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Research Article

Modal Formulation of Segmented Euler-Bernoulli Beams

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We consider the obtention of modes and frequencies of segmented Euler-Bernoulli beams with internal damping and external viscous damping at the discontinuities of the sections. This is done by following a Newtonian approach in terms of a fundamental response of stationary beams subject to both types of damping. The use of a basis generated by the fundamental solution of a differential equation of fourth-order allows to formulate the eigenvalue problem and to write the modes shapes in a compact manner. For this, we consider a block matrix that carries the boundary conditions and intermediate conditions at the beams and values of the fundamental matrix at the ends and intermediate points of the beam. For each segment, the elements of the basis have the same shape since they are chosen as a convenient translation of the elements of the basis for the first segment. Our method avoids the use of the first-order state formulation also to rely on the Euler basis of a differential equation of fourth-order and it allows to envision how conditions will influence a chosen basis.

1. Introduction

The methodology introduced by Tsukazan [1] in terms of a fundamental response [2, 3] is applied here to a triple-span Euler-Bernoulli beam with internal damping of the type Kelvin-Voight and viscous external damping at the discontinuities of the sections.

In the literature, the study of free vibrations of beams of the type Euler-Bernoulli have been sufficiently studied [4–11]. However, the effects of the nonproportional damping has been little studied in terms of modal analysis. Friswell and Lees [12] considered the method of separation of variables for obtaining the eigenvalues of a double-span pinned-pinned nonhomogeneous damped beam without intermediate devices. Chang et al. [13]

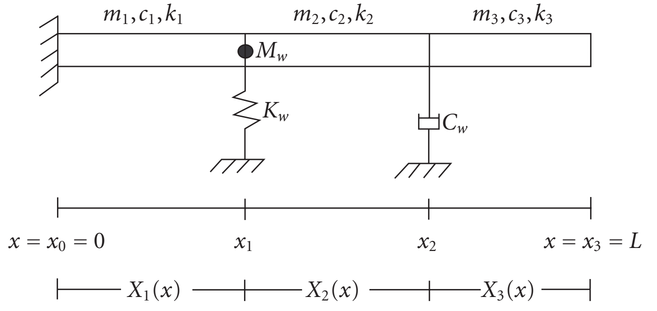


Figure 2.1. A triple-span discontinuous cantilever beam.

uses the Laplace transform for obtaining the natural frequencies of a pinned-pinned uniform Euler-Bernoulli beam, by considering masses, springs, and viscous dampers located in the middle of the beam. Sorrentino et al. [14] obtain the frequencies of the beam by using the state space formulation with a first-order transfer matrix. The obtention of the modes was accomplished by using the Euler basis in connection with fourth-order spatial differential equations, the Laplace transform and with the state-space methodology. Simulations were performed for double-span and four-span beams with several types of damping: internal, external, nonproportional, viscous damping.

Here, we consider the original Newtonian approach by keeping the formulation of a second-order system, that includes damping and stiffness, in each segment of the beam. The coefficients for the displacement boundary conditions and intermediate continuity conditions at discontinuity points of the beam are casted in a convenient block matrix that we refer to as being the coefficient matrix. The values that the elements of the basis at each segment take at the ends of the beam and intermediate discontinuity points give rise to another block matrix called the basis matrix. The introduction of these block matrices allows to formulate the eigenvalue problem in a compact matrix form. By choosing a basis that is generated by a fundamental solution of a fourth-order differential equation, the basis matrix becomes sparse. This approach can also be employed with double- or four-span beams subject to classical and nonclassical boundary conditions. In a forthcoming work, we will discuss multispan beams subject to a elastic coupling and discuss a reduction in the computation of the coefficients of a mode in each segment.

2. Statement of problem

We consider an Euler-Bernoulli beam of length L with two intermediate devices and two discontinuous cross sections, as in Figure 2.1. A flexural movement is represented in the beam by $v_j(t,x)$ in the jth segment $[x_{j-1},x_j]$, $j = 1:3$ with $0 = x_0 \leq x_1 \leq x_2 \leq x_3 = L$.

Here, M_w denotes value of the attached mass, C_w attached damping coefficient, K_w the attached stiffness.

In each segment of the beam, we have the governing equations [6, 15]

$$M_j \frac{\partial^2 v_j(t,x)}{\partial t^2} + C_j \frac{\partial v_j(t,x)}{\partial t} + K_j v_j(t,x) = 0, \quad x_{j-1} < x < x_j,\ j = 1:3, \tag{2.1}$$

where

$$M_j = m_j = \rho_j A_j,$$
$$K_j = \frac{\partial^2}{\partial x^2}\left[k_j(x)\frac{\partial^2}{\partial x^2}\right]. \tag{2.2}$$

The damping coefficient can be considered to be of the form

$$C_j = c_{0j}(x) + \frac{\partial^2}{\partial x^2}\left[c_{4j}(x)\frac{\partial^2}{\partial x^2}\right] \tag{2.3}$$

which includes the case of external viscous damping and internal Kelvin-Voigt damping. In the above, we have the following usual parameter description:

(i) ρ_j denotes density,
(ii) A_j denotes cross-sectional area,
(iii) c_{ij} denotes damping coefficients,
(iv) k_j denotes stiffness coefficients.

