hat{\theta}_2^2)} = \left(\theta_1^0\right)^2, \tag{186}$$

$$\frac{\text{var}(\hat{\theta}_1^2)}{\text{var}(\hat{\theta}_1^1)} = \left(\theta_2^0\right)^2. \tag{187}$$

From the stability constraints we have that $\theta_1^0\theta_2^0 < 1$, which under Assumption 4.2 implies that $[\theta_1^0]^2 = [\theta_2^0]^2 < 1$. Therefore, we have that $\mathrm{var}(\hat{\theta}_2^1) < \mathrm{var}(\hat{\theta}_2^2)$ and $\mathrm{var}(\hat{\theta}_1^2) < \mathrm{var}(\hat{\theta}_1^1)$. Finally, under Assumption 4.3 we have

$$\frac{\mathrm{tr}(P_1)}{\mathrm{tr}(P_2)} = \frac{(\theta_1^0)^2\left((\theta_2^0)^2\left((\theta_2^0)^2+1\right)+(\theta_1^0)^2+1\right)}{(\theta_2^0)^2\left((\theta_2^0)^2+(\theta_1^0)^2\left((\theta_1^0)^2+1\right)+1\right)}\times \\ \frac{\gamma_0^2(\theta_2^0)^2(\theta_1^0)^2+\gamma_0^2(\theta_1^0)^2+\gamma_0^2-2\gamma_0\gamma_2\theta_2^0\theta_1^0-\gamma_2^2(\theta_1^0)^2}{\gamma_0^2(\theta_2^0)^2(\theta_1^0)^2+\gamma_0^2(\theta_2^0)^2+\gamma_0^2-2\gamma_0\gamma_2\theta_2^0\theta_1^0-\gamma_2^2(\theta_2^0)^2}. \tag{188}$$

Now, if $[\theta_1^0]^2 > [\theta_2^0]^2$ then (188) is larger than 1 as

$$\frac{(\theta_1^0)^2\left((\theta_2^0)^2\left((\theta_2^0)^2+1\right)+(\theta_1^0)^2+1\right)}{(\theta_2^0)^2\left((\theta_2^0)^2+(\theta_1^0)^2\left((\theta_1^0)^2+1\right)+1\right)} > 1$$

$$(\theta_1^0)^2-(\theta_2^0)^2+(\theta_1^0)^4-(\theta_2^0)^4+(\theta_2^0\theta_1^0)^2\left((\theta_2^0)^2-(\theta_1^0)^2\right) > 0,$$

since $[\theta_1^0\theta_2^0]^2 < 1$ and $\mid (\theta_1^0)^2-(\theta_2^0)^2 \mid > (\theta_2^0\theta_1^0)^2 \mid (\theta_2^0)^2-(\theta_1^0)^2 \mid$. The last fraction of (188) is also positive if $[\theta_1^0]^2 > [\theta_2^0]^2$. The dual argument is valid for the last item. □

This result is in accordance with the ones obtained from branches. They mean that the selection of the minimal EMP that provides the most accurate estimates depends upon the numeric value of the modules. More specifically, this result shows that the direct modules play a key factor in the selection of the best EMP. Furthermore, the direct modules are estimated more accurately under Assumptions 4.2 and 4.3.

In summary, we have demonstrated that for cycles with two nodes some EMPs provide equal accuracy and for a fully symmetrical scenario that all minimal EMPs provide the same overall accuracy. We have also shown how the numeric values of the direct modules influence the minimal EMPs that provide the most accurate estimates.

4.5.3 3-node cycles

Here we extend the analysis from 2-node cycle networks to cycle networks composed of three nodes. For this class of networks there are three modules to be identified: $G_1^0(q), G_2^0(q)$, and $G_3^0(q)$. From the necessary and sufficient conditions given in Theorem 3.17, we know that at least one node must be both excited and measured, and every other node must be either measured or excited. This produces a total of 12 minimal EMPs, which are described in Table 10.

Let us start the analysis considering Assumptions 4.2 and 4.3. The following result states which EMPs have overall similar accuracy for a three node cycle.

Theorem 4.13. *Consider a 3-node cycle with dynamic network, for which the minimal EMPs from Table 10 apply. Under Assumptions 4.2 and 4.3, the following holds*

Table 10 – The 12 minimal EMPs for a 3-node cycle.

EMP	$(\mathcal{B}, \mathcal{C})$	EMP	$(\mathcal{B}, \mathcal{C})$
EMP_1	$\mathcal{B} = \{1,2,3\}; \mathcal{C} = \{1\}$	EMP_2	$\mathcal{B} = \{1,2,3\}; \mathcal{C} = \{2\}$
EMP_3	$\mathcal{B} = \{1,2,3\}; \mathcal{C} = \{3\}$	EMP_4	$\mathcal{B} = \{1\}; \mathcal{C} = \{1,2,3\}$
EMP_5	$\mathcal{B} = \{2\}; \mathcal{C} = \{1,2,3\}$	EMP_6	$\mathcal{B} = \{3\}; \mathcal{C} = \{1,2,3\}$
EMP_7	$\mathcal{B} = \{1,2\}; \mathcal{C} = \{1,3\}$	EMP_8	$\mathcal{B} = \{1,3\}; \mathcal{C} = \{1,2\}$
EMP_9	$\mathcal{B} = \{2,3\}; \mathcal{C} = \{1,2\}$	EMP_{10}	$\mathcal{B} = \{1,2\}; \mathcal{C} = \{2,3\}$
EMP_{11}	$\mathcal{B} = \{1,3\}; \mathcal{C} = \{2,3\}$	EMP_{12}	$\mathcal{B} = \{2,3\}; \mathcal{C} = \{1,3\}$

1. *EMPs$_{1-6}$ have the same trace of the asymptotic covariance matrix;*

2. *EMPs$_{7,9,11}$ have the same trace of the asymptotic covariance matrix;*

3. *EMPs$_{8,10,12}$ have the same trace of the asymptotic covariance matrix.*

Proof. The proof will be based on similarity among the information matrices. The information matrices of EMPs$_{1-6}$ can be computed by using the partial information matrix as

$$M_1 = M_{11} + M_{12} + M_{13}; M_2 = M_{22} + M_{21} + M_{23}; M_3 = M_{33} + M_{31} + M_{32};$$
$$M_4 = M_{11} + M_{21} + M_{31}; M_5 = M_{22} + M_{12} + M_{32}; M_6 = M_{33} + M_{13} + M_{23}.$$

Under Assumptions 4.2 and 4.3, we have that $M_{11} = M_{22} = M_{33}$. Now, EMP_1 and EMP_5 will have similar covariance matrices if and only if $M_{13} = QM_{32}Q^T$. This follows from the stated assumptions and Lemma 4.3. Similarly, EMP_2 and EMP_6 will have similar covariance matrices if and only if $M_{13} = QM_{21}Q^T$. Using the same reasoning, EMP_3 and EMP_4 will produce similar covariances matrices if and only if $M_{32} = QM_{21}Q^T$. All this follows from Lemma 4.3. Now, for EMP_1 and EMP_2 to produce a similar covariance matrix we would need that $M_{12} = QM_{23}Q^T$ and $M_{13} = QM_{21}Q^T$. The remaining relationships all follow from Lemma 4.3 and Assumptions 4.2 - 4.3:

$$M_{21} \sim M_{13} \sim M_{32},$$
$$M_{31} \sim M_{12} \sim M_{23}.$$

□

This theorem establishes that some minimal EMPs provide similar accuracy under Assumptions 4.2 and 4.3. We should expect that in a more general situation the structure of these minimal EMPs will carry no advantage over the other. In fact, Assumptions 4.2 and 4.3 are not necessary for some results to hold. The next theorem gives conditions under which some minimal EMPs yield similar covariance matrices.

Theorem 4.14. *Consider a 3-node cycle with dynamic network (162), for which the minimal EMPs from Table 10 apply. Under Assumption 4.3, the following holds*

1. *If* $G_1^0(q) \equiv G_2^0(q)$ *then*

$$\begin{aligned}&\operatorname{tr}(P_1) = \operatorname{tr}(P_6), \operatorname{tr}(P_2) = \operatorname{tr}(P_5), \operatorname{tr}(P_3) = \operatorname{tr}(P_4),\\ &\operatorname{tr}(P_7) = \operatorname{tr}(P_{11}), \operatorname{tr}(P_8) = \operatorname{tr}(P_{12}).\end{aligned}$$

2. *If* $G_1^0(q) \equiv G_3^0(q)$ *then*

$$\begin{aligned}&\operatorname{tr}(P_1) = \operatorname{tr}(P_4), \operatorname{tr}(P_2) = \operatorname{tr}(P_6), \operatorname{tr}(P_3) = \operatorname{tr}(P_5),\\ &\operatorname{tr}(P_{10}) = \operatorname{tr}(P_{12}), \operatorname{tr}(P_9) = \operatorname{tr}(P_{11}).\end{aligned}$$

3. *If* $G_3^0(q) \equiv G_2^0(q)$ *then*

$$\begin{aligned}&\operatorname{tr}(P_1) = \operatorname{tr}(P_5), \operatorname{tr}(P_2) = \operatorname{tr}(P_4), \operatorname{tr}(P_3) = \operatorname{tr}(P_6),\\ &\operatorname{tr}(P_8) = \operatorname{tr}(P_{10}), \operatorname{tr}(P_7) = \operatorname{tr}(P_9).\end{aligned}$$

Proof. All results follow from Lemma 4.3 and the following implications.

$$\begin{aligned}
R_{21} = R_{32} &\iff G_1^0(q) = G_2^0(q) \text{ and } \mathrm{SNR}_{21} = \mathrm{SNR}_{32} \implies M_{21} \sim M_{32}\\
R_{13} = R_{32} &\iff G_3^0(q) = G_2^0(q) \text{ and } \mathrm{SNR}_{13} = \mathrm{SNR}_{32} \implies M_{13} \sim M_{32}\\
R_{13} = R_{21} &\iff G_3^0(q) = G_1^0(q) \text{ and } \mathrm{SNR}_{13} = \mathrm{SNR}_{21} \implies M_{13} \sim M_{21}\\
R_{31} = R_{12} &\iff G_1^0(q) = G_3^0(q) \text{ and } \mathrm{SNR}_{31} = \mathrm{SNR}_{12} \implies M_{31} \sim M_{12}\\
R_{23} = R_{12} &\iff G_1^0(q) = G_2^0(q) \text{ and } \mathrm{SNR}_{23} = \mathrm{SNR}_{12} \implies M_{23} \sim M_{12}\\
R_{23} = R_{31} &\iff G_3^0(q) = G_2^0(q) \text{ and } \mathrm{SNR}_{23} = \mathrm{SNR}_{31} \implies M_{23} \sim M_{31}.
\end{aligned}$$

□

Notice that Assumption 4.3 is not necessary for these results to hold. It was only used to unify the conclusions stated in this theorem. To see this, consider EMP_1 and EMP_5. From Lemma 4.3 they will have similar covariance matrices if $G_3^0(q) = G_2^0(q)$, $\mathrm{SNR}_{11} = \mathrm{SNR}_{22}$ and $\mathrm{SNR}_{13} = \mathrm{SNR}_{32}$. Similar conclusions can be derived for the other EMPs. These results reveals a symmetry for a cycle network. If some modules are equal, then some EMPs will be equal due to the distribution of excitations and measurements in the cycle network.

