UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
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# ASYMPTOTIC BEHAVIOUR AND COMPARISON PRINCIPLE FOR SOLUTIONS OF LOCAL AND NON-LOCAL EQUATIONS IN EXTERIOR DOMAINS

Porto Alegre, RS
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Tese apresentada ao Programa de Pós-Graduação em Matemática, Área de Análise, da Universidade Federal do Rio Grande do Sul (UFRGS, RS), como requisito parcial para obtenção do título de **Doutor em Matemática**.

Orientador: Prof. Dr. Leonardo Prange Bonorino

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*"Tenho a impressão de ter sido uma criança brincando à beira-mar, divertindo-me em descobrir uma pedrinha mais lisa ou uma concha mais bonita que as outras, enquanto o imenso oceano da verdade continua misterioso diante de meus olhos."*

Isaac Newton

**ABSTRACT**

**ASYMPTOTIC BEHAVIOUR AND COMPARISON PRINCIPLE FOR SOLUTIONS OF LOCAL AND NON-LOCAL EQUATIONS IN EXTERIOR DOMAINS**

In this work, we establish an asymptotic behaviour theorem for solutions of a class of quasilinear elliptic equations in $\mathbb{R}^n \backslash K$, where $K$ is a compact set, provided the structure of the equation and the dimension $n$ are related. This result is obtained by a combination of the Harnack inequality with symmetrization techniques. Furthermore, with the choice of suitable test functions, a Comparison Principle is obtained for solutions of a class of non-local elliptic equations in $\mathbb{R}^n \backslash K$, where $K$ is a compact set.

**Keywords:** Asymptotic behaviour; Comparison Principle; Exterior domains.

## RESUMO

## COMPORTAMENTO ASSINTÓTICO E PRINCÍPIO DE COMPARAÇÃO PARA SOLUÇÕES DE EQUAÇÕES LOCAIS E NÃO LOCAIS EM DOMÍNIOS EXTERIORES

Neste trabalho, provamos um teorema de comportamento assintótico para soluções de equações elípticas quasilineares definidas em $\mathbb{R}^n \backslash K$, onde $K$ é um conjunto compacto, desde que a estrutura dessa equação e a dimensão $n$ estejam relacionadas. Esse resultado é obtido através da aplicação de uma desigualdade de Harnack associada com técnicas de simetrização. Além disso, com a escolha adequada de funções teste, obtemos um Princípio de Comparação para soluções de uma classe de equações elípticas não locais definidas em $\mathbb{R}^n \backslash K$, sendo $K$ um conjunto compacto.



**Palavras Chave:** Comportamento assintótico; Princípio de Comparação; Domínios exteriores.

## SYMBOL LIST

$\rightharpoonup$ means weak convergence;

$\to$ means strong convergence;

$B(x,r) = B_r(x)$ is the open ball centered at $x$ and with radius $r > 0$ in $\mathbb{R}^n$;

$\partial\Omega$ is the boundary of the set $\Omega$;

$\overline{\Omega}$ represents the closure of $\Omega$;

$\Omega^c$ means $\mathbb{R}^n \backslash \Omega$;

$f^+ = \max\{f, 0\}$ ($f^- = \max\{-f, 0\}$) is the positive (negative) part of $f$;

$|A|$ represents the Lebesgue measure of the set $A$;

a.e. is short for almost everywhere;

$u\mid_A$ is the restriction of the function $u$ to the set $A$ ;

$\inf\limits_X u$ ($\sup\limits_X u$) represents the infimum (supremum) of the function $u$ over the set $X$;

$\operatorname*{osc}\limits_X u = \sup\limits_X u - \inf\limits_X u$ is the oscillation of the function $u$ in $X$;

$\liminf\limits_{x\to x_0} u$ ($\limsup\limits_{x\to x_0} u$) represents the limit inferior (superior) of the function $u$ when $x \to x_0$;

$\|u\|_{L^p(\Omega)} = \|u\|_{p,\Omega} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$;

$\|u\|_{L^\infty(\Omega)} = \inf\{a \geq 0; |\{x \in \Omega; |u(x)| > a\}| = 0\}$;

$L^p(\Omega) = \{u : \Omega \to \mathbb{R}$ is measurable and $\|u\|_{L^p(\Omega)} < \infty\}$, $1 \leq p < \infty$;

$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}; u$ is measurable and $\|u\|_{L^\infty(\Omega)} < \infty\}$;

$L^p_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R}$ is measurable and $f|_K \in L^p(K)$, for every $K \subset \Omega$ compact$\}$;

$\mathrm{supp}(u) = \overline{\{x \in \Omega; u(x) \neq 0\}}$;

$C_c(\Omega) = \{u \in C(\Omega);\ \mathrm{supp}(u) \subset \Omega \text{ is compact}\}$;

$C^k(\Omega) = \{u : \Omega \to \mathbb{R}; u \text{ is k times continuously differentiable}\}$;

$C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega)$;

$C_c^\infty = C^\infty(\mathbb{R}^n) \cap C_c(\Omega)$;

$SO(n)$ is the special orthogonal group in dimension $n$;

$W^{s,p}(\Omega) = \left\{ w \in L^p(\Omega)\ ;\ \frac{w(x) - w(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$, $s \in (0,1)$, $p \in (1, \infty)$;

# Contents

# INTRODUCTION

The theory of Partial Differential Equations (PDE's) had great developments in the last century, mainly due to its applications in several areas, such as engineering, physics, biology, and economics. In this sense, understanding the existence, multiplicity, and the behavior of solutions for different types of PDE's grows in importance, given that several applied problems depend on two or more independent variables.

Partial Differential Equations can be classified by many properties, like, for instance, linearity, order, homogeneity, locality, and, as elliptic, hyperbolic, or parabolic. This work addresses properties for solutions of two particular classes of elliptic PDE's. The first is described by a quasilinear local operator; the second, described by an integro-differential non-local operator.

In the first part of the dissertation, we investigate the behaviour at infinity for solutions $u$ on exterior domains $\mathbb{R}^n \backslash K$, where $K \subset \mathbb{R}^n$ is any compact set, of an equation associated to the quasilinear operator

$$\operatorname{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u\big).$$

In some sense, this generalizes results from Gilbarg and Serrin [15]. We prove the existence of the limit at infinity for such solutions in a non-homogeneous context, considering the case $p < n$, where $p \in (1, +\infty)$. Our main result is the following.

**Theorem.** *Consider $p < n$ and let $u \in C^1(\mathbb{R}^n \setminus K)$ be a bounded weak solution of*

$$-\operatorname{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u\big) \;=\; f$$

*in $\mathbb{R}^n \setminus K$. If*

$$f \in L^r(\mathbb{R}^n \backslash K)\,, \quad \textit{for some} \;\; r \,<\, n/p$$

*and*

$$f \in L^{\frac{n}{p-\theta}}(\mathbb{R}^n \backslash K)\,, \quad \textit{for some} \;\; \theta \in (0,1)$$

*with*

$$\lim_{R \to +\infty} \;\; R^{\theta} \, \| \, f \, \|_{L^{\frac{n}{p-\theta}}(\,\mathbb{R}^n \backslash B_R(0)\,)} \;=\; 0,$$

*then the limit $\lim\limits_{|x| \to \infty} u(x)$ exists.*

We follow some ideas from Gilbarg and Serrin [15] based on the Harnack inequality, but to overcome the non-homogeneity of the PDE, we employ symmetrization results to obtain a sufficient condition for the existence of the limit in terms of the integrability of $f$ at infinity. The main properties of our operator are described in Section 1.1.

In the second half of the last century, several works about quasilinear operators, similar to the one we study here, have been made. We mention considerations on singularities of solutions of second order PDE's by Gilbarg and Serrin [15]. Singularity problems and the behavior of solutions have also been studied by Serrin [30, 31] and Serrin and Weinberger [33]. Some years later, research concerning the regularity of solutions have been developed by Evans [12], Tolksdorf [38], and Lieberman [24].

More recently, Fraas and Pinchover [13] studied singularities of positive solutions of $p-$Laplace type equations, improving results from Serrin [30] and proving a limit result at infinity under suitable conditions. Moreover, Liouville type results and asymptotic behaviour of solutions have been developed by Serrin [32] and Fraas and Pinchover [14]. On exterior domains, this has been addressed by Dancer, Daners, and Hauer [9] and Bonorino, Silva, and Zingano [4].

On the comparison of solutions of PDE's and their Schwarz symmetrizations, we cite classical works from Talenti [36, 37], which, among other outcomes, estimates the spherically symmetric rearrangement of a solution to a PDE by an expression that involves the solution itself. Moreover, results that are similar to Talenti's have been made by Trombetti and Vasquez [39] and Bonorino and Montenegro [3]. It is important to mention that we use the result from Talenti mentioned above (see Section 1.2) in order to prove the limit result.

In the second part of this work, we obtain a Comparison Principle for solutions on exterior domains $\mathbb{R}^n \backslash K$, where $K \subset \mathbb{R}^n$ is a compact set, of an equation involving the general integro-differential operator

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} \mathbf{K}(x,y) \left|u(x) - u(y)\right|^{p-2} \big(u(x) - u(y)\big)\, dxdy.$$

Properties of $\mathcal{L}$, including the spaces of the definition to the function $u$ and properties of the kernel $\mathbf{K}(x,y)$ are discussed in Section 1.3. We restrict ourselves to the case $sp \geq n$. The main difference from this operator to divergent form operators studied in Bonorino, Silva, and Zingano [4] is its non-locality, which requires a different approach in important aspects of the problem. Our main theorem is the following.

**Theorem.** *Let $K$ be a compact set of $\mathbb{R}^n$ and let $u, v \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \backslash K)$ be bounded weak solutions of $\mathcal{L}w = 0$ in $\mathbb{R}^n \backslash K$. Suppose, further, that $sp \geq n$. If $v \geq u$ on $K$, then $v \geq u$ in $\mathbb{R}^n \backslash K$.*

Integro-differential operators of this kind has been extensively studied by several authors in recent years. Initially, the case $p = 2$ and $\mathbf{K}(x, y) = |x - y|^{-n-2s}$, that gives the classical fractional Laplace equation, was studied. For example, we can mention the description of the fractional Laplacian by an extension problem made by Caffarelli and Silvestre [7], the study of solutions by Servadei and Valdinoci [34], and the work by Silvestre [35], that treats the regularity of the associated obstacle problem. A friendly intro to the fractional Sobolev spaces and to the fractional Laplacian is Nezza, Palatucci and Valdinoci [26]. Finally, we also cite the book by Landkof [23] about potential theory and kernels that are used in this class of operators.

Regarding $p$-fractional Laplacian operators, many investigations have been performed in the last decade. In bounded domains, fractional supersolutions, the obstacle problem, and the Perron method were studied by Kovenpää, Kuusi, and Palatucci [19, 20, 21], and Palatucci [27] made contribuitions to the Dirichlet problem and comparison results. For the evolution $p$-fractional Laplacian equation, fundamental solutions and asymptotic behaviour were studied by Vásquez [40].

Regarding symmetrization applied to fractional equations, Park [28] proved a result that states a fractional version of Pólya-Szegö inequality, while Di Blasio and Volzone [11] established, via symmetrization methods, comparison and regularity results for the fractional Laplacian. Another example, for parabolic equations, is Vásquez and Volzone [41], where the authors adress symmetrization for linear and non-linear equations of porous medium type.