In what follows, we will consider the particular case of beams with uniform sections. Then the coefficients in the operators C_j, K_j become constants, that is,

$$K_j = k_j\frac{\partial^4}{\partial x^4} = E_jI_j\frac{\partial^4}{\partial x^4}, \qquad C_j = c_{0j} + c_{4j}\frac{\partial^4}{\partial x^4}, \qquad M_j = m_j, \tag{2.4}$$

where E_j denotes Young's modulus of elasticity, I_j denotes the area moment of inertia.

3. Modal analysis

Free vibrations whose spatial distribution amplitude in each segment is $X_j(x)$,

$$v_j = e^{\lambda t}X_j(x), \quad x \in [x_{j-1}, x_j],\ j = 1:3, \tag{3.1}$$

can be found by substituting them into the above system. It turns out the spatial modal differential equation

$$X_j^{(iv)}(x) - a_j^2(\lambda)\rho_jA_jX_j(x) = 0, \quad x \in [x_{j-1}, x_j],\ j = 1:3, \tag{3.2}$$

for each segment of the beam. Here,

$$a_j^2(\lambda) = -(\alpha_j + \lambda\beta_j)\lambda \tag{3.3}$$

with

$$\alpha_j = \frac{c_{0j}}{\rho_jA_j(E_jI_j + \lambda c_{4j})}, \quad \beta_j = \frac{1}{E_jI_j + \lambda c_{4j}}, \quad j = 1:3. \tag{3.4}$$

The solution for each segment (3.2) can be conveniently written as

$$X_j(x) = d_{1j}\phi_{1j}(x) + d_{2j}\phi_{2j}(x) + +d_{3j}\phi_{3j}(x) + d_{4j}\phi_{4j}(x) = \Psi_j(x)\mathbf{d_j}, \quad j = 1:3, \tag{3.5}$$

where

$$\Psi_j = \Psi_j(x,\lambda) = \left[\phi_{1,j}(x), \phi_{2,j}(x), \phi_{3,j}(x), \phi_{4,j}(x)\right] \tag{3.6}$$

is a solution basis of (3.2) in the segment $[x_{j-1}, x_j]$, $j = 1:3$, and $\mathbf{d_j}$ is the column vector with components d_{1j}, d_{2j}, d_{3j}, d_{4j}. Here we have emphasized that the solution matrix basis Ψ_j depend upon the parameter λ corresponding to a free vibration.

Generic boundary conditions of classical or nonclassical nature can be written as

$$\begin{aligned}
A_{11}X_1(0) + B_{11}X_1'(0) + C_{11}X_1''(0) + D_{11}X_1'''(0) &= 0,\\
A_{12}X_1(0) + B_{12}X_1'(0) + C_{12}X_1''(0) + D_{12}X_1'''(0) &= 0,\\
A_{21}X_3(L) + B_{21}X_3'(L) + C_{21}X_3''(L) + D_{21}X_3'''(L) &= 0,\\
A_{22}X_3(L) + B_{22}X_3'(L) + C_{22}X_3''(L) + D_{22}X_3'''(L) &= 0.
\end{aligned} \tag{3.7}$$

The continuity conditions for the displacement, the inertia moment, the bending moment, and the shear force at the discontinuity point x_j, $j = 1:2$ of the transversal section, including an intermediate device, can be written in general as follows:

$$\begin{aligned}
&E_{11}^{(j)}X_j(x_j) + F_{11}^{(j)}X_j'(x_j) + G_{11}^{(j)}X_j''(x_j) + H_{11}^{(j)}X_j'''(x_j)\\
&\qquad = E_{12}^{(j)}X_{j+1}(x_j) + F_{12}^{(j)}X_{j+1}'(x_j) + G_{12}^{(j)}X_{j+1}''(x_j) + H_{12}^{(j)}X_{j+1}'''(x_j),\\
&E_{21}^{(j)}X_j(x_j) + F_{21}^{(j)}X_j'(x_j) + G_{21}^{(j)}X_j''(x_j) + H_{21}^{(j)}X_j'''(x_j)\\
&\qquad = E_{22}^{(j)}X_{j+1}(x_j) + F_{22}^{(j)}X_{j+1}'(x_j) + G_{22}^{(j)}X_{j+1}''(x_j) + H_{22}^{(j)}X_{j+1}'''(x_j),\\
&E_{31}^{(j)}X_j(x_j) + F_{31}^{(j)}X_j'(x_j) + G_{31}^{(j)}X_1''(x_j) + H_{31}^{(j)}X_1'''(x_j)\\
&\qquad = E_{32}^{(j)}X_{j+1}(x_j) + F_{32}^{(j)}X_{j+1}'(x_j) + G_{32}^{(j)}X_{j+1}''(x_j) + H_{32}^{(j)}X_{j+1}'''(x_j),\\
&E_{41}^{(j)}X_j(x_j) + F_{41}^{(j)}X_j'(x_j) + G_{41}^{(j)}X_j''(x_j) + H_{41}^{(j)}X_j'''(x_j)\\
&\qquad = E_{42}^{(j)}X_{j+1}(x_j) + F_{42}^{(j)}X_{j+1}'(x_j) + G_{42}^{(j)}X_{j+1}''(x_j) + H_{42}^{(j)}X_{j+1}'''(x_j) + F_j, \quad j = 1:2,
\end{aligned} \tag{3.8}$$

where F_j denotes the force exerted by an external device.