We now investigate whether there is a minimal EMP that yields the smallest accuracy among the minimal EMPs. To this end, we consider a state-space 3-node cycle with the following network matrix:

$$G(q, \theta^0) = \begin{bmatrix} 0 & 0 & \theta_3^0 \\ \theta_1^0 & 0 & 0 \\ 0 & \theta_2^0 & 0 \end{bmatrix} q^{-1}. \tag{189}$$

In order to gain some insight into the choice of the best EMP, we have simulated $1,000$ cycle networks under an equally excited scenario – see Assumption 4.3 – with different parameter values drawn from a uniform distribution $\mathcal{U}(-1,1)$. The trace of the asymptotic covariance matrix was used to assess the quality of each EMP. The results of this experiment are shown in Table 11.

Table 11 – Number of times that each EMP was selected as the best from 1,000 randomly generated systems.

EMP	$1-6$	7	8	9	10	11	12
Occurrences	0	236	117	201	105	238	103

As there are many minimal EMPs, one might expect that new features arise for a 3-node cyclic dynamic network when compared to the case of 2 nodes, and this is indeed the case. These results suggest that a better approach is to excite and measure 2 nodes, since neither the EMPs$_{1-3}$ (just one node as excitation source) nor the EMPs$_{4-6}$ (just one node was measured) were selected as the best in any simulation. Additionally, the EMPs$_{7,9,11}$ yielded the best accuracy almost twice as often when compared to EMPs$_{8,10,12}$. What EMPs$_{7,9,11}$ have in common is that one excites (instead of measuring, as in EMPs$_{8,10,12}$) the node situated next to the node that is both measured and excited.

As we have seen for branches and 2-node cycles, the numeric values of the direct modules have a key influence on the most accurate minimal EMP. In order to test whether the largest module plays a crucial role in the selection of the best EMP, we will consider 7 different numerical experiments, described in Table 12.

Table 12 – Parameter values for the numerical experiments.

Exp.	θ_3^0	θ_1^0	θ_2^0	Exp.	θ_3^0	θ_1^0	θ_2^0
I	0.50	0.50	0.50	V	0.25	1.00	0.50
II	1.00	0.50	0.25	VI	0.25	0.50	1.00
III	1.00	0.25	0.50	VII	0.50	0.25	1.00
IV	0.50	1.00	0.25				

From the results with 2-node cycle networks, we conjecture that EMPs for which the largest parameters are direct modules should result in the smaller trace of covariance matrix. The trace of the covariance matrix for EMPs$_{7-12}$ in the different cases are displayed in Table 13 with $\sigma_i^2 = 1$ and $\lambda_j = 0.01$, $\forall j \in \mathcal{C}$ and $\forall i \in \mathcal{B}$. The results indicate that our conjecture holds, since the best EMP was, in all cases, the one in which the direct module is the largest parameter.

Furthermore, as seen in Table 13, in the fully symmetric case these same EMPs$_{7,9,11}$ outperform the other three – EMPs$_{8,10,12}$. This effect does not happen in the two node

Table 13 – Trace of the covariance matrix for the cases in Table 12 and different EMPs, all equally excited and measured.

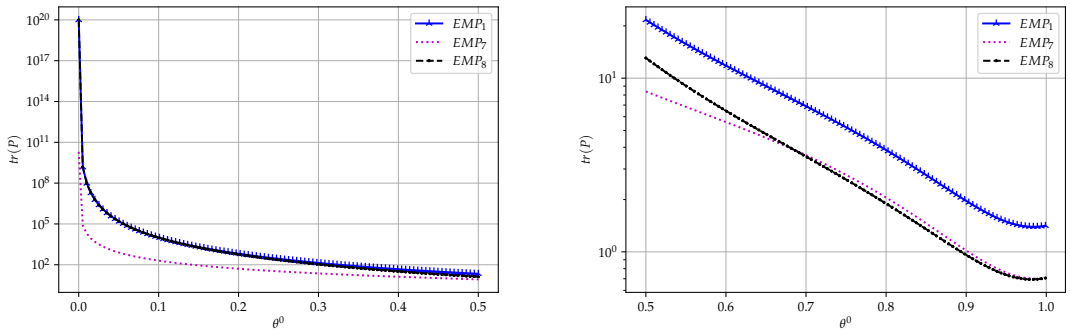
EMP| Exp.	I	II	III	IV	V	VI	VII
1	0.2029	0.1267	0.0393	1.0968	1.3204	0.4695	0.1094
2	0.2029	0.1094	0.4695	0.0393	0.1267	1.0968	1.3204
3	0.2029	1.3204	1.0968	0.4695	0.1094	0.0393	0.1267
4	0.2029	1.0968	1.3204	0.1267	0.0393	0.1094	0.4695
5	0.2029	0.4695	0.1094	1.3204	1.0968	0.1267	0.0393
6	0.2029	0.0393	0.1267	0.1094	0.4695	1.3204	1.0968
7	0.0782	0.3270	0.0930	0.3270	0.0930	0.0255	0.0255
8	0.1189	0.0290	0.1004	0.0290	0.1004	0.5840	0.5840
9	0.0782	0.0255	0.0255	0.0930	0.3270	0.3270	0.0929
10	0.1189	0.5840	0.5840	0.1004	0.0290	0.0290	0.1004
11	0.0782	0.0929	0.3270	0.0255	0.0255	0.0929	0.3270
12	0.1189	0.1004	0.0290	0.5840	0.5840	0.1004	0.0290

cycle network counterpart, where all EMPs lead to the same results for fully symmetric network.

To further investigate this phenomenon, we have simulated under Assumptions 4.2 and 4.3 how the trace of the covariance matrices is distributed for $\theta^0 \in (0, 1)$. In Figure 19 is depicted the trace of the covariance matrices of the EMPs$_{1,7,8}$ in a fully symmetric cycle network. This figure shows that for $|\theta^0| \in (0, 0.664]$ EMP$_7$ yields the most accurate results, whereas for $\theta^0 \in (0.664, 1)$ EMP$_8$ is the most precise EMP. This justifies why we have observed in Table 11 that EMPs$_{7,9,11}$ appeared twice as often as EMPs$_{8,10,12}$. Notice that for small magnitudes of θ^0, EMPs$_{1,8}$ are three times worse than EMP$_7$, whereas for values of θ^0 close to one, these EMPs provide a similar accuracy. Moreover, the high values of the variance near zero are due to the attenuation of the excitation signals in the network caused by the numeric values of the parameters. If the nodes signals become too small, the signal-to-noise ratio of the network approaches zero and deteriorates the quality of the estimates.

In conclusion, for a three-node cycle we have observed that there are some EMPs that provide the most accurate results with respect to the others under Assumptions 4.2 and 4.3. We have shown that these results also depend on the numeric values of the modules, and that, in general, EMPs with equal shares of excitations and measurements provide the most accurate estimates. Balancing the number of excitations and measurements in a cycle is the strategy that tends to yield the best results.

Figure 19 – Comparison of the trace of the covariance matrix for $\text{EMPs}_{1,7,8}$ under Assumptions 4.2 and 4.3 for $\theta^0 = (0, 1)$.



Source: The Author.

4.5.4 4-node cycles

Let us now focus on the class of cycles with four nodes. There are four modules to be identified from a four-node cycle network: $G_1^0(q), G_2^0(q), G_3^0(q)$, and $G_4^0(q)$. For this class of networks, there are only two minimal EMPs to be chosen from Theorem 3.17:

I. $\text{EMP}_1 = (\mathcal{B} = \{1, 3\}, \mathcal{C} = \{2, 4\})$,

II. $\text{EMP}_2 = (\mathcal{B} = \{2, 4\}, \mathcal{C} = \{1, 3\})$.

Similarly to the case of cycles with 2 nodes, when we consider Assumptions 4.2 and 4.3 both minimal EMPs yield the same overall accuracy as the next result formally states.

Theorem 4.15. *Consider a cycle network with four nodes and network matrix (162), for which* $\text{EMPs}_{1,2}$ *apply. Under Assumptions 4.2 and 4.3, both minimal EMPs yield the same trace of the asymptotic covariance matrix.*

Proof. We can write the information matrices of these minimal EMPs as:

$$M_1 = M_{21} + M_{41} + M_{23} + M_{43}, \tag{190}$$

$$M_2 = M_{12} + M_{32} + M_{14} + M_{34}. \tag{191}$$

From Lemma 4.3 we have that $M_{21} \sim M_{32}, M_{43} \sim M_{14}, M_{12} \sim M_{41}$, and $M_{23} \sim M_{34}$. □

This result states that for a fully symmetric network both EMPs yield same overall accuracy. We can then expect that in a more general situation there will be no preferred choice, and that the choice of the best EMP will depend on other factors. The key decider for one EMP or the other will depend on the direct modules and on the SNR_{ji} applied at the direct modules. In order to investigate the influence of the direct modules we will

consider once again a state-space cycle network with four nodes and network matrix as follows:

$$G(q,\theta^0) = \begin{bmatrix} 0 & 0 & 0 & \theta_4^0 \\ \theta_1^0 & 0 & 0 & 0 \\ 0 & \theta_2^0 & 0 & 0 \\ 0 & 0 & \theta_3^0 & 0 \end{bmatrix} q^{-1}. \tag{192}$$

We conducted a numerical experiment with 10,000 systems where the network parameters were randomly selected from a uniform distribution $\mathcal{U}(0,1)$. The following conjecture was tested under Assumption 4.3: if $|\theta_1^0\theta_3^0| > |\theta_2^0\theta_4^0|$ then EMP_1 is the best, otherwise EMP_2. This conjecture comes from the extrapolation of the results observed in the previous networks concerning direct modules: in EMP_1 the modules $\theta_1^0 q^{-1}$ and $\theta_3^0 q^{-1}$ are the direct modules, so one expects that if they are the larger modules then this will be the best EMP. The same point is valid for EMP_2 once the necessary changes have been made. From the $10,000$ systems tested, the conjecture proved correct in 99.64% of them, implying that, those indicators can be used to determine which EMP will have the best accuracy for cyclic networks with four nodes. Once again, it was observed that the gains obtained in choosing the best EMP can be very significant: in 20% of the cases the ratio between the variances of the two EMPs was above 100, and the median of this ratio was found to be 8.7.