We organize the work in three chapters. In Chapter 1, the preliminary theory is developed. We define the operators that are studied in the work, we recall some results about these operators involving symmetrization and, finally, two crucial estimates are proved.

Chapter 2 is devoted to exploring results involving the quasilinear operator. In this chapter, we prove the main theorem involving the limit at infinity to solutions to this problem. Moreover, we provide an example that indicates the optimality of our result and present a few corollaries of our limit theorem.

In Chapter 3, we study the non-local problem. Initially, we obtain properties for the Gagliardo seminorm to solutions to our problem. Using these properties, the main comparison result is proved. In addition, we show a few consequences of this theorem, including a non-homogeneous version of the Comparison Principle.

We emphasize that in addition to the author and advisor of this research, the results of Chapters 2 and 3 were obtained in collaboration with Diego Marcon Farias and Filipe Jung dos Santos.

# Chapter 1

# PRELIMINARIES

In this chapter, we present the preliminary theory that is necessary to a better understanding of the next chapters. We introduce, in Section 1.1, concepts about quasilinear operators, especially the one considered in our result from Chapter 2. In Section 1.2, we recall symmetrization techniques and results usefull to the next chapter. In Section 1.3, we treat fractional Sobolev spaces and non-local operators. In the last section, we present two important estimates for our purposes.

## 1.1 Quasilinear Operator

In this section, we recall properties and results about a quasilinear divergent operator. Classical results about operators that are similar (mostly, more general) to the one that we study here can be found in Ladyzhenskaya and Ural'tseva [22], Gilbarg and Trudinger [16] and Pucci and Serrin [29].

We consider, for $p \in (1, \infty)$, the operator in divergence form

$$-\mathrm{div}\big(|\nabla u|^{p-2}A(|\nabla u|)\nabla u\big), \tag{1.1}$$

defined in the weak sense for functions $u \in W^{1,p}(\Omega)$. Here, the function $A$ is assumed to satisfy

i. $A \in C([0, +\infty))$ and $A(0) > 0$;

ii. $t \mapsto t^{p-1}A(t)$ is strictly increasing for $t > 0$;

iii. $\delta \leq A \leq \Gamma$, for positive constants $\delta, \Gamma$.

The conditions i., ii. and iii. to the function $A$ are based on Serrin [32]. The prototype case for the generalization in (1.1) is the $p$-Laplace equation $-\Delta_p$, where $A(t) \equiv 1$ so that

$$-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

**Remark 1.1.** Note that, if we consider $\eta(t) = t^{p-1}A(t)$, by conditions i., ii. and iii. above, we have that

$$\delta t^{p-1} \leq \eta(t) \leq \Gamma t^{p-1}.$$

Considering now the inverse $\eta^{-1}$, it follows that

$$\left(\frac{t}{\Gamma}\right)^{\frac{1}{p-1}} \leq \eta^{-1}(t) \leq \left(\frac{t}{\delta}\right)^{\frac{1}{p-1}}. \tag{1.2}$$

These inequalities are important for some estimates.

We recall that a function $u \in W^{1,p}_{loc}(\Omega)$ on some domain $\Omega \subseteq \mathbb{R}^n$ is a weak solution of

$$-\mathrm{div}\big(|\nabla u|^{p-2}A(|\nabla u|)\nabla u\big) \;=\; f \;\text{ in }\; \Omega \tag{1.3}$$

if $u$ satisfies

$$\int_\Omega |\nabla u|^{p-2}A(|\nabla u|)\nabla u \cdot \nabla\varphi\, dx = \int_\Omega f\varphi\, dx \tag{1.4}$$

for all $\varphi \in C_0^\infty(\Omega)$.

In the proof of the limit result for this operator, we use a Harnack inequality, which we present below, and symmetrization results that are presented in Section 1.2.

The following statement is derived from Theorem 3.14 of Malý and Ziemer [25] and an earlier version is stated in [31, Theorem 5].

**Theorem 1.2.** *Let $u$ be a non-negative weak solution of* (1.3) *in an open ball $B_R$. Assume that $p \leq n$ and that there exists $\theta \in (0,1)$ such that $f \in L^{\frac{n}{p-\theta}}(B_R)$. Then, for any $\sigma \in (0,1)$,*

$$\sup_{B_{\sigma R}} u \leq C\left(\inf_{B_{\sigma R}} u + K(R)\right), \tag{1.5}$$

*where*

$$K(R) = \left(R^\theta \,\|\, f \,\|_{L^{\frac{n}{p-\theta}}(B_R)}\right)^{\frac{1}{p-1}} \tag{1.6}$$

*and $C$ depends on $n, p, \sigma, \theta, \delta, \Gamma$.*

The result above can be extended with no difficulty to arbitrary compact subsets. We can extract the corollary below which gives the Harnack inequality for solutions on exterior domains over the spheres $S_R$, for all $R$ large, with $C$ independent of $R$.

**Corollary 1.3.** *Let $u$ be a non-negative weak solution of* (1.3) *on $\mathbb{R}^n \backslash B_1$. Assume that $p \leq n$ and that there is some $\theta \in (0,1)$ such that $f \in L^{\frac{n}{p-\theta}}$. Then, for all $R$ sufficiently large,*

$$\sup_{S_R} u \leq C\left(\inf_{S_R} u + K(R)\right), \tag{1.7}$$

*where*

$$K(R) \;=\; \left(\big(R/4\big)^\theta \,\|\, f \,\|_{L^{\frac{n}{p-\theta}}(\mathbb{R}^n \backslash B_{R/4}(0))}\right)^{\frac{1}{p-1}} \tag{1.8}$$

*and $C$ depends on $n, p, \theta, \delta, \Gamma$.*

*Proof.* We can cover $S_R$ with a quantity $N$ of balls $B_i = B_{R/2}(x_i)$ with centers $x_i$ lying on $S_R$, with $N$ not depending on $R$. Ordering these balls so that $B_i \cap B_{i+1} \neq \varnothing$, we have

$$\inf_{B_i} u \leq \sup_{B_{i+1}} u. \tag{1.9}$$

Now for each $i$, we apply Theorem 1.2 on the ball $B_{3R/4}(x_i) \subset \mathbb{R}^n \backslash B_1$, with $\sigma = 2/3$. We obtain

$$\sup_{B_i} u \leq C \left( \inf_{B_i} u + K(R) \right) \tag{1.10}$$

with

$$\begin{aligned} K(R) &= \left( \left(3R/4\right)^\theta \| f \|_{L^{\frac{n}{p-\theta}}(B_{3R/4}(x_i))} \right)^{\frac{1}{p-1}} \\ &\leq 3^{\frac{1}{p-1}} \left( \left(R/4\right)^\theta \| f \|_{L^{\frac{n}{p-\theta}}(\mathbb{R}^n \backslash B_{R/4}(0))} \right)^{\frac{1}{p-1}}. \end{aligned}$$

Then combining inequalities (1.9) and (1.10) yields, for all $i, j \in \{1, \ldots, N\}$,

$$\sup_{B_i} u \leq C \left( \inf_{B_j} u + K(R) \right)$$

after a proper redefinition of $C$ depending only on $N$. This leads to (1.7), as it is clear we can choose $K$ above as in (1.8), redefining $C$ if necessary. $\square$

## 1.2 Symmetrization

In this section we recall important definitions and useful results about symmetrization. For an exhaustive treatment about this topics we refer to Hardy, Littlewood and Pólya [17], Talenti [36], Alvino, Lions and Trombetti [2] and Brothers and Ziemer [5].

First, if $\Omega$ is an open bounded set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ is a measurable function, the distribution function of $u$ is given by

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad \text{for } t \geq 0.$$

The decreasing rearrangement of $u$, also called the *generalized inverse* of $\mu_u$, is defined by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) \geq s\}.$$

If $\Omega^\sharp$ is the open ball in $\mathbb{R}^n$, centered at 0, with the same measure as $\Omega$ and $\omega_n$ is the measure of the unit ball in $\mathbb{R}^n$, the function

$$u^\sharp(x) = u^*(\omega_n |x|^n), \quad \text{for} \quad x \in \Omega^\sharp$$

is the spherically symmetric decreasing rearrangement of $u$. It is also called the Schwarz symmetrization of $u$. The next remark reviews important properties of rearrangements and will be used through this work.

**Remark 1.4.** Let $v, w$ be integrable functions in $\Omega$ and let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing non-negative function. Then

$$\int_{\Omega} g(|v(x)|)\, dx = \int_0^{|\Omega|} g(v^*(s))\, ds = \int_{\Omega^\sharp} g(v^\sharp(x))\, dx.$$

Hence, if $\mu_v(t) \geq \mu_w(t)$ for all $t > t_1 > 0$, it follows that

$$\int_{t_1 < v} g(v(x))\, dx = \int_0^{\mu_u(t_1)} g(v^*(s))\, ds \geq \int_0^{\mu_w(t_1)} g(w^*(s))\, ds = \int_{t_1 < w} g(w(x))\, dx,$$

since $v^*(s) \geq w^*(s)$ for $s \leq \mu_w(t_1)$. Finally, the Pólya-Szegö principle, as you can see, for instance, in Brothers and Ziemer [5], states that

$$\int_{\Omega} |Dv(x)|^2\, dx \;\geq\; \int_{\Omega^\sharp} |Dv^\sharp(x)|^2\, dx, \quad \text{for } v \in H_0^1(\Omega).$$

This inequality also holds if we replace $\Omega$ and $\Omega^\sharp$ by $\{t_1 < v < t_2\}$ and $\{t_1 < v^\sharp < t_2\}$, respectively.

The next Theorem is one of several results that compare solutions of PDE's in general domains with their Schwarz symmetrizations, obtaining estimates that are similar to Pólya-Szegö principle, for example. As references for works in this line of research, we can cite Talenti [36], Trombetti and Vasquez [39], Kesavan [18], Bonorino and Montenegro [3], and Talenti [37], from which we can enunciate, in our context

**Theorem 1.5.** *Consider a solution $u$ of* (1.3) *on a bounded domain $\Omega$, where $A$ satisfies $i - iii$, and assume $f \in L^1$. Then, $u^\sharp$, the spherically symmetric rearrangement of $u$, satisfies*

$$u^\sharp(x) \;\leq\; \sup_{\partial\Omega} |u| \,+\, \int_{\omega_n |x|^n}^{|\Omega|} \eta^{-1}\left( \frac{r^{-1+1/n}}{n\, \omega_n^{1/n}} \int_0^r f^*(s)\, ds \right) \frac{r^{-1+1/n}}{n\, \omega_n^{1/n}}\, dr, \tag{1.11}$$

*where $\eta(t) = t^{p-1} A(t)$.*

*Proof.* [37, Theorem 1]. □

From the Theorem above, we can derive the following statement.