Figure 2.1 shows a cantilever beam with intermediate continuity conditions at the points $x = x_1$ and $x = x_2$ and subject to a concentrated mass, spring, and a dashpot. The boundary conditions at $x = x_0 = 0$ and $x = x_3 = L$ are

$$X_1(0) = X_1'(0) = 0, \qquad X_3''(L) = X_3'''(L) = 0. \tag{3.9}$$

At the intermediate point $x = x_1$, we have

$$\begin{aligned} X_1(x_1) &= X_2(x_1), \\ X_1'(x_1) &= X_2'(x_1), \\ k_2^{-1}k_1X_1''(x_1) &= X_2''(x_1), \\ -k_2^{-1}(M_w\lambda^2 + K_w)X_1(x_1) + k_2^{-1}k_1X_1'''(x_1) &= X_2'''(x_1). \end{aligned} \tag{3.10}$$

Similarly, at the point $x = x_2$, we have

$$\begin{aligned} X_2(x_2) &= X_3(x_2), \\ X_2'(x_2) &= X_3'(x_2), \\ k_3^{-1}k_2X_2''(x_2) &= X_3''(x_2), \\ -k_3^{-1}(C_w\lambda)X_2(x_2) + k_3^{-1}k_2X_2'''(x_2) &= X_3'''(x_2). \end{aligned} \tag{3.11}$$

The substitution of (3.5) into (3.7) and (3.8), the boundary and continuity conditions leads to a linear algebraic system

$$\mathcal{U}(\lambda)\mathbf{d} = \mathbf{0}, \tag{3.12}$$

for the vector $\mathbf{d}$ of order 12×1,

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}, \qquad \mathbf{d_j} = \begin{bmatrix} d_{1j} \\ d_{2j} \\ d_{3j} \\ d_{4j} \end{bmatrix}, \qquad j = 1:2. \tag{3.13}$$

Here, the matrix $\mathcal{U}$ is of order 12×12 and it has the form

$$\mathcal{U} = \mathcal{B}\Phi, \tag{3.14}$$

where $\mathcal{B}$ is a matrix of order 12×24 formed with the coefficients associated to the boundary and continuity conditions and Φ is a matrix of order 24×12 whose components are values of the solution basis at the ends and the conditions at the discontinuity. A detailed description of these block matrices is given in Section 4. Then nonzero solutions of (3.12) are obtained for frequency values λ real or complex that satisfy the characteristic equation

$$\det(\mathcal{U}) = 0. \tag{3.15}$$

In classical conservative mechanical vibration theory, modes are essential for performing a decoupling of the system. However, any real structure with or without intermediate devices is dissipative. This implies the existence of complex modes that not necessarily decouple a damped system [16]. On the other hand, any pair of complex conjugate modes represent a free vibration in which distributed coordinates oscillate and share the same decay rate and frequency but are not synchronous. This later is because it introduced a phase when writing the mode or amplitude was in polar form [15].

4. Block matrix formulation

A detailed description of the matrix $\mathcal{U}$ in terms of the boundary and basis block matrices is given in what follows for a triple-span beam subject to generic conditions. The matrix corresponding to the boundary values can be written as follows:

$$\mathcal{B}_0 = \begin{bmatrix} A_{11} & B_{11} & C_{11} & D_{11} \\ A_{12} & B_{12} & C_{12} & D_{12} \end{bmatrix}, \qquad \mathcal{B}_L = \begin{bmatrix} A_{21} & B_{21} & C_{21} & D_{21} \\ A_{22} & B_{22} & C_{22} & D_{22} \end{bmatrix}. \tag{4.1}$$

The matrix coefficients corresponding to the continuity conditions at $x = x_j$, $j = 1:2$, can be described in terms of the matrices

$$\mathcal{B}_{1j} = \begin{bmatrix} E_{11}^{(j)} & F_{11}^{(j)} & G_{11}^{(j)} & H_{11}^{(j)} \\ E_{21}^{(j)} & F_{21}^{(j)} & G_{21}^{(j)} & H_{21}^{(j)} \\ E_{31}^{(j)} & F_{31}^{(j)} & G_{31}^{(j)} & H_{31}^{(j)} \\ E_{41}^{(j)} & F_{41}^{(j)} & G_{41}^{(j)} & H_{41}^{(j)} \end{bmatrix}, \qquad \mathcal{B}_{2j} = \begin{bmatrix} E_{12}^{(j)} & F_{12}^{(j)} & G_{12}^{(j)} & H_{12}^{(j)} \\ E_{22}^{(j)} & F_{22}^{(j)} & G_{22}^{(j)} & H_{22}^{(j)} \\ E_{32}^{(j)} & F_{32}^{(j)} & G_{32}^{(j)} & H_{32}^{(j)} \\ E_{42}^{(j)} & F_{42}^{(j)} & G_{42}^{(j)} & H_{42}^{(j)} \end{bmatrix}. \tag{4.2}$$