Thus, the principle that EMPs where the larger modules are direct modules provide better accuracy is confirmed once again.

4.5.5 Larger cycles

We now consider cycles with larger cardinalities. Recall that for cycles with $n > 3$ all minimal EMPs have cardinality n and they can be easily spotted by using Corollary 3.4. Any minimal EMP must have at least two excited nodes and two measured nodes, and the distribution of excitations and measurements must not be contiguous.

Let us introduce the following definition that will be useful in the analysis of cycles.

Definition 4.5. *An $\text{EMP}_2 = (\mathcal{B}_2, \mathcal{C}_2)$ is said to be **circular** to $\text{EMP}_1 = (\mathcal{B}_1, \mathcal{C}_1)$ if for a $k \in \mathbb{Z}$ it holds $\mathcal{B}_2 = \{i + k \ (mod\ n) | i \in \mathcal{B}_1\}$ and $\mathcal{C}_2 = \{j + k \ (mod\ n) | j \in \mathcal{C}_1\}$.*

This definition can be seen as just a proper relabeling of the nodes. It is useful for the analysis of cycles because of their symmetry. Most minimal EMPs will be *circular* to other minimal EMPs. A symmetric property with respect to the accuracy of the circular EMPs is formally stated in the following lemma.

Lemma 4.4. *Consider a cycle with network matrix (162) and any minimal EMP_1 that applies. Under Assumptions 4.2 and 4.3, all circular EMPs with respect to EMP_1 yield the same trace of the covariance matrix.*

Proof. Under Assumptions 4.2 and 4.3, all quantities in the cycle are equal. The result follows from just a relabel of the nodes in the cycle. □

This result states that for fully symmetrical cycle networks, we can significantly reduce the number of minimal EMPs under analysis because most EMPs are circular to others and they yield the same overall accuracy under Assumptions 4.2 and 4.3. Another important concept that will be useful in the analysis for cycles is defined in the following.

Definition 4.6. *Consider EMP$_1 = (\mathcal{B}_1, \mathcal{C}_1)$ and EMP$_2 = (\mathcal{B}_2, \mathcal{C}_2)$ both with cardinality n. We say that EMP$_1$ and EMP$_2$ are **symmetrical opposite** to each other if $\mathcal{B}_1 = \mathcal{C}_2$ and $\mathcal{C}_1 = \mathcal{B}_2$.*

EMP$_1$ being symmetrical opposite to EMP$_2$ means that the nodes that are excited in EMP$_1$ are measured in EMP$_2$, while the nodes that are measured in EMP$_1$ are excited in EMP$_2$. We are now going to present a result that is valid for any minimal EMP for cycles with more than three nodes.

Theorem 4.16. *Consider a cycle network with network matrix given in (162) with $n \geq 4$, for which a minimal EMP$_1 = (\mathcal{B}_1, \mathcal{C}_1)$ applies. Under Assumptions 4.2 and 4.3, a minimal EMP$_2 = (\mathcal{B}_2, \mathcal{C}_2)$ such that $\mathcal{B}_2 = \mathcal{C}_1$ and $\mathcal{C}_2 = \mathcal{B}_1$ yields the same trace of the asymptotic covariance matrix.*

Proof. We are going to show that the two minimal EMPs from statement have similar information matrix. Let $\mathcal{B}_1 = \{i_1, i_2, \ldots, i_m\}$ and $\mathcal{C}_1 = \{j_1, j_2, \ldots, j_p\}$ such that $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_p$, without loss of generality. We are going to use the partial information matrices associated with each $i \in \mathcal{B}_1$ and each $j \in \mathcal{C}_1$. Lemma 4.3 states that two partial information matrices are similar if there is constant difference in their indexes, i.e. they are circular. For every partial information matrix M_{ji} there is a total of n similar matrices to M_{ji}: $M_{j+k,i+k}$ for $k = 1, 2, \ldots, n$. We proceed by forming the partial information matrices using the following procedure. Choose all j_l and i_k such that $j_l - i_k = 1$. Repeat the same procedure by iterating $j_l - i_k = c_k$ for $c_k \in \{1, 2, \ldots, n-1\}$. For EMP$_2$ we have exactly the opposite $i_k - j_l = -c_k$. Thus, the difference in EMP$_1$ $j_l - i_k = c_k$ is mapped into EMP$_2$ as $i_k - j_l = n - c_k$. The other way is also true, the difference $i_l - j_k = c_k$ in EMP$_2$ is mapped into $j_k - i_l = -c_k = n - c_k$ for EMP$_1$. Therefore, for every $j_l - i_k = c_k$, there exists an i_{k_2} and j_{l_2} such that $i_{k_2} - j_{l_2} = c_k$ and $j_l = i_{k_2} + k$, $i_k = j_{l_2} + k$, $k = 1, \ldots, n$. It follows from Lemma 4.3 that $M_{j_l,i_k} = Q M_{i_{k_2},j_{k_2}} Q^T$. This establishes the relationship between the two minimal EMPs. □

This theorem reveals a duality that exists between excitations and measurements in cycles. If every excitation is exchanged by a measurement, and vice-versa, the same overall accuracy will be obtained. This means that EMPs that are are symmetrical opposite to each other yield the same overall accuracy. These results will be fundamental in the analysis of

cycles as they will significantly reduce the number of minimal EMPs under analysis. In order to determine which minimal EMPs have some advantage over the others, we will consider cycles with few number of nodes in the following.

5-node cycles.

In order to provide a better understanding of the problem of selecting the most accurate EMP for cycles, let us focus now on cycles with five nodes. From Corollary 3.4 we have all minimal EMPs for a cycle with five nodes, which are displayed in Table 14.

Table 14 – All minimal EMPs for a five nodes loop.

EMP	$\mathcal{B}$	$\mathcal{C}$	EMP	$\mathcal{B}$	$\mathcal{C}$	EMP	$\mathcal{B}$	$\mathcal{C}$
EMP_1	$\{1,2,4\}$	$\{3,5\}$	EMP_2	$\{3,5\}$	$\{1,2,4\}$	EMP_3	$\{1,3,5\}$	$\{2,4\}$
EMP_4	$\{2,4\}$	$\{1,3,5\}$	EMP_5	$\{1,4\}$	$\{2,3,5\}$	EMP_6	$\{2,3,5\}$	$\{1,4\}$
EMP_7	$\{2,4,5\}$	$\{1,3\}$	EMP_8	$\{1,3\}$	$\{2,4,5\}$	EMP_9	$\{2,5\}$	$\{1,3,4\}$
EMP_{10}	$\{1,3,4\}$	$\{2,5\}$						

From Theorem 4.16 it follows that EMPs which are symmetrical opposite to others yield similar accuracy under Assumptions 4.2 and 4.3. This means that even EMPs and odd EMPs in Table 14 yield the same trace of the covariance matrix. Similarly to the four-node case, there is no EMP that yields better accuracy under Assumptions 4.2 and 4.3 as the next theorem shows.

Theorem 4.17. *Consider a five node cycle with network matrix in (162) for $n = 5$. All minimal EMPs for this network yield the same trace of the asymptotic covariance matrix under Assumptions 4.2 and 4.3.*

Proof. It follows from Theorem 4.16 that even EMPs and odd EMPs in Table 14 yield the same trace of covariance matrix. Let us show that the odds (even) EMPs give the same results. Consider $\text{EMPs}_{1,5}$, their information matrix for each EMP can be written as:

$$M_1 = M_{31} + M_{32} + M_{34} + M_{51} + M_{52} + M_{54} \tag{193}$$

$$M_5 = M_{21} + M_{31} + M_{34} + M_{51} + M_{54} + M_{24}. \tag{194}$$

From Lemma 4.3 we have that $M_{21} = QM_{32}Q^T$ and $M_{52} = QM_{24}Q^T$. Now, consider $\text{EMPs}_{4,7}$:

$$M_4 = M_{12} + M_{14} + M_{32} + M_{34} + M_{52} + M_{54} \tag{195}$$

$$M_7 = M_{12} + M_{14} + M_{15} + M_{32} + M_{34} + M_{35}. \tag{196}$$

From Lemma 4.3 we have that $M_{15} = QM_{54}Q^T$ and $M_{35} = QM_{52}Q^T$. Proceeding as before, consider $\text{EMPs}_{5,10}$:

$$M_5 = M_{21} + M_{24} + M_{31} + M_{34} + M_{51} + M_{54} \tag{197}$$

$$M_X = M_{21} + M_{23} + M_{24} + M_{51} + M_{53} + M_{54}. \tag{198}$$

Once again, from Lemma 4.3 we have $M_{23} = QM_{34}Q^T$ and $M_{31} = QM_{53}Q^T$. Finally, consider EMPs$_{1,4}$. From Lemma 4.3 we have $M_{12} = QM_{51}Q^T$ and $M_{31} = QM_{14}Q^T$. From this we conclude that all minimal EMPs yield the same trace of covariance matrix under stated assumptions. □

This result is rather expected, by using the concept of *circular* EMP from Definition 4.5 and Lemma 4.4 we can see that all minimal EMPs are circular to either a minimal EMP with two excitations or an EMP with three excitations. Therefore, under Assumptions 4.2 and 4.3 there are only two minimal EMPs to compare and they yield the same overall accuracy. Notice that for equality among some EMPs, Assumptions 4.2 and 4.3 are not necessary. Common dynamics among some modules is sufficient to result in equal overall accuracy of certain EMPs. The next result shows under which circumstances two minimal EMPs yield similar accuracy.