**Corollary 1.6.** *Under the same hypotheses from Theorem 1.5, $u^\sharp$ satisfies*

$$\sup_{\Omega^\sharp} u^\sharp \leq\; \sup_{\partial\Omega} |u| \,+ \left(\frac{1}{n\, \omega_n\, \delta}\right)^{\frac{1}{p-1}} \int_0^{\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}} \rho^{-\frac{n-1}{p-1}} \left( \int_{B_\rho} f^\sharp(x)\, dx \right)^{\frac{1}{p-1}} d\rho. \tag{1.12}$$

*Proof.* Making the changes of variables

$$s = \omega_n\, t^n\,, \;\; ds = n\, \omega_n\, t^{n-1} dt$$

in (1.11), it follows

$$u^\sharp(x) \leq \sup_{\partial\Omega} |u| + \int_{\omega_n |x|^n}^{|\Omega|} \eta^{-1}\left( \frac{r^{-1+\frac{1}{n}}}{\omega_n^{-1+\frac{1}{n}}} \int_0^{\left(\frac{r}{\omega_n}\right)^{\frac{1}{n}}} f^*(\omega_n\, t^n)\, t^{n-1}\, dt \right) \frac{r^{-1+\frac{1}{n}}}{n\omega_n^{\frac{1}{n}}}\, dr.$$

If we consider now the change of variables

$$r = \omega_n\, \rho^n\,, \quad dr = n\, \omega_n\, \rho^{n-1} d\rho,$$

with

$$r^{-1+1/n} = \omega_n^{-1+1/n}\, \rho^{-n+1},$$

it follows

$$u^\sharp(x) \leq \sup_{\partial\Omega} |u| + \int_{|x|}^{\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}} \eta^{-1}\left( \rho^{-n+1} \int_0^{\rho} f^*(\omega_n\, t^n)\, t^{n-1}\, dt \right) d\rho.$$

Note that, from Remark 1.4, we get

$$\int_0^{\rho} f^*(\omega_n\, t^n)\, t^{n-1}\, dt = \frac{1}{n\omega_n} \int_{B_\rho} f^\sharp(x)\, dx,$$

and so,

$$u^\sharp(x) \leq \sup_{\partial\Omega} |u| + \int_{|x|}^{\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}} \eta^{-1}\left( \frac{\rho^{-n+1}}{n\omega_n} \int_{B_\rho} f^\sharp(x)\, dx \right) d\rho.$$

Using the lower estimate in (1.2), we get

$$u^\sharp(x) \leq \sup_{\partial\Omega} |u| + \left(\frac{1}{n\, \omega_n\, \delta}\right)^{\frac{1}{p-1}} \int_{|x|}^{\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}} \rho^{-\frac{n-1}{p-1}} \left( \int_{B_\rho} f^\sharp(x)\, dx \right)^{\frac{1}{p-1}} d\rho,$$

from which we obtain (1.12). □

## 1.3 Fractional Sobolev Spaces and Non-local Operators

In this section we define the fractional Sobolev spaces and the non-local operators that are important concepts for this work. For references about this topics, we can cite Demengel, Demengel and Erné [10], Bucur and Valdinoci [6], Palatucci [27] and the classical Nezza, Palatucci and Valdinoci [26].

Let $\Omega \subset \mathbb{R}^n$ be an open set. Define, for each $s \in (0,1)$ and $p \in (1,\infty)$, the usual fractional Sobolev space

$$W^{s,p}(\Omega) = \left\{ w \in L^p(\Omega)\ ;\ \frac{w(x) - w(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}.$$

We emphasize that such space is an intermediate set between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, where can be considered the norm

$$\|w\|_{W^{s,p}(\Omega)} = \left(\|w\|^p_{L^p(\Omega)} + [w]^p_{W^{s,p}(\Omega)}\right)^{\frac{1}{p}},$$

where the term

$$[w]_{W^{s,p}(\Omega)} := \left(\int_\Omega\int_\Omega \frac{|w(x)-w(y)|^p}{|x-y|^{n+ps}}\, dxdy\right)^{\frac{1}{p}},$$

is called *Gagliardo seminorm* from $w$.

We can mention that such normed vector space is a Banach space, as you can see in [10, Proposition 4.24].

Like the classical case, where $s$ is an integer, every function in the Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approached by a sequence of infinitely differentiable functions. The next result is stated in [1, Theorem 7.38].

**Proposition 1.7.** *For every $s \in (0,1)$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$.*

Consider $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ functions with the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. It follows from Proposition 1.7 that $W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. But in general, for $\Omega \subset \mathbb{R}^n$, the set $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$, that is, $W_0^{s,p}(\Omega) \neq W^{s,p}(\Omega)$.

Similarly to the usual Sobolev spaces, we have that the fractional space $W^{s,p}(\Omega)$ is of local type, that is

**Proposition 1.8.** *For every $u \in W^{s,p}(\Omega)$ and for every $\varphi \in C_0^\infty(\Omega)$, the product $\varphi u$ belongs to $W^{s,p}(\Omega)$.*

*Proof.* [10, Proposition 4.26]. □

Also, $W^{s,p}_{loc}(\Omega)$ is defined as the subspace of those $u \in L^p_{loc}(\Omega)$ such that

$$[\,u\,]_{W^{s,p}(\Gamma)} := \left(\int_\Gamma\int_\Gamma \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy\right)^{\frac{1}{p}} < \infty,$$

for every $\Gamma \subset\subset \Omega$.

The tail space, denoted by $L^{p-1}_{sp}(\mathbb{R}^n)$, is defined as the space of the functions $u \in L^{p-1}_{loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n\setminus B_1(0)} |u(z)|^{p-1}\, |z|^{-n-sp}\, dz < \infty.$$

We are interested in the general operator $\mathcal{L}$ acting on functions $u \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ described by

$$\mathcal{L}u(x) \,=\, \int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,\big|u(x)-u(y)\big|^{p-2}\big(u(x)-u(y)\big)\,dxdy, \qquad x \in \Omega. \tag{1.13}$$

The assumptions we require on the kernel $\mathbf{K}(x,y)$ are

$$\begin{aligned} &1.\ \mathbf{K}(x,y) \,\geq\, 0 \\ &2.\ \mathbf{K}(x,y) \,=\, \mathbf{K}(y,x) \\ &3.\ \lambda \,\leq\, \mathbf{K}(x,y)\,|x-y|^{n+sp} \,\leq\, \Lambda\,, \quad \text{for constants} \quad \Lambda \,\geq\, \lambda \,>\, 0. \end{aligned} \tag{1.14}$$

If we consider, in (1.13), the particular kernel $\mathbf{K}(x,y) = |x-y|^{-n-sp}$, we have, up to a constant, the very studied $p$-fractional Laplace operator, denoted by $(-\Delta)^s_p$. Moreover, we can mention the very important case where $p = 2$, when the operator $(-\Delta)^s_p$ reduces to the fractional Laplacian, which is denoted by $(-\Delta)^s$.

A typical property of these operators is the non-locality, in the sense that the value $\mathcal{L}u(x)$, in each point $x \in \Omega$, does not depend only on the values of $u$ in the neighborhood of $x$, but the whole $\mathbb{R}^n$. In this sense, and due to the random nature of the process, it is natural to express the Dirichlet condition in $\mathbb{R}^n \setminus \Omega$ rather than $\partial\Omega$.

We remind that the Laplacian operator, $-\Delta$, is linear, as the fractional Laplacian $(-\Delta)^s$. On the other hand, in the general case $p \neq 2$, operators $p-$Laplacian $-\Delta_p$, $p-$fractional Laplacian $(-\Delta)^s_p$ and $\mathcal{L}$ are not linear. However, we highlight that $\mathcal{L}$ is an $(p-1)$-homogeneous operator, that is, for every $w$ and $\alpha > 0$, we have $\mathcal{L}(\alpha w) = \alpha^{p-1}\mathcal{L}w$.

A function $u \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a weak (sub) supersolution to

$$\mathcal{L}u(x) = 0, \qquad x \in \Omega, \tag{1.15}$$

if

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))\,dxdy\,(\leq)\,\geq 0,$$

for all test functions $\varphi \in C^\infty_0(\Omega)$ with $\varphi \geq 0$. Also, if $v \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$, we can say that $\mathcal{L}u \leq \mathcal{L}v$ in $\Omega$ in the weak sense, if

$$\begin{aligned} &\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))\,dxdy \\ \leq &\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,|v(x)-v(y)|^{p-2}\,(v(x)-v(y))\,(\varphi(x)-\varphi(y))\,dxdy, \end{aligned}$$

for all test functions $\varphi \in C^\infty_0(\Omega)$ with $\varphi \geq 0$.

By definition, $u$ is a weak solution to (1.15) if it is both a sub and supersolution to (1.15), in case that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{K}(x,y)\,|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))\,dxdy = 0, \tag{1.16}$$

for all $\varphi \in C_0^\infty(\Omega)$. Moreover, a function $u$ satisfies $\mathcal{L}u(x) = f$ in $\Omega$ in the weak sense if

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{K}(x,y)\,|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)\,dxdy = \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx \tag{1.17}$$

holds for any $\varphi \in C_0^\infty(\Omega)$. Observe that in these definitions of weak solutions, we can consider $\varphi \in W_0^{s,p}(\Omega)$ instead of $\varphi \in C_0^\infty(\Omega)$, since we can use density arguments and that $W_0^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)}$.

Finally, for $\mathbf{K}(x,y) = |x-y|^{-n-sp}$ we can prove the following result concerning the invariance under rotations to the $p$-fractional Laplacian $(-\Delta)_p^s$.

**Theorem 1.9.** *Consider $u \in W^{s,p}(\Omega)$ a weak solution of $(-\Delta)_p^s u = f$ in $\Omega$ and any rotation $R \in SO(n)$. Then, we have that $\tilde{u} := u \circ R$ is a weak solution of $(-\Delta)_p^s \tilde{u} = \tilde{f}$ in $\tilde{\Omega} := R^{-1}(\Omega)$, where $\tilde{f} := f \circ R$.*
*In particular, if $(-\Delta)_p^s u = 0$ in $\mathbb{R}^n$, then also $(-\Delta)_p^s \tilde{u} = 0$ in $\mathbb{R}^n$.*

*Proof.* In fact, let $\varphi \in W_0^{s,p}(\tilde{\Omega})$ a test function. Noticing that $det(JR) = 1$, $R^{-1} \in SO(n)$ is also a rotation and so $|R^{-1}x| = |x|$, for all $x \in \mathbb{R}^n$, we have, by changing variables $w = Rx$, $dw = dx$, $z = Ry$, $dz = dy$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|u(Rx)-u(Ry)\big|^{p-2}\big(u(Rx)-u(Ry)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+sp}}\,dxdy \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|u(w)-u(z)\big|^{p-2}\big(u(w)-u(z)\big)\big(\varphi(R^{-1}w)-\varphi(R^{-1}z)\big)}{|R^{-1}w-R^{-1}z|^{n+sp}}\,dwdz \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|u(w)-u(z)\big|^{p-2}\big(u(w)-u(z)\big)\big(\varphi(R^{-1}w)-\varphi(R^{-1}z)\big)}{|w-z|^{n+sp}}\,dwdz \\
=&\int_{\mathbb{R}^n} f(x)\,\varphi(R^{-1}x)\,dx,
\end{aligned}$$

where the last equality is valid since $\varphi \circ R^{-1} \in W_0^{1,p}(\Omega)$ is a test function in $\Omega$. Changing variables again in the last expression, making $w = R^{-1}x$, $dw = dx$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|u(Rx)-u(Ry)\big|^{p-2}\big(u(Rx)-u(Ry)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+sp}}\,dxdy \\
=&\int_{\mathbb{R}^n} f(Rw)\,\varphi(w)\,dw,
\end{aligned}$$

as we wanted. $\square$

## 1.4 Auxiliary Estimates

In this section, we obtain estimates that are usefull to the proof of the results in Chapter 3. The first Lemma shows that we can control the Gagliardo seminorm from a family of functions with interesting properties.