The values of the basis solutions at the ends of the beam x_0, x_3, and at each discontinuity point x_k, $k = 1,2$, can be written in terms of the Wronskian matrices in each segment

$$\mathbf{\Phi_j}(x) = \begin{bmatrix} \phi_{1j}(x) & \phi_{2j}(x) & \phi_{3j}(x) & \phi_{4j}(x) \\ \phi'_{1j}(x) & \phi'_{2j}(x) & \phi'_{3j}(x) & \phi'_{4j}(x) \\ \phi''_{1j}(x) & \phi''_{2j}(x) & \phi''_{3j}(x) & \phi''_{4j}(x) \\ \phi'''_{1j}(x) & \phi'''_{2j}(x) & \phi'''_{3j}(x) & \phi'''_{4j}(x) \end{bmatrix}, \quad j = 1:3. \tag{4.3}$$

For a triple-span beam, we will have the block matrices

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{11} & -\mathcal{B}_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{B}_{12} & -\mathcal{B}_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{B}_3 \end{bmatrix}, \tag{4.4}$$

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi_1}(0) & 0 & 0 \\ \mathbf{\Phi_1}(x_1) & 0 & 0 \\ 0 & \mathbf{\Phi_2}(x_1) & 0 \\ 0 & \mathbf{\Phi_2}(x_2) & 0 \\ 0 & 0 & \mathbf{\Phi_3}(x_2) \\ 0 & 0 & \mathbf{\Phi_3}(x_3) \end{bmatrix}. \tag{4.5}$$

In the above, 0 denotes null matrices with appropriate dimensions, that is, 2×4 or 4×4.

4.1. A cantilever triple-span beam subject to damping. For the triple-span cantilever beam of Figure 2.1, the corresponding blocks for the coefficients of the boundary conditions are

$$\mathcal{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathcal{B}_L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.6}$$

while the blocks for the continuity conditions at the intermediate discontinuous sections are

$$\mathcal{B}_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k_2^{-1}k_1 & 0 \\ -k_2^{-1}(M_w\lambda^2 + K_w) & 0 & 0 & k_2^{-1}k_1 \end{bmatrix}, \qquad \mathcal{B}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{B}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k_3^{-1}k_2 & 0 \\ -k_3^{-1}(C_w\lambda) & 0 & 0 & k_3^{-1}k_2 \end{bmatrix}, \qquad \mathcal{B}_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.7}$$

5. The fundamental basis

The classical or spectral Euler basis of the fourth-order equation,

$$X^{(iv)}(x) - \varepsilon^4 X(x) = 0, \tag{5.1}$$

is constructed by using the roots $\pm\varepsilon$, $\pm i\varepsilon$ of the characteristic polynomial $s^4 - \varepsilon^4 = 0$, that is,

$$\Psi = [\sin(\varepsilon x), \cos(\varepsilon x), \sinh(\varepsilon x), \cosh(\varepsilon x)]. \tag{5.2}$$

However, among all possible bases that we can choose, it would be convenient to choose the basis that makes (4.5) as sparse as possible. In this work, this is accomplished by choosing in each segment a *fundamental* basis that is a translation of a fixed basis that is generated by an initial-value solution in the first segment. This later solution can be found in the work of Timoshenko et al. [17] literature without the systematic treatment considered in [2, 3, 18]. We will consider the basis for the first segment that is constituted by the solution $h(x)$ of the initial value problem

$$\begin{gathered} h^{(iv)}(x) - \varepsilon^4 h(x) = 0, \\ h(0) = 0, \qquad h'(0) = 0, \qquad h''(0) = 0, \qquad h'''(0) = 1, \end{gathered} \tag{5.3}$$

and its first three derivatives $h'(x)$, $h''(x)$, $h'''(x)$. With respect to the spectral Euler basis, the fundamental solution $h(x)$ has the following representation:

$$h(x) = \frac{\sinh(\varepsilon x) - \sin(\varepsilon x)}{2\varepsilon^3}. \tag{5.4}$$

By defining

$$\phi_{jk}^{(i-1)}(x) = h^{(j+i-2)}(x - x_{k-1}, \varepsilon_k), \quad i, j = 1:4,\ k = 1:3, \\ \varepsilon_k^4 = a_k^2(\lambda)\rho_k A_k, \tag{5.5}$$

where we have emphasized the dependence of the solution of (5.3) upon the parameter ε in each segment of the beam, it turns out

$$\mathbf{\Phi}_j(x_{j-1}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad j = 1:3. \tag{5.6}$$

By taking into account the initial values of $h(x,\epsilon)$, the matrix (4.5) becomes more sparse and it is given by

$$\Phi = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{11}(x_1) & \phi_{21}(x_1) & \phi_{31}(x_1) & \phi_{41}(x_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi'_{11}(x_1) & \phi'_{21}(x_1) & \phi'_{31}(x_1) & \phi'_{41}(x_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi''_{11}(x_1) & \phi''_{21}(x_1) & \phi''_{31}(x_1) & \phi''_{41}(x_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi'''_{11}(x_1) & \phi'''_{21}(x_1) & \phi'''_{31}(x_1) & \phi'''_{41}(x_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi_{12}(x_2) & \phi_{22}(x_2) & \phi_{32}(x_2) & \phi_{42}(x_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi'_{12}(x_2) & \phi'_{22}(x_2) & \phi'_{32}(x_2) & \phi'_{42}(x_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi''_{12}(x_2) & \phi''_{22}(x_2) & \phi''_{32}(x_2) & \phi''_{42}(x_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi'''_{12}(x_2) & \phi'''_{22}(x_2) & \phi'''_{32}(x_2) & \phi'''_{42}(x_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{13}(L) & \phi_{23}(L) & \phi_{33}(L) & \phi_{43}(L) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi'_{13}(L) & \phi'_{23}(L) & \phi'_{33}(L) & \phi'_{43}(L) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi''_{13}(L) & \phi''_{23}(L) & \phi''_{33}(L) & \phi''_{43}(L) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi'''_{13}(L) & \phi'''_{23}(L) & \phi'''_{33}(L) & \phi'''_{43}(L)
\end{bmatrix}. \tag{5.7}$$