Corollary 4.2. *Consider a cycle with network matrix in (162) with $n = 5$ nodes, for which EMPs in Table 14 apply. The following results hold.*

1. *if $G_1^0(q) = G_2^0(q)$, $G_3^0(q) = G_5^0(q)$ and SNR$_{21}$ = SNR$_{32}$, SNR$_{52}$ = SNR$_{24}$ then* $\mathrm{tr}(P_1) = \mathrm{tr}(P_5)$*;*

2. *if $G_1^0(q) = G_3^0(q)$, $G_4^0(q) = G_5^0(q)$ and SNR$_{15}$ = SNR$_{54}$, SNR$_{35}$ = SNR$_{52}$ then* $\mathrm{tr}(P_4) = \mathrm{tr}(P_8)$*;*

3. *if $G_1^0(q) = G_4^0(q)$, $G_2^0(q) = G_3^0(q)$ and SNR$_{23}$ = SNR$_{34}$, SNR$_{31}$ = SNR$_{53}$ then* $\mathrm{tr}(P_5) = \mathrm{tr}(P_{10})$*;*

4. *if $G_1^0(q) = G_5^0(q)$, $G_2^0(q) = G_4^0(q)$ and SNR$_{12}$ = SNR$_{51}$, SNR$_{31}$ = SNR$_{14}$ then* $\mathrm{tr}(P_1) = \mathrm{tr}(P_7)$*;*

Proof. All results follow by using Definition 4.3 and Lemma 4.3. □

Many other results are possible by simple checking the partial information matrix – see Definition 4.3 – and by using Lemma 4.3. For cycles, in general, in order for two minimal EMPs to yield similar results, it is sufficient to have some common dynamic among the modules and that some SNR$_{ji}$ are the same at some nodes.

In conclusion, we have observed that for fully symmetric networks all minimal EMPs for a 5-node cycles yield the same overall accuracy. This is expected since most EMPs are *circular* to EMPs that either have three excitations or three measurements. In Section 4.5.6 we are going to investigate how to choose between a minimal EMP with three excitations or one with three measurements in different scenarios.

6-node cycles.

Let us now analyze whether the results for a 5-node cycle remain valid for a cycle with six nodes, that is, if all minimal EMPs yield the same overall accuracy. For a cycle with

six nodes there are six modules to be identified and a total of 32 minimal EMPs one can choose. All minimal EMPs for a cycle with six nodes are displayed in Table 15.

Table 15 – All minimal EMPs for a cycle network with six nodes.

EMP	$(\mathcal{B},\mathcal{C})$	EMP	$(\mathcal{B},\mathcal{C})$	EMP	$(\mathcal{B},\mathcal{C})$	EMP	$(\mathcal{B},\mathcal{C})$
1	({1, 2, 3, 5}; {4, 6})	2	({1, 2, 4, 5}; {3, 6})	3	({1, 2, 4, 6}; {3, 5})	4	({1, 2, 4}; {3, 5, 6})
5	({1, 2, 5}; {3, 4, 6})	6	({1, 3, 4, 5}; {2, 6})	7	({1, 3, 4, 6}; {2, 5})	8	({1, 3, 4}; {2, 5, 6})
9	({1, 3, 5, 6}; {2, 4})	10	({1, 3, 5}; {2, 4, 6})	11	({1, 3, 6}; {2, 4, 5})	12	({1, 3}; {2, 4, 5, 6})
13	({1, 4, 5}; {2, 3, 6})	14	({1, 4, 6}; {2, 3, 5})	15	({1, 4}; {2, 3, 5, 6})	16	({1, 5}; {2, 3, 4, 6})
17	({2, 3, 4, 6}; {1, 5})	18	({2, 3, 5, 6}; {1, 4})	19	({2, 3, 5}; {1, 4, 6})	20	({2, 3, 6}; {1, 4, 5})
21	({2, 4, 5, 6}; {1, 3})	22	({2, 4, 5}; {1, 3, 6})	23	({2, 4, 6}; {1, 3, 5})	24	({2, 4}; {1, 3, 5, 6})
25	({2, 5, 6}; {1, 3, 4})	26	({2, 5}; {1, 3, 4, 6})	27	({2, 6}; {1, 3, 4, 5})	28	({3, 4, 6}; {1, 2, 5})
29	({3, 5, 6}; {1, 2, 4})	30	({3, 5}; {1, 2, 4, 6})	31	({3, 6}; {1, 2, 4, 5})	32	({4, 6}; {1, 2, 3, 5})

As we have shown in Lemma 4.4 and Theorem 4.16, most minimal EMPs yield the same overall accuracy under Assumptions 4.2 and 4.3. From now on, unless otherwise stated, we will consider that Assumptions 4.2 and 4.3 hold for the rest of the section. Recall that for the branch networks case, the principles derived under these assumptions are extendable to some degree to more general scenarios. We are going to show that the same holds for cycles in Section 4.5.6.

Let us focus on minimal EMPs that are not circular to each other in Table 15. For this purpose, we remove all circular EMPs and those EMPs that are symmetrical opposite to others. Recall that under Assumptions 4.2 and 4.3 EMPs that are circular or symmetrical opposite to others yield the same overall accuracy. For a cycle with 6 nodes there are only four minimal EMPs that are not circular to each other, listed below.

I. $\text{EMP}_1 = (\{1, 2, 3, 5\}, \{4, 6\})$;

II. $\text{EMP}_2 = (\{1, 2, 4, 5\}, \{3, 6\})$;

III. $\text{EMP}_4 = (\{1, 2, 4\}, \{3, 5, 6\})$;

IV. $\text{EMP}_{10} = (\{1, 3, 5\}, \{2, 4, 6\})$.

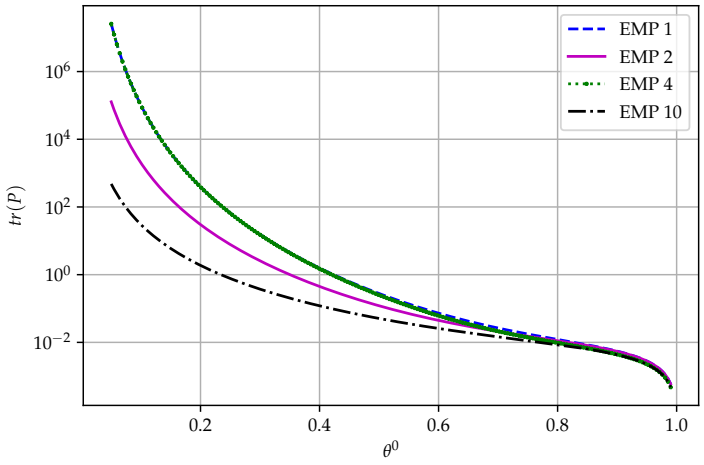
All other minimal EMPs can be obtained from these four EMPs by a proper relabel of the nodes or by exchanging the excitations for measurements and vice-versa. By using Lemma 4.4 and Theorem 4.16 the number of EMPs under analysis reduced from 32 to only 4 minimal EMPs. We are now interested in verifying whether there is one EMP among these four minimal EMPs that yields the most accurate parameter estimates. Let us now

consider a state space network matrix (163) for $n = 6$ with just one parameter:

$$G(q, \theta^0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \theta^0 \\ \theta^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta^0 & 0 \end{bmatrix} q^{-1}. \tag{199}$$

We can compute the information matrices for each minimal EMP based on Lemma 4.1. From the stability constraints we must have that $\theta^0 < 1$. Therefore, the analysis region for this particular state space network must be constrained to this region. Figure 20 depicts the comparison among the trace of the covariance matrices for the four minimal EMPs in the stability region $\theta^0 \in (0, 1)$.

Figure 20 – Comparison among minimal EMPs for a state-space cycle network with six nodes under Assumptions 4.2 and 4.3.



Source: The Author.

This figure shows that EMP_{10} is the one that yields most accurate estimates in the stability region for cycles with six nodes under Assumptions 4.2 and 4.3. $\text{EMPs}_{1,4}$ are tied with the worst performance among the minimal EMPs, while EMP_2 is the runner-up EMP with most accurate estimates. The large difference in the precision obtained by the different EMPs are prominent for small values of θ^0. For very small values of θ^0 the difference is in the order of five times the magnitude. This difference drastically reduces when θ^0 approaches 1. For $\theta^0 = 0.2$, the variance of $\text{EMPs}_{1,4}$ are more than 200 times the variance obtained by EMP_{10}, while for $\theta^0 = 0.4$ this difference reduces to thirteen times. Contrary to the case of branches, where one should choose EMPs where the first nodes are excited and the last ones are measured, a principle that emerges for large cycles is to alternately excite and measure their nodes.

Differently from the cycles with five nodes, where all minimal EMPs yield the same overall accuracy under Assumptions 4.2 and 4.3, there is a minimal EMP that yields the most accurate results for cycles with six nodes. This minimal EMP obeys the principle of equal shares between excitations and measurements and it is characterized by nodes that are alternately excited and measured. The latter feature is what makes the best EMP to standout among the others. Thus, for a cycle with six nodes there is a structural property in how the excitations and measurements should be placed in order to improve the quality obtained for the parameter estimates. We conjecture that this principle is more general and extends to larger cycles with an arbitrary number of nodes. In order to test this hypothesis we are going to consider cycles with seven nodes.

7-node cycles.

We now consider minimal EMPs for cycles with seven nodes. By considering this class of networks, we aim to verify whether a minimal EMP where we alternatively excite and measure the nodes of the cycle is the one which yields better accuracy. To this end, we are going to perform the same analysis provided to cycles with six nodes. For the class of cycle networks with seven nodes, there are seven modules to be identified. According to Theorem 3.17, there is a total of 84 minimal EMPs for which the user can choose. All minimal EMPs for a cycle network with seven nodes are displayed in Table 16.

Table 16 – All minimal EMPs for a 7-node cycle network.