**Lemma 1.10.** *Consider the open sets $V \subset\subset U \subset\subset B_{R_0}(0)$ and, for every $R \geq R_0 \geq 1$, $\psi = \psi_R \in C^1(\mathbb{R}^n)$ such that*

$$\begin{cases} \psi = 0 & \text{in } V \\ \psi = 1 & \text{in } B_R(0) \backslash U \\ \psi = 0 & \text{in } \mathbb{R}^n \backslash B_{2R}(0) \\ |\nabla \psi| \leq m, & \text{in } U \backslash V, \text{ where } m \text{ is a constant depending only on } U \text{ and } V \\ |\nabla \psi| \leq 2/R, & \text{in } B_{2R}(0) \backslash B_R(0). \end{cases}$$

*Then, if $sp \geq n$, the functions $\psi_R$ satisfy*

$$\sup_R [\, \psi_R \,]_{W^{s,p}(\mathbb{R}^n)} \; < \; \infty.$$

*Proof.* Notice we may write $\psi_R = \psi^1 + \psi_R^2$, where $\psi^1$, $\psi_R^2 \in C^1(\mathbb{R}^n)$,

$$|\psi^1| \leq 1\,, \quad supp\, \psi^1 \subseteq B_{R_0}(0)\,, \quad |\nabla \psi^1| \leq m;$$

$$|\psi_R^2| \leq 1\,, \quad supp\, \psi_R^2 \subseteq B_{2R}(0)\,, \quad |\nabla \psi_R^2| \leq \frac{2}{R}.$$

It follows then by the triangle inequality for the seminorm that

$$[\, \psi_R \,]_{W^{s,p}(\mathbb{R}^n)} \leq [\, \psi^1 \,]_{W^{s,p}(\mathbb{R}^n)} + [\, \psi_R^2 \,]_{W^{s,p}(\mathbb{R}^n)},$$

so, as we know $[\, \psi^1 \,]_{W^{s,p}} < \infty$ and does not depend on $R$, we need only to estimate $[\, \psi_R^2 \,]_{W^{s,p}}$. Observe that, although it is not really the case, each $\psi_R^2$ is essentially obtained by a change of scale in the function $\psi_1^2$, namely, $\psi_R^2(x) = \psi_1^2(\frac{x}{R})$, and so, $\psi_R^2(Rw) = \psi_1^2(w)$, considering $x = Rw$. Therefore,

$$\begin{aligned} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\psi_R^2(x) - \psi_R^2(y)|^p}{|x-y|^{n+sp}}\, dxdy \;&=\; \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\psi_R^2(Rw) - \psi_R^2(Rz)|^p}{R^{n+sp}\, |w-z|^{n+sp}}\, R^{2n}\, dxdy \\ &=\; R^{n-sp} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\psi_1^2(w) - \psi_1^2(z)|^p}{|w-z|^{n+sp}}\, dxdy \\ &\leq\; R_0^{n-sp}\, [\psi_1^2]^p_{W^{s,p}(\mathbb{R}^n)}, \end{aligned}$$

as we wanted. □

In the next Lemma, we consider the function $L : \mathbb{R} \to \mathbb{R}$ given by $L(x) = x|x|^{p-2}$, and we get some inequalities that will be helpfull to the proof of the Comparison Principle. This result uses ideas from Damascelli [8].

**Lemma 1.11.** *For $p \geq 2$, there is a positive constant $c_1$, depending on $p$, such that, for all $s \leq t \in \mathbb{R}$,*

$$L(t) - L(s) \geq c_1 \big(|t| + |s|\big)^{p-2} (t - s).$$

*In particular, for all $s \leq t \in \mathbb{R}$,*

$$L(t) - L(s) \geq c_1 |t - s|^{p-2} (t - s), \tag{1.18}$$

*and for all $s,\, t \in \mathbb{R}$,*

$$\big(L(t) - L(s)\big)(t - s) \geq c_1 |t - s|^{p}. \tag{1.19}$$

*Proof.* The proof comes by using that $L'(t) = (p-1)\,|t|^{p-2}$ and writing

$$\begin{aligned} L(t) - L(s) &= \int_0^1 L'\big(s + \tau(t-s)\big)\,(t-s)\,d\tau \\ &= (p-1)\Big(\int_0^1 |s + \tau(t-s)|^{p-2}\,d\tau\Big)\,(t-s). \end{aligned}$$

The remaining integral is estimated at [8, Lemma 2.1], for $p > 2$, as

$$\int_0^1 |s + \tau(t-s)|^{p-2}\,d\tau \geq c\,(|t| + |s|)^{p-2},$$

for all $s, t$, by some positive constant $c$ depending on $p$. $\square$

# Chapter 2

# ASYMPTOTIC BEHAVIOUR

In this section, we obtain some limit results for the solutions in exterior domains to the equation (1.3) in the case $p < n$. The proof follows ideas from Gilbarg and Serrin [15] using the Harnack inequality, but in our case, they were combined with symmetrization results, since our problem is non-homogeneous.

On the matter of the limit at infinity for solutions, there is no loss of generality in assuming $K = B_1$. Our main theorem then reads as follows.

**Theorem 2.1.** *Consider $p < n$ and let $u \in C^1(\mathbb{R}^n \setminus B_1)$ be a bounded weak solution of*

$$-\mathrm{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u\big) \;=\; f$$

*in $\mathbb{R}^n \setminus B_1$. If*

$$f \in L^r(\mathbb{R}^n \backslash B_1)\,, \quad \textit{for some} \;\; r \;<\; n/p \tag{2.1}$$

*and*

$$f \in L^{\frac{n}{p-\theta}}(\mathbb{R}^n \backslash B_1)\,, \quad \textit{for some} \;\; \theta \in (0,1) \tag{2.2}$$

*with*

$$\lim_{R\to+\infty} \;\; R^\theta \, \| \, f \, \|_{L^{\frac{n}{p-\theta}}(\,\mathbb{R}^n \backslash B_R(0)\,)} \;=\; 0, \tag{2.3}$$

*then the limit $\lim\limits_{|x|\to\infty} u(x)$ exists.*

*Proof.* Let $u$ be a bounded weak solution of

$$-\mathrm{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u\big) \;=\; f$$

on $\mathbb{R}^n \backslash B_1$ and set $m \;=\; \liminf\limits_{|x|\to\infty} u$. For a given $\varepsilon > 0$ there is some $R_0 > 0$ such that

$$u(x) \;>\; m - \varepsilon \;\text{ for all } x \text{ such that } |x| \geq R_0\,,$$

so that the function

$$v \;=\; u - m + \varepsilon$$

is a positive solution on $\mathbb{R}^n \backslash B_{R_0}$. We pick up a sequence $(x_k)$ with $|x_k| \to \infty$ and $R_0 < |x_k| < |x_{k+1}|$ such that

$$u(x_k) \,\leq\, m + \epsilon,$$

hence

$$v(x_k) \,\leq\, 2\epsilon. \tag{2.4}$$

Now let $R_k = |x_k|$, $S_{R_k} = \partial B_{R_k}(0)$. By applying Corollary 1.3 to $v$ we get

$$\sup_{S_{R_k}} v \,\leq\, C\left(\,\inf_{S_{R_k}} v + K(R_k)\right)$$

for a positive constant $C$ independent of $k$, with

$$K(R_k) = \left(\left(R_k/4\right)^{\theta} \| \, f \, \|_{\frac{n}{p-\theta}, \mathbb{R}^n \backslash B_{R_k/4}(0)}\right)^{\frac{1}{p-1}}.$$

By hypothesis (2.3), $K(R) \to 0$ as $R \to \infty$, so that $K(R_k) \leq \epsilon$, for all $k$ sufficiently large. Hence, using (2.4), it follows

$$\sup_{S_{R_k}} v \,\leq\, C\,\epsilon\,, \;\; \text{for all } k \text{ sufficiently large},$$

and, consequently,

$$\sup_{\partial A(R_k, R_{k+1})} v \quad \leq\, C\,\epsilon\,, \;\; \text{for all } k \text{ sufficiently large}. \tag{2.5}$$

In the sequence, we obtain a bound for $v$ on the interior of the annuli $A(R_k, R_{k+1})$. For that we apply Corollary 1.6 to $v$ on $\Omega = A(R_k, R_{k+1})$, with

$$f_k \,=\, f\,\big|_{A(R_k, R_{k+1})}.$$

As $|\Omega| = \omega_n\,(\,R_{k+1}^n - R_k^n\,) \leq \omega_n\, R_{k+1}^n$ we obtain

$$\sup_{A(R_k,R_{k+1})^\sharp} v^\sharp \;\leq\; \sup_{\partial A(R_k,R_{k+1})} v \;\; + \left(\frac{1}{n\,\omega_n\,\delta}\right)^{\frac{1}{p-1}} \int_0^{R_{k+1}} \rho^{-\frac{n-1}{p-1}} \left(\int_{B_\rho} f_k^\sharp(x)\,dx\right)^{\frac{1}{p-1}} d\rho, \tag{2.6}$$

where $A(R_k, R_{k+1})^\sharp$ is the ball centered at 0 with the same measure as $A(R_k, R_{k+1})$. Let us split the integral above as

$$\begin{aligned}
&\int_0^{R_{k+1}} \rho^{-\frac{n-1}{p-1}} \left(\int_{B_\rho} f_k^\sharp(x)\,dx\right)^{\frac{1}{p-1}} d\rho \\
=\;\; &\int_0^{1} \rho^{-\frac{n-1}{p-1}} \left(\int_{B_\rho} f_k^\sharp(x)\,dx\right)^{\frac{1}{p-1}} d\rho \;+ \int_1^{R_{k+1}} \rho^{-\frac{n-1}{p-1}} \left(\int_{B_\rho} f_k^\sharp(x)\,dx\right)^{\frac{1}{p-1}} d\rho.
\end{aligned} \tag{2.7}$$

By the Hölder inequality we have, for any $q \geq 1$,

$$\int_{B_\rho} f_k^\sharp(x)\,dx \,\leq\, \left(\int_{B_\rho} (f_k^\sharp)^q(x)\,dx\right)^{1/q} \left(\omega_n\,\rho^n\,\right)^{\frac{1}{q'}},$$

and using that

$$\left(\int_{B_\rho} \left(f_k^\sharp\right)^q(x)\,dx\right)^{1/q} \;\le\; \|f\|_{q\,,\,A(R_k,R_{k+1})},$$

we get

$$\int_{B_\rho} f_k^\sharp(x)\,dx \;\le\; \omega_n^{\frac{1}{q'}}\,\rho^{\frac{n}{q'}}\,\|f\|_{q\,,\,A(R_k,R_{k+1})}. \tag{2.8}$$

Using this with $q$ chosen as $s = \frac{n}{p-\theta} > \frac{n}{p}$ we have for the first integral in (2.7)

$$\begin{aligned}
\int_0^1 \rho^{-\frac{n-1}{p-1}}\left(\int_{B_\rho} f_k^\sharp(x)\,dx\right)^{\frac{1}{p-1}} d\rho \;&\le\; \omega_n^{\frac{1}{s'(p-1)}}\int_0^1 \rho^{-\frac{n-s}{s(p-1)}}\,d\rho\,\|f\|^{\frac{1}{p-1}}_{s\,,\,A(R_k,R_{k+1})}\\
&= \omega_n^{\frac{1}{s'(p-1)}}\;\frac{s(p-1)}{sp-n}\;\|f\|^{\frac{1}{p-1}}_{s\,,\,A(R_k,R_{k+1})}.
\end{aligned}$$