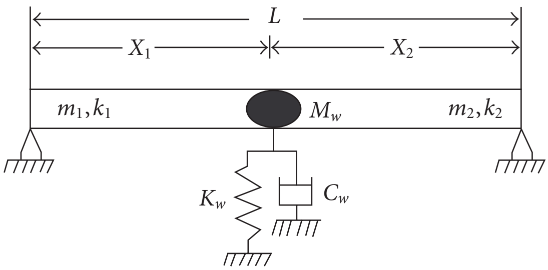


Figure 6.1. A double-span discontinuous cantilever beam.

The fundamental response $h(x,\varepsilon)$, has the same shape for each segment, but depends on different values for the involved physical parameters.

6. Numerical examples

6.1. Double-span beam. We first consider the case of a pinned-pinned double-span beam of length L as Figure 6.1 that was studied in Sorrentino et al. [14] and Chang et al. [13].

The spatial modal differential equation to double-span beam can be expressed in the form

$$X_j^{(iv)}(x) - a_j^2(\lambda)\rho_j A_j X_j(x) = 0, \quad x \in [x_{j-1}, x_j],\ j = 1:2, \tag{6.1}$$

for each segment of the beam, where a_j, $j = 1:2$ are given in (3.3).

The boundary conditions to beam above at $x = x_0 = 0$ and $x = x_2 = L$ are

$$X_1(0) = X_1''(0) = 0, \qquad X_2(L) = X_2''(L) = 0. \tag{6.2}$$

We have the intermediate continuity conditions at the point $x = x_1$,

$$\begin{gathered} X_1(x_1) = X_2(x_1), \\ X_1'(x_1) = X_2'(x_1), \\ k_2^{-1}k_1X_1''(x_1) = X_2''(x_1), \\ -k_2^{-1}(M_w\lambda^2 + C_w\lambda + K_w)X_1(x_1) + k_2^{-1}k_1X_1'''(x_1) = X_2'''(x_1). \end{gathered} \tag{6.3}$$

For a double-span beam, the blocks that correspond to the boundary conditions are

$$\mathcal{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathcal{B}_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{6.4}$$

At the intermediate points, where continuity conditions are to be held, we have

$$\mathcal{B}_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k_2^{-1}k_1 & 0 \\ -k_2^{-1}(M_w\lambda^2 + C_w\lambda + K_w) & 0 & 0 & k_2^{-1}k_1 \end{bmatrix}, \qquad \mathcal{B}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.5}$$

Thus, the coefficient block matrix of the given double-span beam is

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_0 & 0 & 0 \\ 0 & \mathcal{B}_{11} & -\mathcal{B}_{21} \\ 0 & 0 & \mathcal{B}_L \end{bmatrix} \tag{6.6}$$

or expanded

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{k_1}{k_2} & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \Gamma & 0 & 0 & \dfrac{k_1}{k_2} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{6.7}$$

where $\Gamma = -k_2(M_w\lambda^2 + C_w\lambda + K_w)$.

For constructing the basis matrix, that carries the values of the generic solution basis at the ends of the beam and at the discontinuity points of a double-span beam, we consider

$$\phi_{jk}^{(i-1)}(x) = h^{(j+i-2)}(x - x_{k-1}, \varepsilon_k), \quad i, j = 1:4, \ k = 1:2, \tag{6.8}$$

where $h(x)$ is the solution of (5.3). Then, the basis matrix is given by

$$\Phi = \begin{bmatrix} \mathbf{\Phi_1}(0) & 0 \\ \mathbf{\Phi_1}(x_1) & 0 \\ 0 & \mathbf{\Phi_2}(x_1) \\ 0 & \mathbf{\Phi_2}(L) \end{bmatrix} \tag{6.9}$$

Table 6.1. Parameter values of a double-span beam.

Parameter	Numeric value	Unit
$m_1 = m_2$	1.6363×10^4	kg/m
$k_1 = k_2$	1.6669×10^{11}	Nm^2
L	15.24	m

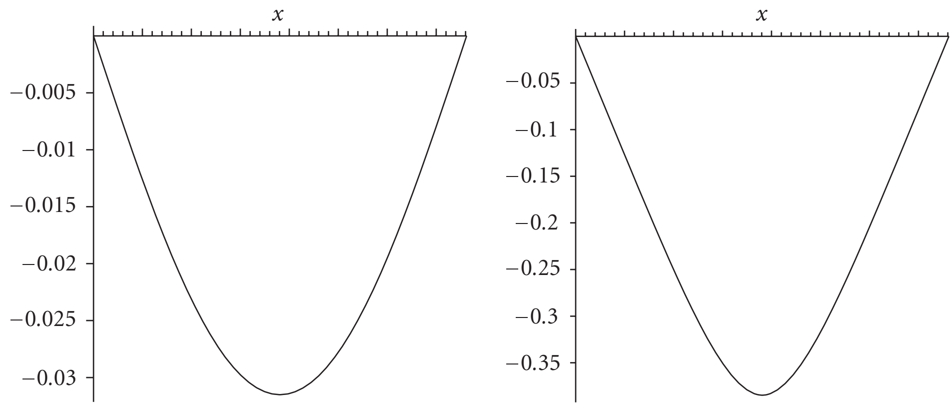
Table 6.2. Eigenvalues (rad/s) to double-span beam.