EMP	$(\mathcal{B},\mathcal{C})$	EMP	$(\mathcal{B},\mathcal{C})$	EMP	$(\mathcal{B},\mathcal{C})$	EMP	$(\mathcal{B},\mathcal{C})$
1	({1, 2, 3, 4, 6}; {5, 7})	2	({1, 2, 3, 5, 6}; {4, 7})	3	({1, 2, 3, 5, 7}; {4, 6})	4	({1, 2, 3, 5}; {4, 6, 7})
5	({1, 2, 3, 6}; {4, 5, 7})	6	({1, 2, 4, 5, 6}; {3, 7})	7	({1, 2, 4, 5, 7}; {3, 6})	8	({1, 2, 4, 5}; {3, 6, 7})
9	({1, 2, 4, 6, 7}; {3, 5})	10	({1, 2, 4, 6}; {3, 5, 7})	11	({1, 2, 4, 7}; {3, 5, 6})	12	({1, 2, 4}; {3, 5, 6, 7})
13	({1, 2, 5, 6}; {3, 4, 7})	14	({1, 2, 5, 7}; {3, 4, 6})	15	({1, 2, 5}; {3, 4, 6, 7})	16	({1, 2, 6}; {3, 4, 5, 7})
17	({1, 3, 4, 5, 6}; {2, 7})	18	({1, 3, 4, 5, 7}; {2, 6})	19	({1, 3, 4, 5}; {2, 6, 7})	20	({1, 3, 4, 6, 7}; { 2, 5})
21	({1, 3, 4, 6}; { 2, 5, 7})	22	({1, 3, 4, 7}; { 2, 5, 6})	23	({1, 3, 4}; { 2, 5, 6, 7})	24	({1, 3, 5, 6, 7}; { 2, 4})
25	({1, 3, 5, 6}; { 2, 4, 7})	26	({1, 3, 5, 7}; { 2, 4, 6})	27	({1, 3, 5}; { 2, 4, 6, 7})	28	({1, 3, 6, 7}; { 2, 4, 5})
29	({1, 3, 6}; { 2, 4, 5, 7})	30	({1, 3, 7}; { 2, 4, 5, 6})	31	({1, 3}; { 2, 4, 5, 6, 7})	32	({1, 4, 5, 6}; { 2, 3, 7})
33	({1, 4, 5, 7}; { 2, 3, 6})	34	({1, 4, 5}; { 2, 3, 6, 7})	35	({1, 4, 6, 7}; { 2, 3, 5})	36	({1, 4, 6}; { 2, 3, 5, 7})
37	({1, 4, 7}; { 2, 3, 5, 6})	38	({1, 4}; { 2, 3, 5, 6, 7})	39	({1, 5, 6}; { 2, 3, 4, 7})	40	({1, 5, 7}; { 2, 3, 4, 6})
41	({1, 5}; { 2, 3, 4, 6, 7})	42	({1, 6}; { 2, 3, 4, 5, 7})	43	({2, 3, 4, 5, 7}; { 1, 6})	44	({2, 3, 4, 6, 7}; { 1, 5})
45	({2, 3, 4, 6}; { 1, 5, 7})	46	({2, 3, 4, 7}; { 1, 5, 6})	47	({2, 3, 5, 6, 7}; { 1, 4})	48	({2, 3, 5, 6}; { 1, 4, 7})
49	({2, 3, 5, 7}; { 1, 4, 6})	50	({2, 3, 5}; { 1, 4, 6, 7})	51	({2, 3, 6, 7}; { 1, 4, 5})	52	({2, 3, 6}; { 1, 4, 5, 7})
53	({2, 3, 7}; { 1, 4, 5, 6})	54	({2, 4, 5, 6, 7}; { 1, 3})	55	({2, 4, 5, 6}; { 1, 3, 7})	56	({2, 4, 5, 7}; { 1, 3, 6})
57	({2, 4, 5}; { 1, 3, 6, 7})	58	({2, 4, 6, 7}; { 1, 3, 5})	59	({2, 4, 6}; {1, 3, 5, 7})	60	({2, 4, 7}; {1, 3, 5, 6})
61	({2, 4}; {1, 3, 5, 6, 7})	62	({2, 5, 6, 7}; {1, 3, 4})	63	({2, 5, 6}; {1, 3, 4, 7})	64	({2, 5, 7}; {1, 3, 4, 6})
65	({2, 5}; {1, 3, 4, 6, 7})	66	({2, 6, 7}; {1, 3, 4, 5})	67	({2, 6}; {1, 3, 4, 5, 7})	68	({2, 7}; {1, 3, 4, 5, 6})
69	({3, 4, 5, 7}; {1, 2, 6})	70	({3, 4, 6, 7}; {1, 2, 5})	71	({3, 4, 6}; {1, 2, 5, 7})	72	({3, 4, 7}; {1, 2, 5, 6})
73	({3, 5, 6, 7};{1, 2, 4})	74	({3, 5, 6}; {1, 2, 4, 7})	75	({3, 5, 7}; {1, 2, 4, 6})	76	({3, 5}; {1, 2, 4, 6, 7})
77	({3, 6, 7}; {1, 2, 4, 5})	78	({3, 6}; {1, 2, 4, 5, 7})	79	({3, 7}; {1, 2, 4, 5, 6})	80	({4, 5, 7}; {1, 2, 3, 6})
81	({4, 6, 7}; {1, 2, 3, 5})	82	({4, 6}; {1, 2, 3, 5, 7})	83	({4, 7}; {1, 2, 3, 5, 6})	84	({5, 7}; {1, 2, 3, 4, 6})

We can use the results of Lemma 4.4 and Theorem 4.16 to reduce the number of candidates EMPs in a similar fashion as we did for the class of cycles with six nodes. Recall that these results are related to the concepts of circular and symmetrical opposite

EMPs – see Definitions 4.5 and 4.6. All EMPs that are circular to a particular EMP_1 yield the same accuracy under Assumptions 4.2 and 4.3 to EMP_1. The same holds for the symmetrical opposite (EMP_{84} in Table 16) of EMP_1. All minimal EMPs in Table 16 can be generated by the following six minimal EMPs using either Lemma 4.4 or Theorem 4.16:

I. $\text{EMP}_1 = (\{1, 2, 3, 4, 6\}; \{5, 7\})$;

II. $\text{EMP}_2 = (\{1, 2, 3, 5, 6\}; \{4, 7\})$;

III. $\text{EMP}_4 = (\{1, 2, 3, 5\}; \{4, 6, 7\})$;

IV. $\text{EMP}_5 = (\{1, 2, 3, 6\}; \{4, 5, 7\})$;

V. $\text{EMP}_8 = (\{1, 2, 4, 5\}; \{3, 6, 7\})$;

VI. $\text{EMP}_{10} = (\{1, 2, 4, 6\}; \{3, 5, 7\})$.

By using these results, we are able to reduce the number of EMPs under analysis, more precisely, this number of candidates reduced from 84 minimal EMPs in Table 16 to only six possible minimal EMPs that could yield different accuracy results. Notice that EMP_{10} is the EMP that is closer to be the "equivalent" to an EMP with alternate excitations and measurements for a cycle with an odd number of nodes. This means that if the principle derived for the network with six nodes is valid for cycles with larger number of nodes, then we expect that EMP_{10} would overcome the other EMPs with respect to the accuracy of the parameter estimates. In order to analyze if this is the case for cycles with seven nodes, we will analyze once more a state-space network matrix (163) with a single parameter:

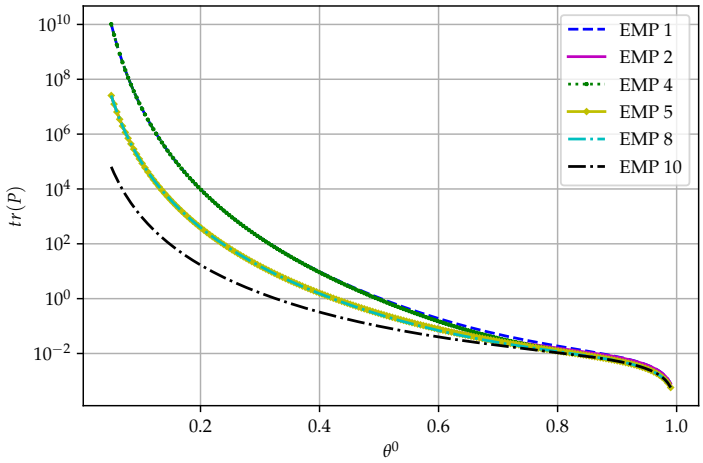
$$G(q, \theta^0) = \begin{bmatrix} \mathbf{0} & \theta^0 \\ diag(\theta^0, \ldots, \theta^0) & 0 \end{bmatrix} q^{-1}.$$

Recall that there is a stability constraint for this network: $\theta^0 < 1$. We proceed as before by computing the information matrices of each minimal EMP using the partial information matrix approach as in Lemma 4.1. Figure 21 depicts the trace of the covariance matrix for the minimal $\text{EMPs}_{1,2,4,5,8,10}$.

This figure shows that minimal EMP_{10} is the one that yields the most accurate estimates for state-space cycle networks with seven nodes. Thus, the principle that it is better to alternatively excite and measure the nodes of a cycle is confirmed for this network cardinality. Similar to the case of a cycle with six nodes, for "small" values of the true parameter θ^0 the difference between the best EMP and the other EMPs is significant. For θ^0 close to zero the difference between EMP_1 and EMP_{10} is about five orders of magnitude. This gap drastically reduces for larger magnitudes of the true parameter θ^0, typically close to 1. The difference between EMP_1 and EMP_{10} for $\theta^0 = 0.95$ is 40%.

In summary, we have shown that for larger cycles there is a significant number of minimal EMPs that are circular to each other. We have shown that circular EMPs yield

Figure 21 – Trace of covariance matrices for minimal EMPs$_{1,2,4,5,8,10}$ for a state-space cycle network with 7 nodes under Assumptions 4.2 and 4.3.



Source: The Author.

the same overall accuracy under Assumptions 4.2 and 4.3. Yet under these assumptions, we have also shown that if we exchange the set of excited nodes for the set of measured nodes for any minimal EMP, then the trace of the asymptotic covariance matrix is not changed, i.e. symmetrical opposite EMPs produce similar accuracy. For a general cycle we have demonstrated that equality between the EMPs depend on SNR$_{ji}$ at some nodes and on the equality among some modules. Therefore, for a cycle network we can relax Assumptions 4.2 and 4.3, and still get the same overall accuracy for some EMPs. For larger cycles we have verified that the principle derived for cycles with a small number of nodes still holds, that is, it is better to choose EMPs with equal shares of excitations and measurements. Furthermore, we have observed that the EMPs in which the nodes are alternatively excited and measured tend to yield the most accurate outcomes. As in the case of branch networks, there is a crucial difference in the magnitude of the resulting accuracy. For small magnitude of the parameters, the gains in precision can be typically large, while for larger magnitudes different EMPs could yield a similar overall accuracy.