For the second integral in (2.7), we use (2.8) with $q = r < n/p$ to get

$$\begin{aligned}
\int_1^{R_{k+1}} \rho^{-\frac{n-1}{p-1}}\left(\int_{B_\rho} f_k^\sharp(x)\,dx\right)^{\frac{1}{p-1}} d\rho \;&\le\; \omega_n^{\frac{1}{r'(p-1)}}\int_1^{R_{k+1}} \rho^{-\frac{n-r}{r(p-1)}}\,d\rho\,\|f\|^{\frac{1}{p-1}}_{r\,,\,A(R_k,R_{k+1})}\\
&\le\; \omega_n^{\frac{1}{r'(p-1)}}\left(\frac{r(p-1)}{rp-n}\rho^{\frac{rp-n}{r(p-1)}}\right)\Bigg|_1^{R_{k+1}}\;\|f\|^{\frac{1}{p-1}}_{r\,,\,A(R_k,R_{k+1})}\\
&\le\; \omega_n^{\frac{1}{r'(p-1)}}\;\frac{r(p-1)}{n-pr}\;\|f\|^{\frac{1}{p-1}}_{r\,,\,A(R_k,R_{k+1})}.
\end{aligned}$$

Putting these estimates together in (2.7), we obtain as a result from (2.6) that

$$\sup_{A(R_k,R_{k+1})^\sharp} v^\sharp \;\le\; \sup_{\partial A(R_k,R_{k+1})} v \;+\; C\left(\|f\|^{\frac{1}{p-1}}_{r\,,\,A(R_k,R_{k+1})} + \|f\|^{\frac{1}{p-1}}_{s\,,\,A(R_k,R_{k+1})}\right), \tag{2.9}$$

for some constant $C$ depending only on $n, p, \delta, \Gamma, r, s$. Now by the hypotheses (2.1), (2.2), we have

$$\|f\|^{\frac{1}{p-1}}_{r\,,\,A(R_k,R_{k+1})} \;+\; \|f\|^{\frac{1}{p-1}}_{s\,,\,A(R_k,R_{k+1})} \;\le\; \epsilon\,,\;\; \text{for all } k \text{ sufficiently large.}$$

Then using (2.5), (2.9) yields

$$\sup_{A(R_k,R_{k+1})^\sharp} v^\sharp \;\le\; C\,\epsilon\,,\;\; \text{for all } k \text{ sufficiently large,}$$

with a constant $C$ depending only on $n, p, \delta, \Gamma, r, s$. Now, since

$$\sup_{A(R_k,R_{k+1})} v \;=\; \sup_{A(R_k,R_{k+1})^\sharp} v^\sharp\,,$$

it follows that

$$\sup_{A(R_k,R_{k+1})} v \;\le\; C\,\epsilon\,,\quad \text{for all } k \text{ sufficiently large,}$$

which amounts to say that

$$v(x) \;\le\; C\,\epsilon\,,\;\; \text{for all } |x| \text{ sufficiently large.}$$

Therefore, by definition of $v$, we have

$$u(x) - m \le C\,\epsilon\,, \text{ for all } |x| \text{ sufficiently large,}$$

and by arbitrariness of $\epsilon$ follows

$$\limsup_{|x|\to\infty} u \le m,$$

proving that $\lim_{|x|\to\infty} u(x) = m\,.$ □

If we consider, in particular, a pointwise limitation for the function $f$, we get the next corollary.

**Corollary 2.2.** *Consider $p < n$ and let $u \in C^1(\mathbb{R}^n \setminus B_1)$ be a bounded weak solution of*

$$-\mathrm{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u\big) = f$$

*in $\mathbb{R}^n \setminus B_1$. If $f$ satisfies*

$$|f(x)| \le \frac{C}{|x|^{p+\varepsilon}}, \tag{2.10}$$

*for $\varepsilon > 0$ and $x \in \mathbb{R}^n \setminus B_1$, then the limit $\lim_{|x|\to\infty} u(x)$ exists.*

*Proof.* If $f$ satisfies (2.10), then the conditions (2.1), (2.2), and (2.3) from Theorem 2.1 are valid, as long as we choose $r > \max\left\{1, \frac{n-\varepsilon}{p}\right\}$ for condition (2.1). □

**Remark 2.3.** This result is the best possible with respect to the conditions (2.1), (2.2), and (2.3), in the sense that considering the counterexample given by the function

$$u(x) = \cos(\log\log|x|), \text{ for } |x| > 1,$$

clearly $u$ does not attain a limit at infinity and satisfies

$$\Delta_p u(x) = f$$

with $f$ such that

$$|f(x)| \le C\,(\log|x|)^{-p+1}\,|x|^{-p}, \text{ for all } |x| \ge 2$$

for some positive constant $C$. In case $\frac{p}{p-1} < n$, $f$ satisfies conditions (2.2) and (2.3), $f \in L^{\frac{n}{p}}(\mathbb{R}^n \backslash B_2(0))$ but fails to satisfy (2.1), for any $r < n/p$.

The next corollary is a consequence from Theorem 2.1 that states a kind of Maximum Principle to solutions of (1.3).

**Corollary 2.4.** *Consider $p < n$ and $u \in C^1(\mathbb{R}^n \setminus B_1)$ a bounded weak solution of*

$$-\mathrm{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u\big) = f$$

*in* $\mathbb{R}^n \setminus B_1$. *If* $f$ *satisfies the assumptions* (2.1), (2.2), *and* (2.3), *then*

$$\sup_{\mathbb{R}^n\setminus B_{R_k}(0)} u \;\leq\; \max\{\sup_{\partial B_{R_k}(0)} u\,, m\} \;+\; C\left(\|f\|^{\frac{1}{p-1}}_{r\,,\mathbb{R}^n\setminus B_{R_k}(0)} \;+\; \|f\|^{\frac{1}{p-1}}_{\frac{n}{p-\theta}\,,\mathbb{R}^n\setminus B_{R_k}(0)}\right),$$

*and*

$$\inf_{\mathbb{R}^n\setminus B_{R_k}(0)} u \;\geq\; \min\{\inf_{\partial B_{R_k}(0)} u\,, m\} \;-\; C\left(\|f\|^{\frac{1}{p-1}}_{r\,,\mathbb{R}^n\setminus B_{R_k}(0)} \;+\; \|f\|^{\frac{1}{p-1}}_{\frac{n}{p-\theta}\,,\mathbb{R}^n\setminus B_{R_k}(0)}\right),$$

*where* $m = \lim_{|x|\to\infty} u(x)$ *and* $C = C(n, p, \delta, \Gamma, r, \theta)$. *In particular, if* $\inf_{\partial B_{R_k}(0)} u \leq m \leq \sup_{\partial B_{R_k}(0)} u$, *we have*

$$\operatorname*{osc}_{\mathbb{R}^n\setminus B_{R_k}(0)} u \;\leq\; \operatorname*{osc}_{\partial B_{R_k}(0)} u \;+\; C\left(\|f\|^{\frac{1}{p-1}}_{r\,,\mathbb{R}^n\setminus B_{R_k}(0)} \;+\; \|f\|^{\frac{1}{p-1}}_{\frac{n}{p-\theta}\,,\mathbb{R}^n\setminus B_{R_k}(0)}\right).$$

*Proof.* Consedering the equation (2.9), using that $v = u - m + \epsilon$, and

$$\sup_{A(R_k,R_{k+1})} v \;=\; \sup_{A(R_k,R_{k+1})^\sharp} v^\sharp\;,$$

we obtain

$$\sup_{A(R_k,R_{k+1})} u \;\leq\; \sup_{\partial A(R_k,R_{k+1})} u \;+\; C\left(\|f\|^{\frac{1}{p-1}}_{r\,,A(R_k,R_{k+1})} \;+\; \|f\|^{\frac{1}{p-1}}_{\frac{n}{p-\theta}\,,A(R_k,R_{k+1})}\right). \tag{2.11}$$

Now, observe that

$$\sup_{\partial A(R_k,R_{k+1})} u \;= \max\{\sup_{\partial B_{R_k}(0)} u,\; \sup_{\partial B_{R_{k+1}}(0)} u\;\}.$$

Then, making $R_{k+1} \to \infty$ in (2.11), we have

$$\sup_{\mathbb{R}^n\setminus B_{R_k}(0)} u \;\leq\; \max\{\sup_{\partial B_{R_k}(0)} u\,, m\} \;+\; C\left(\|f\|^{\frac{1}{p-1}}_{r\,,\mathbb{R}^n\setminus B_{R_k}(0)} \;+\; \|f\|^{\frac{1}{p-1}}_{\frac{n}{p-\theta}\,,\mathbb{R}^n\setminus B_{R_k}(0)}\right).$$

The result for the infimun follows considering the solution $-u$.

If $\inf_{\partial B_{R_k}(0)} u \leq m \leq \sup_{\partial B_{R_k}(0)} u$, we obtain

$$\max\{\sup_{\partial B_{R_k}(0)} u\,, m\} = \sup_{\partial B_{R_k}(0)} u \quad \text{and} \quad \min\{\inf_{\partial \mathrm{B}_{\mathrm{R_k}}(0)} \mathrm{u}\,, \mathrm{m}\} = \inf_{\partial \mathrm{B}_{\mathrm{R_k}}(0)} \mathrm{u}\,,$$

and the result for the oscillation follows when we subtract the estimates. □

As an important application from Theorem 2.1, we can consider the case of the mean curvature operator.

**Corollary 2.5.** *Consider* $p = 2$, $n \geq 3$, *and* $u \in C^1(\mathbb{R}^n \setminus B_1)$ *a bounded weak solution of*

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \;=\; f(x)$$

*in* $\mathbb{R}^n \setminus B_1$. *If* $\nabla u$ *is uniformly limited in* $\mathbb{R}^n \setminus B_1$, *and* $f$ *satisfies conditions* (2.1), (2.2), (2.3), *then the limit* $\lim_{|x|\to\infty} u(x)$ *exists.*

*Proof.* Considering the limitation for $\nabla u$, we have that $A$, defined by

$$A(t) = \frac{1}{\sqrt{1+t^2}},$$

satisfies the condition iii., when restricted to $[\,0, \sup|\nabla u|\,]$. In this case, since the prescribed mean curvature equation can be written as $-\mathrm{div}(A(|\nabla u|)\nabla u) = f(x)$, all hypotheses that validate Theorem 2.1 can be verified. □

The next corollary is a generalization for Theorem 2.1:

**Corollary 2.6.** *Consider $p < n$ and $u \in C^1(\mathbb{R}^n \setminus B_1)$ a bounded weak solution of*

$$-\mathrm{div}\big(|\nabla u|^{p-2}A(|\nabla u|)\nabla u\big) \;=\; g(x,u)$$

*in $\mathbb{R}^n \setminus B_1$. If $|g(x,t)| \le h_1(x)h_2(t)$, where $h_1$ satisfies conditions (2.1), (2.2), (2.3), and $h_2 \in L^\infty_{loc}(\mathbb{R})$, then the limit $\lim\limits_{|x|\to\infty} u(x)$ exists.*

*Proof.* The result follows considering

$$f(x) := g(x,u),$$

and observing that $f$ satisfies conditions (2.1), (2.2), and (2.3) from Theorem 2.1. □

As an application of Corollary 2.6, we can consider the next corollary, which shows the limit result for an eigenvalue problem.