Mode (n)	Proposed method	[14]
1	−11.25426117 ± 135.0795544 I	−11.30627 ± 135.1799 I
2	.5512552857e-7 ±542.5166750 I	0 ± 542.5144 I
3	−8.442911066 ± 1128.708193 I	−8.482803 ± 1128.716 I

or expanded

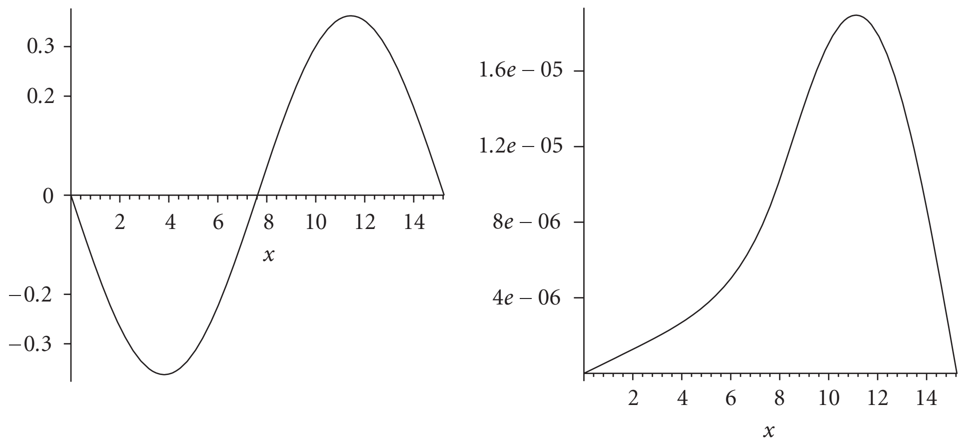
$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_{11}(x_1) & \phi_{21}(x_1) & \phi_{31}(x_1) & \phi_{41}(x_1) & 0 & 0 & 0 & 0 \\ \phi'_{11}(x_1) & \phi'_{21}(x_1) & \phi'_{31}(x_1) & \phi'_{41}(x_1) & 0 & 0 & 0 & 0 \\ \phi''_{11}(x_1) & \phi''_{21}(x_1) & \phi''_{31}(x_1) & \phi''_{41}(x_1) & 0 & 0 & 0 & 0 \\ \phi'''_{11}(x_1) & \phi'''_{21}(x_1) & \phi'''_{31}(x_1) & \phi'''_{41}(x_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{12}(L) & \phi_{22}(L) & \phi_{32}(L) & \phi_{42}(L) \\ 0 & 0 & 0 & 0 & \phi'_{12}(L) & \phi'_{22}(L) & \phi'_{32}(L) & \phi'_{42}(L) \\ 0 & 0 & 0 & 0 & \phi''_{12}(L) & \phi''_{22}(L) & \phi''_{32}(L) & \phi''_{42}(L) \\ 0 & 0 & 0 & 0 & \phi'''_{12}(L) & \phi'''_{22}(L) & \phi'''_{32}(L) & \phi'''_{42}(L) \end{bmatrix}. \tag{6.10}$$

Numerical simulations with the proposed method are presented by using the data in Table 6.1. The parameter values at the discontinuity point $x = x_1 = (L/2)$ of beam used are $M_w = 0.1mL$, $K_w = 0.1mLw_1^2$, and $C_w = 0.1mLw_1$ where $m = m_1 = m_2$ and w_1 is the first natural frequency of the beam without added mass and spring [13]. In Table 6.2, the first three eigenvalues of the beam were obtained by solving the characteristic equation (3.15) with an approximation of $h(x)$ and compared with the ones obtained in [14]. We observed a good agreement among the two methods. In Figures 6.2, 6.3, and 6.4 are showed the modes shapes corresponding to the first three eigenvalues of the beam, where (a) indicates the real part of the mode and (b) the imaginary part of the mode.



(a) Real part (b) Imaginary part

Figure 6.2. First mode to double-span beam.



(a) Real part (b) Imaginary part

Figure 6.3. Second mode to double-span beam.

6.2. Triple-span beam. We now consider the triple-span beam given in Figure 2.1. First, we assume that the beam is uniform with parameter values given in Table 6.3. The viscous damping at the point of discontinuity $x = x_2$ is given by $Cw = 0.1mLw_1$, where $x_1 = 4m$, $x_2 = 10m$, and w_1 is the first natural frequency of the beam without added mass and spring [13].

In Table 6.4, we have the values of the first three eigenvalues of the beam and in Figures 6.5, 6.6, and 6.7 the correspondent modes shapes, where (a) it indicates the real part of the mode and (b) the imaginary part of the mode.

Table 6.3. System parameters to beam uniform triple-span.