4.5.6 Numerical analysis

In this section we provide a numerical analysis to verify if the principles derived so far still hold in more general situations. Here, we are interested in determining to which extent the relevant principles developed for smaller cycles also hold for larger ones. For this evaluation, we are going to focus on aspects related to the “size” of the direct modules and the structural advantages of different EMPs. To this end, we will consider state-space

network matrices (163) of the form:

$$G(q,\theta^0) = \begin{bmatrix} \mathbf{0} & \theta_n^0 \\ \text{diag}(\theta_1^0,\ldots,\theta_{n-1}^0) & \mathbf{0} \end{bmatrix} q^{-1}. \tag{200}$$

We are going to consider a number of different experiment scenarios to evaluate the performance of the minimal EMPs for each network cardinality. We first try to understand the role the numeric values of the modules play in the selection of the best EMP. In order to evaluate the factors that are crucial in the selection of the most accurate EMP, we consider an experiment setting in which the network is uniformly excited – under Assumption 4.3. This analysis makes it possible to shed some light on the influence of the parameters in the accuracy of the EMPs. For other scenarios we are going to evaluate the influence of the module magnitudes in the selection of the most accurate EMP.

For this numeric experiment, we are going to simulate different cycle networks with cardinalities ranging from 5 nodes to 8 nodes. For each network cardinality, we simulate a total of $5,000$ random cycles with network matrices (200), such that each parameter is drawn from a uniform distribution.

There are four types of experiment listed below.

1. Scenario $\mathbb{S}_0$ - The cycle is equally excited (Assumption 4.3) and every parameter is drawn from a uniform distribution $\mathcal{U}(0,1)$.

2. Scenario $\mathbb{S}_k$ - The cycle is equally excited, and the parameter magnitude of module k is at least twice than any other module: $\theta_i \sim \mathcal{U}(0,1)$ for $i \in \mathcal{W} \setminus \{k\}$ and $\theta_k = 2\max_{i\in\mathcal{W}\setminus\{k\}} \theta_i$.

3. Scenario $\mathbb{S}_{kj}$ - The cycle is equally excited, and module k and j have the same parameter magnitude, which is at least twice than any other module: $\theta_i \sim \mathcal{U}(0,1)$ for $i \in \mathcal{W} \setminus \{k,j\}$ and $\theta_{k,j} = 2\max_{i\in\mathcal{W}\setminus\{k,j\}} \theta_i$.

4. Scenario $\mathbb{S}_*$ - Modules are selected as in Scenario $\mathbb{S}_0$ with all input and noise variances randomly selected from a uniform distribution: $\sigma_i^2 \sim \mathcal{U}(60,80)$, for $i \in \mathcal{B}$ and $\lambda_j \sim \mathcal{U}(30,50)$ for $j \in \mathcal{C}$.

Let us start the numerical analysis with cycles with five nodes. Consider a cycle with five nodes, with network matrix in (200), with $n = 5$. Recall that all minimal EMPs for a cycle with 5 nodes are given in Table 14. For a cycle with five nodes under the Assumptions 4.2 and 4.3, we have shown that all minimal EMPs provide similar accuracy. We then expect that there is no preferred EMP in a more general setting when the SNR_{ji}'s are equal at the nodes. The results of the numerical analysis for each experiment are depicted in Table 17.

This table shows that for experiment $\mathbb{S}_0$, there is no preferred choice among the EMPs. This is exactly the expected result since all EMPs yield similar accuracy under

Table 17 – How often a minimal EMP of a 5 nodes cycles was selected as the most accurate in different experimental scenarios.

Scenario	EMPs (%)									
	1	2	3	4	5	6	7	8	9	10
$\mathbb{S}_0$	11.48	10.54	9.34	9.74	10.06	10.34	9.66	9.34	10.08	9.42
$\mathbb{S}_1$	1.82	1.38	65.16	0	25.9	0	0.4	2.5	0.48	2.36
$\mathbb{S}_2$	66.06	0	0.48	2.42	1.34	1.32	2.64	0.4	25.34	0
$\mathbb{S}_3$	0.48	2.3	2.38	0.34	0	26.26	0	65.16	1.64	1.44
$\mathbb{S}_4$	2.7	0.26	0	66.78	2.24	0.24	1.48	1.44	0	24.86
$\mathbb{S}_5$	0	23.8	1.52	1.56	0.54	2.48	67.28	0	2.44	0.38
$\mathbb{S}_{25}$	0	0	0.06	3.9	0.14	3.58	55.56	0	36.76	0
$\mathbb{S}_{35}$	0	52.92	3.42	0.1	0	39.9	0	0	3.58	0.08
$\mathbb{S}_{13}$	0.14	3.32	56.28	0	0	0	0	36.74	0.1	3.42
$\mathbb{S}_{14}$	3.8	0.12	0	0	53.28	0	0.1	3.12	0	39.58
$\mathbb{S}_{24}$	57.98	0	0	35.06	3.44	0.1	3.38	0.04	0	0

Assumptions 4.2 and 4.3 as shown in Theorem 4.17. For other experiments, there is a clear preference for the most often selected EMPs in more than half of the runs. The key factor here is the influence of the direct modules. $\text{EMPs}_{3,5}$ are the most often selected EMPs in experiment $\mathbb{S}_1$ because $G_1(q, \theta^0)$ is the largest module and it is also a direct module for these EMPs. Exactly the same happens in all other $\mathbb{S}_k$ experiments. Notice that for scenarios $\mathbb{S}_{1,2,5}$ the most often selected EMP is the one with three excitations, while for scenarios $\mathbb{S}_{3,4}$ the EMPs with two excitations were the most often selected. The scenarios $\mathbb{S}_{ji}$ come from the extrapolation of the results for 4-node cycles, where the product of the direct modules was the key decider. In scenario $\mathbb{S}_{25}$ the largest modules are $G_2(q, \theta^0)$ and $G_5(q, \theta^0)$, both are direct modules for $\text{EMPs}_{7,9}$. As we can see from this table, these are the most often selected EMPs, followed by the EMPs that have at least one of them as direct module. The same is observed for all other experiments $\mathbb{S}_{ji}$. Similar to the experiments $\mathbb{S}_k$, we have that for scenarios $\mathbb{S}_{ji}$ EMPs with three excitations were the most often selected in three out of the five experiments. Once again, we have shown that the direct modules have a key influence in the selection of the most accurate EMP.

The relevant question is now to decide whether it is better to choose an EMP with three excitations or one with three measurements. This question can be answered based on the direct modules of the EMPs. As in the four node case, the best EMPs are those that have the largest modules as direct modules. In the same spirit of Table 17, we compared the number of times a particular EMP with three excited nodes was more accurate than another EMP with only two excitations given that the product of their direct modules was larger than its contestant. This hold true in 98.5% of the 5,000 network simulations. This

means that the magnitudes of the direct modules are a key factor in the decision of the best EMP. If the user has some prior knowledge about the module magnitudes, then it should be better to choose an EMP whose "larger" modules are direct modules.

Let us now focus on cycles with six nodes and network matrix (200). All minimal EMPs for this class of networks are given in Table 15. For this specific class of network, we have shown that there is a particular EMP that yields most accurate estimates under Assumptions 4.2 and 4.3. The results for the numeric analysis of $5,000$ cycle networks are displayed in Table 18.

Table 18 – How often the nine six minimal EMPs were the most selected in a $5,000$ simulation runs for a cycle with six nodes in different experiment scenarios.

$\mathbb{S}_0$	EMP	10	23	5	29	22	8	14	31	26
	(%)	22.26%	21.54%	5.12%	4.76%	4.62%	4.6%	4.48%	4.18%	3.98%
$\mathbb{S}_1$	EMP	10	14	7	15	9	16	13	11	8
	(%)	24.2%	23.20%	9.82%	8.64%	6.94%	6.9%	6.34%	6.16%	5.90%
$\mathbb{S}_2$	EMP	5	23	26	2	3	27	25	4	22
	(%)	24.26%	23.9%	9.68%	8.68%	6.92%	6.58%	6.42%	6.0%	5.78%
$\mathbb{S}_3$	EMP	10	20	18	31	1	11	19	12	29
	(%)	24.36%	22.72%	10.04%	9.62%	7.08%	6.76%	5.96%	5.94%	5.78%
$\mathbb{S}_4$	EMP	8	23	7	15	24	4	17	28	14
	(%)	24.42%	23.14%	10.34%	8.76%	7.36%	6.5%	6.2%	5.92%	5.82%
$\mathbb{S}_5$	EMP	10	22	26	2	30	6	5	13	19
	(%)	24.9%	23.84%	9.6%	9.14%	6.84%	6.76%	6.28%	5.64%	5.54%
$\mathbb{S}_6$	EMP	29	23	31	18	21	25	32	28	20
	(%)	25.78%	24.08%	12.48%	11.42%	11.4%	10.84%	10.22%	9.64%	7.7%
$\mathbb{S}_*$	EMP	23	10	22	8	29	26	5	31	14
	(%)	12.9%	12.02%	4.0%	3.72%	3.67%	3.66%	3.52%	3.44%	3.38%

This table shows that the principle derived for six node cycles remains valid in more general scenarios. $\text{EMPs}_{10,23}$ from Table 15 are the ones in which their nodes are alternately excited and measured. For scenario $\mathbb{S}_0$ these EMPs are the most often selected EMPs, together they account for almost half of the total. They were four times more often selected than any other minimal EMP. For the other Scenarios $\mathbb{S}_i$ we see that these two minimal EMPs also stand out as the top two most selected EMPs. Once again, the main difference in how often those EMPs were selected is influenced by the magnitude of the direct modules. In all scenarios the two most selected EMPs are the ones in which the largest module is a direct module. Finally, the Scenario $\mathbb{S}_*$ where all quantities are randomly selected, $\text{EMPs}_{10,23}$ are still the ones that yield the best results. In this scenario, these EMPs are four times more often selected than other candidates.

Let us now focus on cycles with seven nodes. We proceed with the analysis in the very same fashion as we did for cycles with five and six nodes. The results for this cycle are displayed in Table 19.

Table 19 – How often the top nine most accurate minimal EMPs were selected for a cycle with 7 nodes in different experimental scenarios.