**Corollary 2.7.** *Consider $p < n$ and $u \in C^1(\mathbb{R}^n \setminus B_1)$ a bounded weak solution of*

$$-\mathrm{div}\big(|\nabla u|^{p-2}A(|\nabla u|)\nabla u\big) \;=\; g(x,u)$$

*in $\mathbb{R}^n \setminus B_1$, where*

$$g(x,t) = \sum_{i=1}^{k} V_i(x)\, t\, |t|^{p_i-2} + h(x).$$

*If the functions $h$ and $V_i$, for each $i = 1, 2, \cdots, k$, satisfy the conditions (2.1), (2.2), and (2.3), we have the existence of the limit $\lim\limits_{|x|\to\infty} u(x)$.*

# Chapter 3

# COMPARISON PRINCIPLE

In this chapter, we obtain results concerning the operator (1.13) in exterior domains. The main result, Theorem 3.3, is a Comparison Principle for the solutions of (1.15) in exterior domains. The proof follows some steps made by Bonorino, Silva and Zingano [4] for an operator similar to (1.1).

The first result is a limitation for a Gagliardo seminorn of $u$ by the supremum of $u$.

**Theorem 3.1.** *Let $K \subset \mathbb{R}^n$ be a compact set and $u \in W^{s,p}_{loc}(\mathbb{R}^n \backslash K) \cap L^\infty(\mathbb{R}^n)$ be a weak solution of $\mathcal{L}u(x) = 0$ in $\mathbb{R}^n \backslash K$. Suppose that $sp \geq n$. Then, for any open set $U \subset \mathbb{R}^n$ such that $K \subset U$, it holds*

$$\left( \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n \backslash U} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}} \leq C \sup |u|, \tag{3.1}$$

*with $C$ depending on $n, s, p, \lambda, \Lambda, K$ and $U$.*

*Proof.* In what follows, let us write simply $B_r$ for denoting the ball of radius $r$ whenever it is centered at the origin, making explicit the center point if necessary. We start assuming with no loss of generality that $U$ is bounded. In this case, there is some $R_0 > 0$ such that $U \subseteq B_{R_0}$ and we consider for every $R \geq R_0$ functions $\psi = \psi_R \in C^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} \psi = 0 & \text{in } V, \text{ where } V \text{ is an open set such that } K \subset V \subset\subset U \\ \psi = 1 & \text{in } B_R \backslash U \\ \psi = 0 & \text{in } \mathbb{R}^n \backslash B_{2R} \\ |\nabla \psi| \leq m, & \text{in } U \backslash V, \text{ where } m \text{ is a constant depending only on } U \text{ and } V \\ |\nabla \psi| \leq 2/R, & \text{in } B_{2R} \backslash B_R \end{cases}$$

Using $\varphi = \varphi_R := \psi_R^p u \in W^{s,p}_0(\mathbb{R}^n \backslash K)$ as test function in (1.16) and splitting the integral as the sum of the integrals over $B_{4R}\times B_{4R}$ and its complement on $\mathbb{R}^n\times\mathbb{R}^n$ we obtain

$$\begin{aligned}
&\int_{B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))\,dxdy\,+\\
&\quad+\;2\int_{\mathbb{R}^n\setminus B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))\varphi(x)\,dxdy=0.
\end{aligned}\tag{3.2}$$

Using that

$$\varphi(x)-\varphi(y)=(u(x)-u(y))\psi(x)^p+(\psi(x)^p-\psi(y)^p)u(y),$$

we have

$$\begin{aligned}
&\int_{B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))\,dxdy\\
=&\int_{B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p}\psi(x)^p\,dxdy\;+\\
&\quad+\int_{B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)^p-\psi(y)^p)u(y)\,dxdy.
\end{aligned}$$

Taking it to (3.2) and isolating the first integral above we get

$$\begin{aligned}
&\int_{B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p}\psi(x)^p\,dxdy\\
=&-\int_{B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)^p-\psi(y)^p)u(y)\,dxdy\\
&\quad-\;2\int_{\mathbb{R}^n\setminus B_{4R}}\int_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))\varphi(x)\,dxdy\\
\le&\,\Lambda\int_{B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}|\psi(x)^p-\psi(y)^p|\,|u(y)|\,dxdy\\
&\quad+\;2\,\Lambda\int_{\mathbb{R}^n\setminus B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}|\varphi(x)|\,dxdy\\
:=&\,(i)\;+\;(ii).
\end{aligned}\tag{3.3}$$

**Estimate of** $(i)$**.**

Applying the mean value theorem to $t\mapsto t^p$ we can estimate, for some $0<\theta<1$,

$$\begin{aligned}
|\psi(x)^p-\psi(y)^p| &= p\,|\theta\,\psi(x)+(1-\theta)\,\psi(y)|^{p-1}\,|\psi(x)-\psi(y)|\\
&\le p\,2^{p-1}\left(\psi(x)^{p-1}+\psi(y)^{p-1}\right)|\psi(x)-\psi(y)|,
\end{aligned}\tag{3.4}$$

where we also used that $(a+b)^{p-1} \leq 2^{p-1}(a^{p-1}+b^{p-1})$, for $a, b \geq 0$. Then, we have

$$
\begin{aligned}
&\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}|\psi(x)^p-\psi(y)^p|\,|u(y)|\,dxdy \\
&\leq\ p\ 2^{p-1} \sup|u| \int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\big(\psi(x)^{p-1}+\psi(y)^{p-1}\big)|\psi(x)-\psi(y)|\,dxdy \\
&=\ p\,2^p \sup|u| \int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\,\psi(x)^{p-1}|\psi(x)-\psi(y)|\,dxdy \\
&=\ p\,2^p \sup|u| \int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{(n+sp)(p-1)/p}}\,\psi(x)^{p-1}\frac{|\psi(x)-\psi(y)|}{|x-y|^{(n+sp)/p}}\,dxdy \\
&\leq p\,2^p \sup|u| \left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} \left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|\psi(x)-\psi(y)|^{p}}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{1}{p}} \\
&\leq p\,2^p \sup|u| \left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} [\,\psi\,]_{W^{s,p}(\mathbb{R}^n)}.
\end{aligned}
$$

From Lemma 1.10, the $\psi_R$ seminorm on the right stays bounded with respect to $R$, so it follows the estimate

$$
(i)\ \leq\ C\ \sup|u| \left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}},
$$

for some positive constant $C$ depending on $n, s, p, \Lambda$, and $U$.

**Estimate of** $(ii)$**.**

Observing that $\varphi = \psi^p\,u$ and $\operatorname{supp}\psi \subset B_{2R}$ we have

$$
\begin{aligned}
&2 \int\limits_{\mathbb{R}^n\setminus B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\,|\varphi(x)|\,dxdy \\
=\ &2 \int\limits_{\mathbb{R}^n\setminus B_{4R}}\int\limits_{B_{2R}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\,\psi(x)^p|u(x)|\,dxdy \\
\leq\ &(2\ \sup|u|)^p \int\limits_{\mathbb{R}^n\setminus B_{4R}}\int\limits_{B_{2R}} \frac{dxdy}{|x-y|^{n+sp}}.
\end{aligned}
$$

Switching the order of integration and making the change of variables $y = x - z$, with

$z \in \mathbb{R}^n \backslash B_{4R}(x)$, gives

$$\begin{aligned}\int\limits_{\mathbb{R}^n\backslash B_{4R}}\int\limits_{B_{2R}} \frac{dxdy}{|x-y|^{n+sp}} &= \int\limits_{B_{2R}}\int\limits_{\mathbb{R}^n\backslash B_{4R}} \frac{dy}{|x-y|^{n+sp}}\,dx \\ &= \int\limits_{B_{2R}}\int\limits_{\mathbb{R}^n\backslash B_{4R}(x)} \frac{dz}{|z|^{n+sp}}\,dx.\end{aligned}$$

Observe now that as $x \in B_{2R}(0)$, then $B_{2R}(0) \subset B_{4R}(x)$, so that $\mathbb{R}^n\backslash B_{4R}(x) \subset \mathbb{R}^n\backslash B_{2R}(0)$. Consequently,

$$\begin{aligned}\int\limits_{B_{2R}}\int\limits_{\mathbb{R}^n\backslash B_{4R}(x)} \frac{dz}{|z|^{n+sp}}\,dx &\leq \int\limits_{B_{2R}}\int\limits_{\mathbb{R}^n\backslash B_{2R}} \frac{dz}{|z|^{n+sp}}\,dx \\ &= \int\limits_{B_{2R}}\int\limits_{2R}^{\infty} r^{-1-sp}dr\ \ n\,\omega_n\,dx \\ &= \int\limits_{B_{2R}} \frac{n\,\omega_n}{sp}\,(2R)^{-sp}\,dx \\ &= \frac{n\,\omega_n^2}{sp}\,(2R)^{n-sp} \leq \frac{n\,\omega_n^2}{sp}\,(2R_0)^{n-sp},\end{aligned}$$

and we have obtained the estimate

$$(ii) \leq C\,(\sup|u|)^p,$$

for some positive constant $C$ depending on $n, s, p$ and $\Lambda$.

Using the estimates for $(i)$ and $(ii)$ in (3.3) it follows that

$$\begin{aligned}&\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\psi(x)^p\,dxdy \\ \leq\ & \frac{1}{\lambda}\int\limits_{B_{4R}}\int\limits_{B_{4R}} \mathbf{K}(x,y)|u(x)-u(y)|^p\,\psi(x)^p\,dxdy \\ \leq\ & C\sup|u| \left[\left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} + (\sup|u|)^{p-1}\right],\end{aligned} \tag{3.5}$$

with $C$ depending on $n, p, s, \lambda, \Lambda$ and $U$.

Next we look the following cases:

**1.** If

$$(\sup|u|)^{p-1} \leq \left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}},$$

(3.5) gives

$$\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\psi(x)^p\,dxdy \leq\; 2\,C\sup|u|\left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}},$$

which leads to

$$\left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{1}{p}} \leq\; 2\,C\,\sup|u|.$$

**2.** If

$$\left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} \leq\; (\sup|u|)^{p-1},$$

then (3.5) readily gives

$$\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\psi(x)^p\,dxdy \;\leq\; 2\,C\,(\sup|u|)^p.$$

It follows in both cases that

$$\left(\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi_(x)^p\,dxdy\right)^{\frac{1}{p}} \leq\; C\,\sup|u|,$$

for some new constant. Now considering $\psi\equiv 1$ in $B_R\backslash U$ we have

$$\int\limits_{B_{4R}}\int\limits_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy \;\geq\; \int\limits_{B_{4R}}\int\limits_{B_R\backslash U} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy$$

and then

$$\left(\int\limits_{B_{4R}}\int\limits_{B_R\backslash U} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{1}{p}} \leq\; C\,\sup|u|$$

holds, for all $R$. Letting $R\to\infty$, this concludes the proof. □

A similar result to that of Theorem 3.1 can be stated for solutions of the non-homogeneous equation as well. This result is goint to be usefull to prove the Comparison Principle for the non-homogeneous case, applying some similar ideas as in [4].