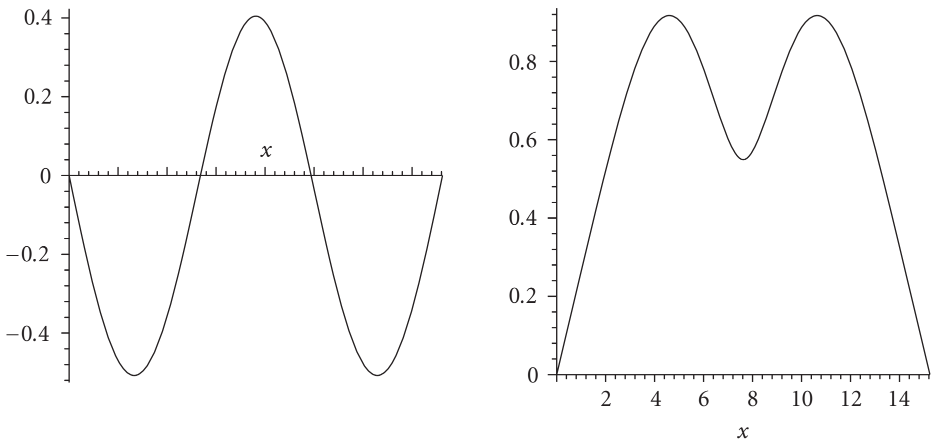
Parameter	Numeric value	Unit
$m_1 = m_2 = m_3 = m$	1.6363×10^4	kg/m
$k_1 = k_2 = k_3$	1.6669×10^{11}	Nm^2
L	15.24	m

Table 6.4. Eigenvalues of a uniform triple-span beam.

Mode (n)	Eigenvalues
1	$-1.355547843 \pm 48.31860604$ I
2	$-12.43829644 \pm 302.5538405$ I
3	$-8.226875796 \pm 847.5582997$ I

Table 6.5. System parameters to triple-span beam.

	Segment first	Segment second	Segment third	Unit
Mass	1.6363×10^4	$0.8 \times m_1$	$0.8 \times m_1$	kg/m
Stiffness	1.6669×10^{11}	$1.4 \times k_1$	$0.6 \times k_1$	Nm^2
Damping	5×10^{-1}	$0.5 \times c_1$	$11.7 \times c_1$	Ns/m^2
Length (L)	4	6	5.24	m



(a) Real part

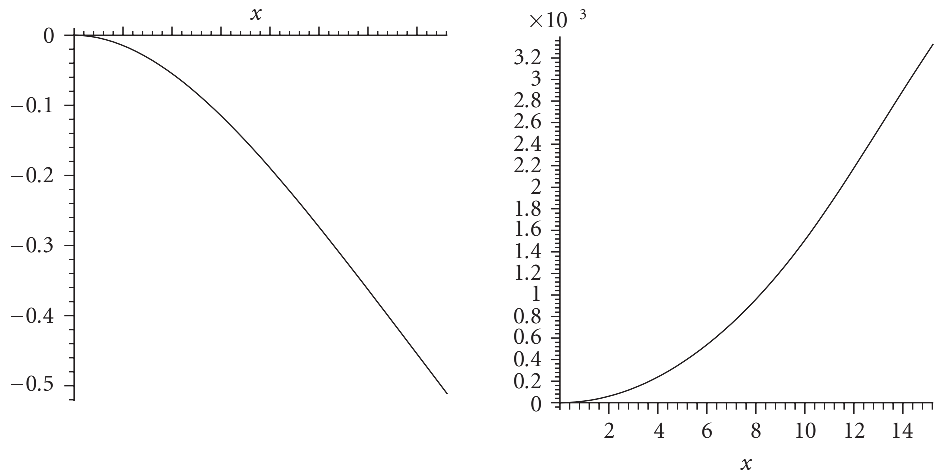
(b) Imaginary part

Figure 6.4. Third mode to double-span beam.

For the second case, we consider that the cantilever beam in Figure 2.1 is nonuniform. Its parameters values are given in Table 6.5. The first three eigenvalues of the beam are listed in Table 6.6.

Table 6.6. Eigenvalues (rad/s) to triple-span beam.

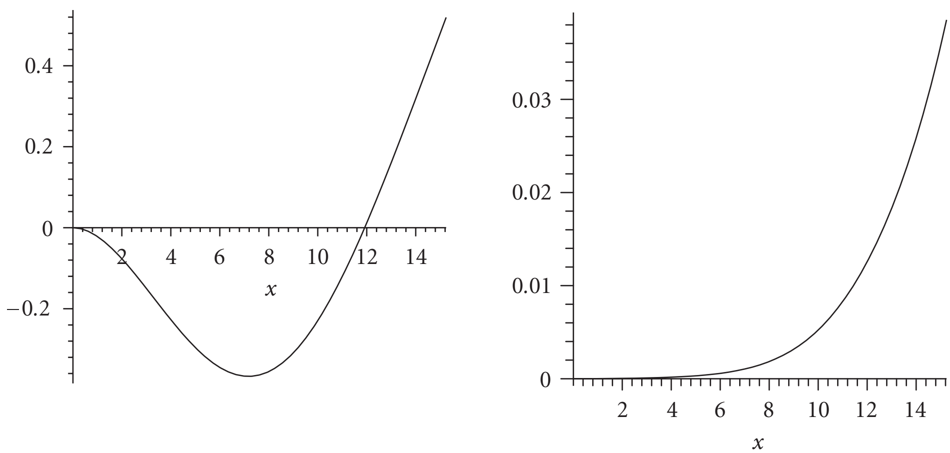
Mode (n)	Eigenvalues
1	−.4314672830 ± 49.72926784 I
2	−8.906552965 ± 278.6470011 I
3	−16.73690547 ± 797.5457311 I