$\mathbb{S}_0$	EMP	59	58	36	75	60	29	27	56	49
	(%)	6.4%	5.94%	5.92%	5.80%	5.72%	5.68%	5.54%	5.52%	5.48%
$\mathbb{S}_1$	EMP	26	36	25	29	35	40	21	27	33
	(%)	27.90%	26.68%	5.80%	5.06%	4.26%	4.04%	3.84%	3.34%	2.74%
$\mathbb{S}_2$	EMP	64	10	58	60	59	16	56	14	13
	(%)	27.92%	25.84%	5.84%	5.26%	4.26%	3.70%	3.62%	3.54%	3.02%
$\mathbb{S}_3$	EMP	49	29	27	26	53	25	5	75	51
	(%)	28.30%	26.26%	5.2%	4.54%	4.10%	3.96%	3.94%	3.92%	3.26%
$\mathbb{S}_4$	EMP	60	21	59	10	58	46	23	36	72
	(%)	28.44%	26.22%	5.60%	5.06%	4.36%	4.0%	3.82%	3.70%	2.86%
$\mathbb{S}_5$	EMP	27	56	49	75	19	57	64	26	8
	(%)	28.44%	25.82%	5.36%	5.02%	4.2%	4.02%	4.02%	3.66%	2.90%
$\mathbb{S}_6$	EMP	59	25	21	36	55	29	10	74	63
	(%)	29.44%	25.26%	5.42%	5.12%	3.86%	3.8%	3.66%	3.40%	3.08%
$\mathbb{S}_7$	EMP	75	58	56	64	73	49	81	60	70
	(%)	28.40%	27.06%	4.98%	4.96%	4.02%	3.62%	3.52%	3.48%	2.94%
$\mathbb{S}_7$	EMP	75	59	49	26	27	25	60	56	29
	(%)	6.12%	5.92%	5.82%	5.60%	5.56%	5.52%	5.48%	5.44%	5.44%

This table shows that for Scenario $\mathbb{S}_0$ there is no EMP that stands out among the other candidates. Recall that there are 84 minimal EMPs in Table 16 for this particular class of network. However, minimal $\text{EMPs}_{59,58,36,75,60,29,27,56,49}$ share something in common: they are the equivalent of EMPs with nodes that are alternately excited and measured. Recall that EMP_{10} from Figure 21 is equivalent to EMP_{59} under Assumptions 4.2 and 4.3. In fact, $\text{EMPs}_{21,25,26,49,56,58}$ are circular to EMP_{10} and $\text{EMPs}_{27,29,36,60,64,75}$ are circular to EMP_{59}. These were the fourteen most often selected minimal EMPs, together they account for 77.8% of the selections out of the total simulation. Furthermore, they were on average five times more likely to be selected than any other EMP that was not alternately excited and measured. This means that the most accurate EMPs are those EMPs that obey the principle of alternately exciting and measuring the nodes.

For Scenarios $\mathbb{S}_i$ we reach the same conclusions with respect to the influence of the direct modules. That is, the EMPs for which the largest module is a direct module are

more likely to be selected as the best EMP in each run. However, in this case not all EMPs for which the largest module is a direct module is equally likely to be selected. The key factor here is the distribution of excitations and measurements in the cycle. Notice that the two best minimal EMPs for each $\mathbb{S}_i$ are a circular EMP either to EMP_{10} or EMP_{59}. Finally, as for Scenario $\mathbb{S}_*$ all minimal EMPs in the top nine are circular to $\text{EMPs}_{10,59}$ as in Scenario $\mathbb{S}_0$. This confirms once again that it is better to choose an EMP where the nodes are alternately excited and measured.

In conclusion, we have shown that the principles derived for larger cycles under Assumptions 4.2 and 4.3 remain valid in more general scenarios. More specifically, for cycles with state-space network matrix. Exactly the same have been shown for branches, but with modules represented by general transfer functions. For larger cycles the principles derived for small cycles are still valid. That is, EMPs where the largest module is a direct module and with an equal shares of excitations and measurements are the most likely to yield the best results. Moreover, we have shown that the EMPs where the nodes are alternately excited and measured are the most likely to provide the most accurate results for larger cycles.

4.6 Conclusion

In this chapter we have analyzed the accuracy of the parameter estimates for two classes of dynamic networks, namely branches and cycles. Focus was given upon selecting the EMP which provided the most accurate parameter estimates. In this way, one could select the best experimental setting for identification of all modules of the dynamic network. We have shown that there is a structural property regarding the allocation of excitations and measurements that yield the most accurate estimates. For branches, we have shown that there are some EMPs that are better than others. Specifically, those EMPs for which the excitation signals are applied to the first half of the network, while the last half are measured. In the case of cycles, we have derived some structural principles for the selection of the best EMP. Fisrt, EMPs with equal shares of excitations and measurements were most likely to give the most accurate results. Second, for larger cycles – typically with more than 4 nodes – the best EMP is the one in which the nodes are alternately excited and measured. These structural properties are not exclusive factors for the selection of the most accurate EMP. We have also shown that the magnitude of the direct modules plays a key role in the decision of the best EMP. Moreover, the signal-to-noise ratio at some nodes also have an impact on the best outcomes, specially the $\text{SNR}_{i+1,i}$ of direct modules. It has been observed that the gains in choosing an appropriate EMP can be hundred of times better than choosing alternative EMPs. Therefore, it is fundamental that the choice of the experimental setting for the identification task should not be chosen arbitrarily.

5 CONCLUSIONS

Dynamic networks are models of interconnected linear time invariant systems that can be used in a wide range of applications, from geological cycles to the electrical power grid. Such models can improve the quality of many engineering systems and they could provide a deep understanding of many phenomena in different disciplines. An advanced network theory can be a fundamental cornerstone for the advancement of scientific knowledge across many fields. However, in order to take maximum advantage of this modeling approach it is necessary to first obtain such models. This thesis addressed the problem of identifying dynamic networks models from data. Obtaining network models from data pose several additional challenges with respect to the identification of single systems. In particular, there is a considerable degree of flexibility in dealing with such a complex system. To mention just a few examples, networks are full of feedback mechanisms and parallel processes, which induces correlation among most signals involved. Furthermore, the high flexibility with respect to the experiment setup, which add some freedom to the user choice, and can impose some constraints in how to obtain network data.

In this thesis, we have paid special attention to two identification problems related to dynamic networks. The first one is related to how to distinguish different network models based on network data. This ability is of fundamental importance since if two different network models generate exactly the same data, we could not hope to tell which one produced the data, and therefore we could not identify a unique model. Hence, in order to obtain consistent estimates is necessary that the corresponding models are identifiable from data.

An important aspect to determine whether a dynamic network is identifiable or not is the experimental setup in which the network is subjected. It turns out that this is a crucial aspect to the identifiability of a dynamic network. This characteristic inspired the concept of Excitation and Measurement Pattern (EMP) of a network, which can determine whether a unique model can be obtained from network data. Our aim was to provide conditions based on the distribution of the inputs and measurements, that is conditions on the EMP, over the network such to render a network generically identifiable. For a given topology, the conditions on the EMP took the form of indication conditions on the nodes

that must be excited and on the nodes that must be measured. This approach also makes it possible to provide an identification scheme for the modules of the network. Typically, the conditions depend upon the topology of the network under analysis. For this reason, we have investigated identifiability conditions for some classes of networks.

Using the concept of EMP we have provided necessary and sufficient conditions for the network identifiability problem for some topologies. We have investigated the role of parallel paths in the identifiability of acyclic dynamic networks. In a fully connected acyclic dynamic network is necessary and sufficient to excite and measure all nodes, except for sources that could be not measured and sinks that could not be excited. For more general acyclic structures we have developed necessary conditions based on the neighborhood of the nodes. Whether a node within an acyclic network must always be excited or measured depends on its neighbors. Furthermore, we have developed necessary and sufficient conditions that define all minimal EMPs for isolated cycle networks. All these conditions can be combined to develop algorithms to analyze the identifiability of more complex networks. An important feature of these conditions is that they are based on the EMP applied at the network. Thus, the developed conditions are easy to verify.

Another contribution to the identifiability of dynamic network was to consider a divide and conquer scheme. By recognizing known topologies within a large network, one could divide a network into two subnetworks with known identifiability conditions, each of which subjected to a valid EMP – one that renders a network generically identifiable. We have then determined what are the additional conditions on the EMP for the identifiability of the whole network given that the two subnetworks were known to be identifiable. Once more, these conditions took the form of indicative conditions on the nodes that must be excited and the ones that must be measured.

Identification of high quality models is crucial to the development of new technologies. To obtain reliable models we must ensure that the identification method yields accurate models. Improving such accuracy of the identification method can be performed by the design of an appropriate experiment. There are many factors that play a role in the tuning of an experiment for identification. In this thesis, the second problem we have tackled was the allocating of inputs signals and the measurements from a network, that is, to choose the distribution of inputs and measurements in the network. This is a structural problem since it only deals with how the excitations and measurements are distributed in the network.

While identifiability conditions offer a number of different possible EMPs for the user to choose from, there are no clear guidelines on how to select among all possible EMPs. A framework for the selection of the EMP was introduced based on the number of excitations and measurements, and on the trace of the asymptotic covariance matrix. We have studied two different classes of dynamic networks, namely branches and cycles. From the analysis of these classes of networks emerged key principles that can be used as guidelines for the user to choose the most accurate EMP. First, we have shown that there is a structural

feature in the selection of the best EMP. Not only the numeric values of the modules and the signal-to-noise ratio at some nodes play a role in the selection of the EMP, but the distribution of excitations and measurements can drastically improve or constrain the accuracy obtained by the identified model. This is of fundamental importance, since one could have significant gains simply by allocating the excitations and measurements in the network.

For branches and cycles we have demonstrated that there exists a trade-off between excitations and measurements. For branches we have established that EMPs where the first half of the network is excited and the other half is measured yield the most accurate parameter estimates. For cycles we have shown that EMPs with equal shares of excitation and measurements tend to give the best results. Moreover, differently from branches, it is better to alternately excite and measure the nodes of larger cycles.

Another important factor in the selection of the best EMP was the influence of the direct modules. This is related to the magnitude of these modules. EMPs where the largest module is a direct module tend to provide more accurate estimates. In addition, we have also shown that the signal-to-noise ratio of the direct modules is a key factor in the selection of the EMPs. A crucial aspect in the selection of the EMPs is the gain in the accuracy when compared to candidate EMPs. In many cases, the best EMP was more than a hundred times better than other EMPs. This attests to the importance of selecting an EMP appropriately according to the network topology at hand.

This work can be extended in various directions. The selection of the most accurate EMP is an interesting line of research for more complex network topologies, such as trees and general acyclic dynamic networks (DAGs). To analyze in what conditions the principles developed in this thesis also holds for these complex networks is certainly a future topic of research. Another interesting topic is to analyze the influence of process noise in the parameter estimates for networks. This could be complemented by the analysis of dynamic networks with more specific model structures for the modules, e.g. FIR, ARX, ARMAX, etc. Investigating the problem of selection an EMP that yields most accurate estimates for a single module embedded in a network is also a promising topic.