**Corollary 3.2.** *Let $K\subset\mathbb{R}^n$ be a compact set and $u\in W^{s,p}_{loc}(\mathbb{R}^n\backslash K)\cap L^\infty(\mathbb{R}^n)$ a weak solution of*

$$\mathcal{L}u(x) \;=\; f \quad in \quad \mathbb{R}^n\backslash K\,,$$

*where $f\in L^1(\mathbb{R}^n\backslash U)$, for any open set $U\supset K$. Suppose that $sp\geq n$. Then, for any open set $U\subset\mathbb{R}^n$ such that $K\subset U$ it holds*

$$\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n\backslash U} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{1}{p}} \;<\; \infty.$$

*Proof.* Following the steps in the proof of the previous theorem, instead of (3.5), we obtain

$$\begin{aligned}
&\int_{B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\psi(x)^p\,dxdy \\
&\le\; C\,\sup|u|\left[\left(\int_{B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} +\;(\sup|u|)^{p-1}\right] + \\
&\quad +\;\int_{\mathbb{R}^n} f(x)\,\psi^p(x)\,u(x)\,dx \\
&\le\; C\,\sup|u|\left[\left(\int_{B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} +\;(\sup|u|)^{p-1}\;+\right. \\
&\quad \left. +\;\|\,f\,\|_{L^1(\mathbb{R}^n\setminus U)}\right].
\end{aligned}$$

If

$$\|\,f\,\|_{L^1(\mathbb{R}^n\setminus U)}\;\le\;\left(\int_{B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\psi(x)^p\,dxdy\right)^{\frac{p-1}{p}} +\;(\sup|u|)^{p-1}$$

for large $R$, then we fall back on estimate (3.5), from which the conclusion follows the same way as before. Otherwise, we simply get

$$\int_{B_{4R}}\int_{B_{4R}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\psi(x)^p\,dxdy\;\le\;C\,\|\,f\,\|_{L^1(\mathbb{R}^n\setminus U)}\,\sup|u|\;<\;\infty,$$

which also leads to the conclusion. □

The next theorem is the main result of this chapter and states a Comparison Principle for solutions to the equation (1.15) in exterior domains.

**Theorem 3.3.** *Let $K$ be a compact set of $\mathbb{R}^n$ and let $u,v\in C(\mathbb{R}^n)\cap W^{s,p}_{loc}(\mathbb{R}^n\setminus K)$ be bounded weak solutions of $\mathcal{L}w = 0$ in $\mathbb{R}^n\setminus K$. Suppose, further, that $sp\ge n$. If $v\ge u$ on $K$, then $v\ge u$ in $\mathbb{R}^n\setminus K$.*

*Proof.* Let $\epsilon>0$. Since $v\ge u$ in $K$, by continuity there exists a bounded open set $U\supset K$ such that $v-u+\epsilon>0$ in $U\setminus K$. Let $R_0\ge 1$ such that $U\subset B_{R_0}$ and consider for $R\ge R_0$ the $\psi=\psi_R$ associated to $U$ as described in the previous theorem. Using the definition of $\psi$ and the fact that $u,v\in W^{s,p}_{loc}(\mathbb{R}^n\setminus K)$, we have that

$$\varphi\;:=\;(v-u+\epsilon)^-\,\psi^p$$

has a compact support in $\mathbb{R}^n\backslash K$ and $\varphi \in W_0^{s,p}(\mathbb{R}^n\backslash K)$. Then using $\varphi$ as test function for the solutions $u$ and $v$, and considering $L(x) = x|x|^{p-2}$, we have

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\, L(v(x)-v(y))\,(\varphi(x)-\varphi(y))\,dxdy = 0,$$
$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\, L(u(x)-u(y))\,(\varphi(x)-\varphi(y))\,dxdy = 0,$$

which subtracted result

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\,(\varphi(x)-\varphi(y))\,dxdy = 0. \tag{3.6}$$

Observing that

$$\begin{aligned}\varphi(x)-\varphi(y) = &\big[(v-u+\epsilon)^-(x) - (v-u+\epsilon)^-(y)\big]\,\psi^p(x) + \\ &+\ (v-u+\epsilon)^-(y)\big(\psi^p(x)-\psi^p(y)\big),\end{aligned}$$

it follows

$$\begin{aligned}&-\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\cdot \\ &\qquad\qquad \cdot\ \big[(v-u+\epsilon)^-(x) - (v-u+\epsilon)^-(y)\big]\,\psi^p(x)\,dxdy \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\cdot \\ &\qquad\qquad \cdot\ (v-u+\epsilon)^-(y)\,\big(\psi^p(x)-\psi^p(y)\big)\,dxdy.\end{aligned} \tag{3.7}$$

We estimate each side of this equality separately.

**Left-hand side estimate:**

Let us denote $G_\epsilon := \operatorname{supp}(v-u+\epsilon)^-$. The left-hand side then splits like

$$\begin{aligned}&-\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \mathbf{K}(x,y)\,\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\cdot \\ &\qquad\qquad \cdot\ \big[(v-u+\epsilon)^-(x) - (v-u+\epsilon)^-(y)\big]\,\psi^p(x)\,dxdy \\ &= \int\limits_{G_\epsilon}\int\limits_{G_\epsilon} \mathbf{K}(x,y)\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\big(v(x)-v(y)-u(x)+u(y)\big)\psi^p(x)\,dxdy \\ &\quad + \int\limits_{G_\epsilon}\int\limits_{\mathbb{R}^n\backslash G_\epsilon} \mathbf{K}(x,y)\,\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\,(v-u+\epsilon)^-(y)\,\psi^p(x)\,dxdy \\ &\quad - \int\limits_{\mathbb{R}^n\backslash G_\epsilon}\int\limits_{G_\epsilon} \mathbf{K}(x,y)\,\big[L(v(x)-v(y)) - L(u(x)-u(y))\big]\,(v-u+\epsilon)^-(x)\,\psi^p(x)\,dxdy.\end{aligned}$$

For the first integral above, observe that from (1.19) in Lemma 1.11, we have

$$\begin{aligned}&\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\big(v(x)-v(y)-u(x)+u(y)\big)\\ &\qquad\geq\ c_1\,|v(x)-v(y)-u(x)+v(y)|^p,\end{aligned}$$

with $c_1>0$. Then using $\lambda\leq\mathbf{K}(x,y)\,|x-y|^{n+sp}$ comes

$$\begin{aligned}&\int\limits_{G_\epsilon}\int\limits_{G_\epsilon}\mathbf{K}(x,y)\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\big(v(x)-v(y)-u(x)+u(y)\big)\psi^p(x)\,dxdy\\ &\qquad\geq\ c_1\,\lambda\int\limits_{G_\epsilon}\int\limits_{G_\epsilon}\frac{|v(x)-u(x)-v(y)+u(y)|^p}{|x-y|^{n+sp}}\,\psi^p(x)\,dxdy\\ &\qquad=\ c_1\,\lambda\int\limits_{G_\epsilon}\int\limits_{G_\epsilon}\frac{|(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\,\psi^p(x)\,dxdy.\end{aligned}$$

Let us look now for the last integral

$$\begin{aligned}&-\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\int\limits_{G_\epsilon}\mathbf{K}(x,y)\,\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\,(v-u+\epsilon)^-(x)\,\psi^p(x)\,dxdy\\ &=\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\int\limits_{G_\epsilon}\mathbf{K}(x,y)\,\big[L(u(x)-u(y))-L(v(x)-v(y))\big]\,(v-u+\epsilon)^-(x)\,\psi^p(x)\,dxdy.\end{aligned}$$

Noticing that $x\in G_\epsilon$ and $y\in\mathbb{R}^n\setminus G_\epsilon$, we have $v(x)-u(x)+\epsilon\leq 0$, $v(y)-u(y)+\epsilon\geq 0$, and so $u(x)-u(y)\geq v(x)-v(y)$. Then using (1.18) we get

$$\begin{aligned}&L(u(x)-u(y))-L(v(x)-v(y))\\ &\geq\ c_1\,\big|u(x)-u(y)-v(x)+v(y)\big|^{p-2}(u(x)-u(y)-v(x)+v(y))\\ &=\ c_1\,\big|u(x)-u(y)-v(x)+v(y)\big|^{p-1}\\ &=\ c_1\,\big|-(v(x)-u(x)+\epsilon)+v(y)-u(y)+\epsilon\big|^{p-1}\\ &\geq\ c_1\,\big|(v-u+\epsilon)^-(x)\big|^{p-1},\end{aligned}$$

the last inequality being valid since $v(y)-u(y)+\epsilon\geq 0$. Therefore

$$\begin{aligned}&\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\int\limits_{G_\epsilon}\mathbf{K}(x,y)\,\big[L(u(x)-u(y))-L(v(x)-v(y))\big]\,(v-u+\epsilon)^-(x)\,\psi^p(x)\,dxdy\\ &\qquad\geq\ c_1\,\lambda\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\int\limits_{G_\epsilon}\frac{|(v-u+\epsilon)^-(x)|^p}{|x-y|^{n+sp}}\,\psi^p(x)\,dxdy\\ &\qquad=\ c_1\,\lambda\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\int\limits_{G_\epsilon}\frac{|(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\,\psi^p(x)\,dxdy.\end{aligned}$$

A similar argument holds for the integral in the middle and yields

$$\begin{aligned}&\int\limits_{G_\epsilon}\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\mathbf{K}(x,y)\,\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\,(v-u+\epsilon)^-(y)\,\psi^p(x)\,dxdy\\ &\qquad\geq\ c_1\,\lambda\int\limits_{G_\epsilon}\int\limits_{\mathbb{R}^n\setminus G_\epsilon}\frac{|(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\,\psi^p(x)\,dxdy.\end{aligned}$$

By adding these inequalities we obtain the following estimate for the right-hand side

$$
\begin{aligned}
-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\mathbf{K}(x,y)\left[L(v(x)-v(y))-L(u(x)-u(y))\right]\cdot \\
&\cdot\left[(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)\right]\psi^p(x)\,dxdy \\
\geq\ c_1\,\lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{|(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\,\psi^p(x)\,dxdy.
\end{aligned}
$$

Still, noting that $\psi\equiv 1$ in $B_R\backslash U$, then

$$
\begin{aligned}
-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\mathbf{K}(x,y)\left[L(v(x)-v(y))-L(u(x)-u(y))\right]\cdot \\
&\cdot\left[(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)\right]\psi^p(x)\,dxdy \\
\geq\ c_1\,\lambda\int_{\mathbb{R}^n}\int_{B_R\backslash U} &\frac{|(v-u+\epsilon)^-(x)-(v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\,dxdy.
\end{aligned}
$$

**Right-hand side estimate:**

For the right-hand side of (3.7) observe that

$$
\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\mathbf{K}(x,y)\left[L(v(x)-v(y))-L(u(x)-u(y))\right]\cdot \\
&\cdot\,(v(y)-u(y)+\epsilon)^-\big(\psi^p(x)-\psi^p(y)\big)\,dxdy \\
=\int_{G_\epsilon}\int_{\mathbb{R}^n} &\mathbf{K}(x,y)\left[L(v(x)-v(y))-L(u(x)-u(y))\right]\cdot \\
&\cdot\,(v(y)-u(y)+\epsilon)^-\big(\psi^p(x)-\psi^p(y)\big)\,dxdy.
\end{aligned}
$$

Let us write

$$
(\dots):=\mathbf{K}(x,y)\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\,(v(y)-u(y)+\epsilon)^-\big(\psi^p(x)-\psi^p(y)\big)
$$

for an easier and shorter notation. Then, we split the integral as

$$
\begin{aligned}
\int_{G_\epsilon}\int_{\mathbb{R}^n}(\dots)\,dxdy &= \int_{G_\epsilon\backslash B_R}\int_{\mathbb{R}^n}(\dots)\,dxdy + \int_{G_\epsilon\cap B_R}\int_{\mathbb{R}^n}(\dots)\,dxdy \\
&= \int_{G_\epsilon\backslash B_R}\int_{\mathbb{R}^n}(\dots)\,dxdy + \int_{G_\epsilon\cap B_R}\int_{B_R}(\dots)\,dxdy + \int_{G_\epsilon\cap B_R}\int_{\mathbb{R}^n\backslash B_R}(\dots)\,dxdy.
\end{aligned}
\tag{3.8}
$$

**Claim 1.**

$$
\int_{G_\epsilon\cap B_R}\int_{B_R}(\dots)\,dxdy\ \leq 0. \tag{3.9}
$$

To prove it we write

$$\int_{G_\epsilon\cap B_R}\int_{B_R}(\dots)\,dxdy \;=\; \int_{G_\epsilon\cap B_R}\int_{U}(\dots)\,dxdy \;+\; \int_{G_\epsilon\cap B_R}\int_{B_R\setminus U}(\dots)\,dxdy.$$

In the first integral on the right, since $U\subset\mathbb{R}^n\setminus G_\epsilon$, then $x\in U$ and $y\in G_\epsilon$ imply that $v(x)-v(y)\geq u(x)-u(y)$, and so

$$L(v(x)-v(y))-L(u(x)-u(y))\;\geq\;0.$$

On the other hand, $\psi^p(x)-\psi^p(y)\leq 0$, since $\psi\leq 1$ and $\psi\equiv 1$ in $B_R\setminus U\supseteq G_\epsilon\cap B_R$. This gives

$$\int_{G_\epsilon\cap B_R}\int_{U}\mathbf{K}(x,y)\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\cdot$$

$$\cdot\;(v(y)-u(y)+\epsilon)^-\big(\psi^p(x)-\psi^p(y)\big)dxdy\leq 0.$$

In the last integral we note that $x\in B_R\setminus U$ and $y\in G_\epsilon\cap B_R$ gives $\psi^p(x)=\psi^p(y)=1$, so that

$$\int_{G_\epsilon\cap B_R}\int_{B_R\setminus U}\mathbf{K}(x,y)\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\cdot$$

$$\cdot\;(v(y)-u(y)+\epsilon)^-\big(\psi^p(x)-\psi^p(y)\big)dxdy=0,$$

proving (3.9).

Now back to estimate (3.8), (3.9) then gives

$$\int_{G_\epsilon}\int_{\mathbb{R}^n}(\dots)\,dxdy\;\leq\;\int_{G_\epsilon\setminus B_R}\int_{\mathbb{R}^n}(\dots)\,dxdy\;+\;\int_{G_\epsilon\cap B_R}\int_{\mathbb{R}^n\setminus B_R}(\dots)\,dxdy\,.$$

**Claim 2.**

$$\int_{G_\epsilon\setminus B_R}\int_{\mathbb{R}^n}(\dots)\,dxdy\;\to 0,\quad \int_{G_\epsilon\cap B_R}\int_{\mathbb{R}^n\setminus B_R}(\dots)\,dxdy\;\to 0\;\text{ as }\;R\to\infty. \tag{3.10}$$

To prove the first limit we denote $M\;:=\;\sup(v-u+\epsilon)^-$, use inequality (3.4), and that $\mathbf{K}(x,y)\,|x-y|^{n+sp}\leq\Lambda$ to estimate

$$\int_{G_\epsilon\setminus B_R}\int_{\mathbb{R}^n}\mathbf{K}(x,y)\big[L(v(x)-v(y))-L(u(x)-u(y))\big]\cdot$$

$$\cdot\;(v(y)-u(y)+\epsilon)^-\big(\psi^p(x)-\psi^p(y)\big)\,dxdy$$

$$\leq\;M\,2^p p\,\Lambda\int_{G_\epsilon\setminus B_R}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{p-1}+|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\,|\psi(x)-\psi(y)|\,dxdy$$

$$\leq\;M\,2^p p\,\Lambda\int_{G_\epsilon\setminus B_R}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{p-1}+|u(x)-u(y)|^{p-1}}{|x-y|^{\frac{(n+sp)(p-1)}{p}}}\,\frac{|\psi(x)-\psi(y)|}{|x-y|^{\frac{n+sp}{p}}}\,dxdy.$$

Distributing the product and applying Hölder's inequality, we get

$$\leq M\, 2^p p\, \Lambda \left[ \left( \int\limits_{G_\epsilon \setminus B_R} \int\limits_{\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\, dxdy \right)^{\frac{p-1}{p}} + \left( \int\limits_{G_\epsilon \setminus B_R} \int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy \right)^{\frac{p-1}{p}} \right] \cdot$$

$$\cdot \left( \int\limits_{G_\epsilon \setminus B_R} \int\limits_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\, dxdy \right)^{\frac{1}{p}}.$$

We have

$$\left( \int\limits_{G_\epsilon \setminus B_R} \int\limits_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\, dxdy \right)^{\frac{1}{p}} \leq [\,\psi\,]_{W^{s,p}(\mathbb{R}^n)},$$

and, by Lemma 1.10, this seminorm is bounded with $R$. Since $G_\epsilon \subset \mathbb{R}^n \setminus U$, the previous theorem ensures the integrals on the brackets go to zero as $R \to \infty$, then it follows the first limit in (3.10). By an analogous argument we can also prove the second one and conclude the claim.

Putting together the estimates so obtained for the members in (3.7) it follows that the expression

$$\int\limits_{\mathbb{R}^n} \int\limits_{B_R \setminus U} \frac{|(v-u+\epsilon)^-(x) - (v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\, dxdy$$

is bounded from above by a quantity that goes to zero as $R \to \infty$. As $B_R \setminus U \to \mathbb{R}^n \setminus U$ with $R \to \infty$ we obtain

$$\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n \setminus U} \frac{|(v-u+\epsilon)^-(x) - (v-u+\epsilon)^-(y)|^p}{|x-y|^{n+sp}}\, dxdy = 0.$$

Hence $(v-u+\epsilon)^-$ is constant in $\mathbb{R}^n$ and, as $(v-u+\epsilon)^- = 0$ in $K$, we then have $(v-u+\epsilon)^- = 0$ in $\mathbb{R}^n$, concluding the result. $\square$

The Comparison Principle yields some consequences. The first, Corollary 3.4, is a uniqueness result that follows immediately from the Comparison Principle.

**Corollary 3.4.** *Let $K$ be a compact set of $\mathbb{R}^n$ and let $u, v \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \setminus K)$ be bounded solutions of* (1.15) *in $\mathbb{R}^n \setminus K$. In case $sp \geq n$, if $v = u$ on $K$, then $v = u$ in $\mathbb{R}^n$.*

The next result is a consequence of Theorem 3.3 that compares a solution $u$ of (1.15) to a continuous function in $K$.

**Corollary 3.5.** *Let $K$ be a compact set of $\mathbb{R}^n$ and $f$ be a continuous function in $K$. Suppose that $sp \geq n$ and $u \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \setminus K)$ is a bounded solution of* (1.15) *in $\mathbb{R}^n \setminus K$. If $u = f$ on $K$, then*

$$\min f \leq\ u \leq \max f \quad \textit{in} \quad \mathbb{R}^n \setminus K.$$

*Proof.* Observe that $v \equiv \max f$ is a solution of (1.15) and satisfies $v \geq u$ in $K$. Then, from Theorem 3.3, we get $u \leq \max f$. By a similar argument, $\min f \leq u$. $\square$

Considering new assumptions, we are able to prove a new version of the Comparison Principle.

**Theorem 3.6.** *Let $K$ be a compact set of $\mathbb{R}^n$ and let $u, v \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \backslash K)$ be bounded functions such that $\mathcal{L}u \leq \mathcal{L}v$ in $\mathbb{R}^n \backslash K$ in the weak sense. Suppose that $sp \geq n$,*

$$\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \backslash U} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}} < \infty$$

*and*

$$\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \backslash U} \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}} < \infty,$$

*for any open set $U$ with $K \subset U$, as in (3.1). If $v \geq u$ in $K$, then $v \geq u$ in $\mathbb{R}^n \backslash K$.*

*Proof.* Since $u$ and $v$ satisfies $\mathcal{L}u \leq \mathcal{L}v$ in $\mathbb{R}^n \backslash K$ in the weak sense, we have, instead of (3.6) in Theorem 3.3,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{K}(x,y) \big[ L(v(x) - v(y)) - L(u(x) - u(y)) \big] \left( \varphi(x) - \varphi(y) \right) dxdy \geq 0,$$

for any non-negative test function $\varphi \in W^{s,p}_0(\mathbb{R}^n \backslash K)$. Hence, if we assume the hypotheses on $u$ and $v$ and use them to prove the convergences in (3.10), we can follow the same steps as Theorem 3.3 and obtain the result. $\square$

The next Corollary is a non-homogeneous version of the Comparison Principle that we get from the previous Theorem.

**Corollary 3.7.** *Let $K$ be a compact set of $\mathbb{R}^n$ and let $u, v \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \backslash K)$ be bounded functions such that*

$$\mathcal{L}u = f_1 \leq f_2 = \mathcal{L}v \quad in \quad \mathbb{R}^n \backslash K$$

*in the weak sense, where $f_1, f_2 \in L^1(\mathbb{R}^n \backslash U)$, for any open set $U$ with $K \subset U$. Suppose that $sp \geq n$. If $v \geq u$ in $K$, then $v \geq u$ in $\mathbb{R}^n \backslash K$.*

*Proof.* Since $f_1$ and $f_2$ satisfy the hypotheses of Corollary 3.2, we have that $u$ and $v$ suit the conditions for the Gagliardo seminorm on Theorem 3.6, for any open set $U$ with $K \subset U$. Therefore, from Theorem 3.6, we get the result. $\square$

In particular, for the classical kernel $\mathbf{K}(x,y) = |x-y|^{-n-sp}$, the operator is invariant under rotations by Theorem 1.9 and so, we have

**Corollary 3.8.** *Let $R_0 > 0$ and $u \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \backslash B_{R_0}(0))$ be a bounded solution of*

$$(-\Delta)^s_p u = 0 \ \text{ in } \ \mathbb{R}^n \backslash B_{R_0}(0).$$

*In case $sp \geq n$, if $u$ is radially symmetric on $B_{R_0}(0)$, then $u$ is radially symmetric in $\mathbb{R}^n$.*

*Proof.* Considering any rotation $T \in SO(n)$, we have that $\tilde{u} := u \circ T$ is also a bounded solution in $\mathbb{R}^n \backslash B_{R_0}(0)$. Since $u$ is radially symmetric in $B_{R_0}(0)$, $\tilde{u} = u$ in $B_{R_0}(0)$, then the previous corollary gives $\tilde{u} = u$ in $\mathbb{R}^n$. $\square$

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