For the identifiability problem, there are many interesting research directions. An algorithm to provide the synthesis for excitations and measurements using the knowledge of the local subnetworks is very appealing. Extending the conditions about the excitation or measurement of some nodes based on their neighborhood to more general networks is also crucial. Development of an algorithm that combines the conditions for parallel paths, trees, multitrees, and the local topology of the nodes to provide a minimal EMP for general acyclic network is also a very interesting topic.

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APPENDIX A COVARIANCE FORMULAE FOR CYCLIC NETWORKS

In this Appendix we present the information matrix of state-space cycle networks for the minimal EMPs. We provide the formulas for the covariance matrices for cycle with two and three nodes.

A.1 Cycles with 2 nodes

Here we consider state-space cycle networks with two nodes. This kind of cycle network is characterized by the following network matrix

$$G(q,\theta^0) = \begin{bmatrix} 0 & \theta_2^0 \\ \theta_1^0 & 0 \end{bmatrix} q^{-1}. \tag{201}$$

We use the prediction error approach to compute the information matrix based on the gradient of the optimal predictors under Assumption 4.1. Recall that for two-node cycles there are four minimal EMPs that will be analyzed: $\text{EMP}_1 = (\{1,2\},\{1\})$ $\text{EMP}_2 = (\{1,2\},\{2\})$, $\text{EMP}_3 = (\{1\},\{1,2\})$ $\text{EMP}_4 = (\{2\},\{1,2\})$. The optimal one-step ahead predictors are as follows:

$$\hat{y}_1(t|t-1) = \frac{r_1(t)}{1-\theta_2\theta_1 q^{-2}} + \frac{\theta_2 r_2(t-1)}{1-\theta_2\theta_1 q^{-2}},$$
$$\hat{y}_2(t|t-1) = \frac{r_2(t)}{1-\theta_2\theta_1 q^{-2}} + \frac{\theta_1 r_1(t-1)}{1-\theta_2\theta_1 q^{-2}}.$$

For EMP_1 the gradient of the predictor will be:

$$\psi_1(t) = \frac{1}{\Delta}\begin{bmatrix} r_2(t-1) + \theta_1 r_1(t-2) \\ \theta_2\left(r_1(t-2) + \theta_2 r_2(t-3)\right) \end{bmatrix},$$

where $\Delta = 1 - 2\theta_2\theta_1 q^{-2} + {\theta_2}^2{\theta_1}^2 q^{-4}$. Then, the information matrix can be obtained as

$$M_1 = \frac{N}{\lambda_1}\begin{bmatrix} \gamma_0\left(\sigma_1^2{\theta_1^0}^2 + \sigma_2^2\right) & \theta_2^0\left(\sigma_1^2\theta_1^0\gamma_0 + \sigma_2^2\theta_2^0\gamma_2\right) \\ \theta_2^0\left(\sigma_1^2\theta_1^0\gamma_0 + \sigma_2^2\theta_2^0\gamma_2\right) & {\theta_2^0}^2\gamma_0\left(\sigma_1^2 + \sigma_2^2{\theta_2^0}^2\right) \end{bmatrix}, \tag{202}$$

where $\gamma_k^i \triangleq \mathbb{E}\, r_i^F(t) r_i^F(t-k)/\sigma_i^2$ with $r_i^F(t) \triangleq r_i(t)/\Delta$, for $i = 1, 2$. We drop the superscript of γ_k^i since both inputs are filtered by the same filter and they share the same second-order statistical properties. The autocovariance γ_k, for $k = 0, 2$ are as follows:

$$\gamma_0 = -\frac{\theta_2^{0^2}\theta_1^{0^2} + 1}{\theta_2^{0^6}\theta_1^{0^6} - 3\theta_2^{0^4}\theta_1^{0^4} + 3\theta_2^{0^2}\theta_1^{0^2} - 1}, \tag{203}$$

$$\gamma_2 = \frac{2\theta_2^{0^3}\theta_1^{0^3}}{\theta_2^{0^6}\theta_1^{0^6} - 3\theta_2^{0^4}\theta_1^{0^4} + 3\theta_2^{0^2}\theta_1^{0^2} - 1}. \tag{204}$$

One can obtain the variance of the parameter estimates from the information matrix as:

$$\mathrm{var}(\hat{\theta}_2^1) = \frac{\lambda_1\gamma_0}{Nd_1}\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right), \tag{205}$$

$$\mathrm{var}(\hat{\theta}_1^1) = \frac{\lambda_1\gamma_0}{N\theta_2^{0^2}d_1}\left(\theta_1^{0^2}\sigma_1^2 + \sigma_2^2\right), \tag{206}$$

where $d_1 = \left(\gamma_0^2\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right)\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right) - \left(\gamma_0^2\sigma_1^2\theta_1^0 + \gamma_2\sigma_2^2\theta_2^0\right)^2\right)$. Now, for EMP$_2$ we can compute the gradient as follows:

$$\psi_2(t,\theta) = \frac{1}{\Delta}\begin{bmatrix}\theta_1\left(\theta_1 r_1(t-3) + r_2(t-2)\right)\\ r_1(t-1) + \theta_2 r_2(t-2)\end{bmatrix},$$

which yields the following information matrix

$$M_2 = \frac{N}{\lambda_2}\begin{bmatrix}\theta_1^{0^2}\gamma_0\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right) & \theta_1^0\left(\sigma_2^2\theta_2^0\gamma_0 + \sigma_1^2\theta_1^0\gamma_2\right)\\ \theta_1^0\left(\sigma_2^2\theta_2^0\gamma_0 + \sigma_1^2\theta_1^0\gamma_2\right) & \gamma_0\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right)\end{bmatrix}. \tag{207}$$

Similarly, the variance of the parameter estimates are:

$$\mathrm{var}(\hat{\theta}_2^2) = \frac{\lambda_2\gamma_0}{N\theta_1^{0^2}d_2}\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right), \tag{208}$$

$$\mathrm{var}(\hat{\theta}_1^2) = \frac{\lambda_2\gamma_0}{Nd_2}\left(\theta_1^{0^2}\sigma_1^2 + \sigma_2^2\right), \tag{209}$$

where $d_2 = \left(\gamma_0^2\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right)\left(\sigma_1^2 + \sigma_2^2\theta_2^{0^2}\right) - \left(\gamma_0^2\sigma_2^2\theta_1^0 + \gamma_2\sigma_1^2\theta_2^0\right)^2\right)$. When EMP$_3$ is considered, the gradient of the optimal predictor has the following form:

$$\psi_3(t,\theta) = \frac{1}{\Delta}\begin{bmatrix}\frac{\theta_1 r_1(t-2)}{\sqrt{\lambda_1}} & \frac{{\theta_1}^2 r_1(t-3)}{\sqrt{\lambda_2}}\\ \frac{\theta_2 r_1(t-2)}{\sqrt{\lambda_1}} & \frac{r_1(t-1)}{\sqrt{\lambda_2}}\end{bmatrix}.$$

This gradient yields the following information matrix:

$$M_3 = N\begin{bmatrix}\sigma_1^2\gamma_0\theta_1^{0^2}\left(\frac{\theta_1^{0^2}}{\lambda_2} + \frac{1}{\lambda_1}\right) & \sigma_1^2\theta_1^0\left(\frac{\theta_1^0\gamma_2}{\lambda_2} + \frac{\theta_2^0\gamma_0}{\lambda_1}\right)\\ \sigma_1^2\theta_1^0\left(\frac{\theta_1^0\gamma_2}{\lambda_2} + \frac{\theta_2^0\gamma_0}{\lambda_1}\right) & \sigma_1^2\gamma_0\left(\frac{1}{\lambda_2} + \frac{\theta_2^{0^2}}{\lambda_1}\right)\end{bmatrix}. \tag{210}$$

Thus, the variance of the parameter estimates are:

$$\text{var}(\hat{\theta}_2^3) = \frac{\lambda_1\lambda_2\gamma_0}{\theta_1^{0\,2} d_3}\left(\lambda_1 + \lambda_2\theta_2^{0\,2}\right) \tag{211}$$

$$\text{var}(\hat{\theta}_1^3) = \frac{\lambda_1\lambda_2\gamma_0}{d_3}\left(\theta_1^{0\,2}\lambda_1 + \lambda_2\right), \tag{212}$$

with $d_3 = \sigma_1^2\left(\gamma_0^2\left(\lambda_1 + \lambda_2\theta_2^{0\,2}\right)\left(\theta_1^{0\,2}\lambda_1 + \lambda_2\right) - \left(\gamma_0^2\lambda_2\theta_2^0 + \gamma_2\lambda_1\theta_1^0\right)^2\right)$. Finally, EMP_4 has the following predictor gradient:

$$\psi_4(t,\theta) = \frac{1}{\Delta}\begin{bmatrix} r_2(t-1) & \theta_1^0 r_2(t-2) \\ \theta_2^{0\,2} r_2(t-3) & \theta_2^0 r_2(t-2) \end{bmatrix}, \tag{213}$$

which gives the following information matrix:

$$M_4 = N\begin{bmatrix} \sigma_2^2\gamma_0\left(\frac{\theta_1^{0\,2}}{\lambda_2} + \frac{1}{\lambda_1}\right) & \sigma_2^2 b_{12}^0\left(\frac{\theta_1^0\gamma_0}{\lambda_2} + \frac{\theta_2^0\gamma_2}{\lambda_1}\right) \\ \sigma_2^2\theta_2^0\left(\frac{\theta_1^0\gamma_0}{\lambda_2} + \frac{\theta_2^0\gamma_2}{\lambda_1}\right) & \sigma_2^2\gamma_0\theta_2^{0\,2}\left(\frac{1}{\lambda_2} + \frac{\theta_2^{0\,2}}{\lambda_1}\right) \end{bmatrix}. \tag{214}$$

From that, one can recover the variance of the parameter estimates as:

$$\text{var}(\hat{\theta}_2^4) = \frac{\lambda_1\lambda_2\gamma_0}{N d_4}\left(\lambda_1 + \lambda_2\theta_2^{0\,2}\right), \tag{215}$$

$$\text{var}(\hat{\theta}_1^4) = \frac{\lambda_1\lambda_2\gamma_0}{N\theta_2^{0\,2} d_4}\left(\theta_1^{0\,2}\lambda_1 + \lambda_2\right), \tag{216}$$

with $d_4 = \sigma_2^2\left(\gamma_0^2\left(\lambda_1 + \lambda_2\theta_2^{0\,2}\right)\left(\theta_1^{0\,2}\lambda_1 + \lambda_2\right) - \left(\gamma_0^2\lambda_2\theta_2^0 + \gamma_2\lambda_1\theta_1^0\right)^2\right)$.