(a) Real part

(b) Imaginary part

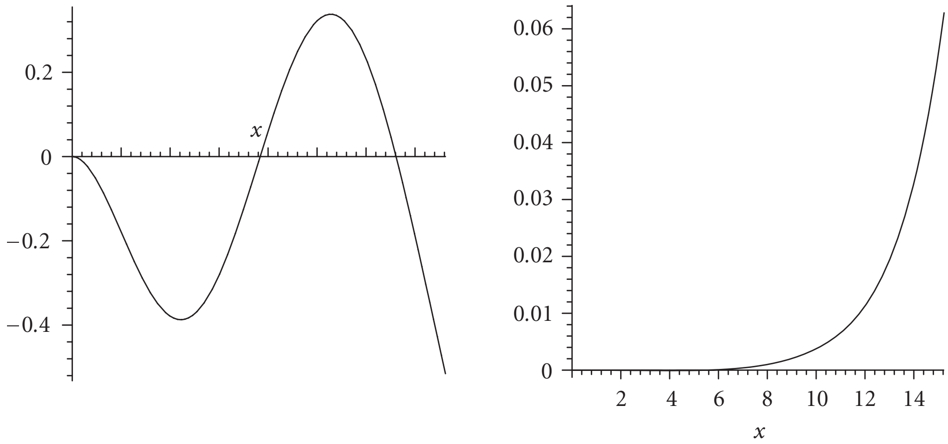
Figure 6.5. First mode of a uniform triple-span beam.



(a) Real part

(b) Imaginary part

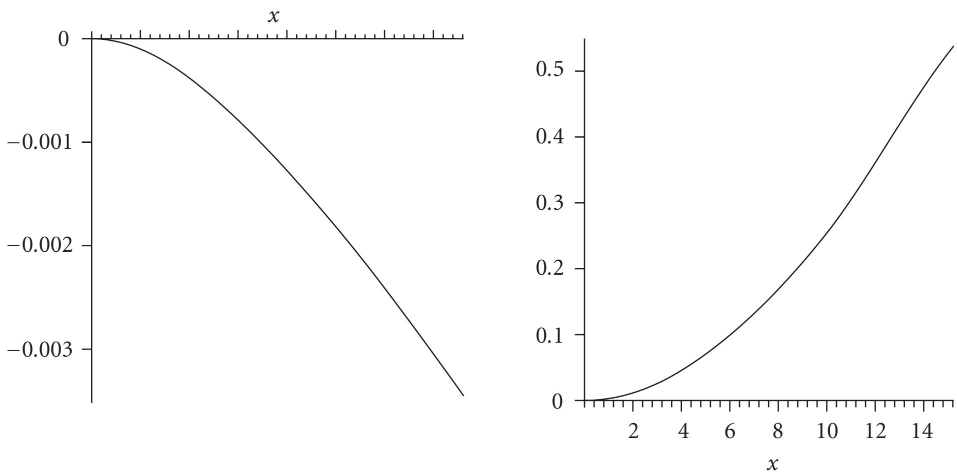
Figure 6.6. Second mode of a uniform triple-span beam.



(a) Real part

(b) Imaginary part

Figure 6.7. Third mode of a uniform triple-span beam.



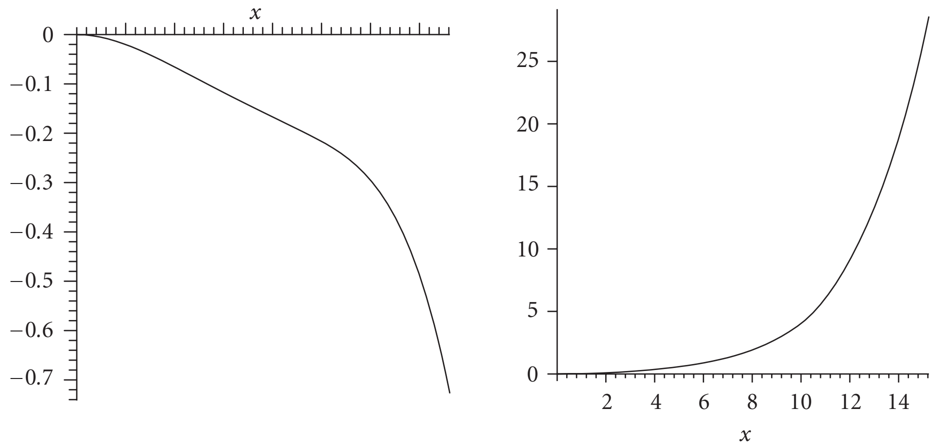
(a) Real part

(b) Imaginary part

Figure 6.8. First mode to triple-span beam.

In Figures 6.8, 6.9, and 6.10 are plotted the first three shape modes corresponding to the first three eigenvalues of the beam, where (a) it indicates the real part of the mode and (b) the imaginary part of the mode.

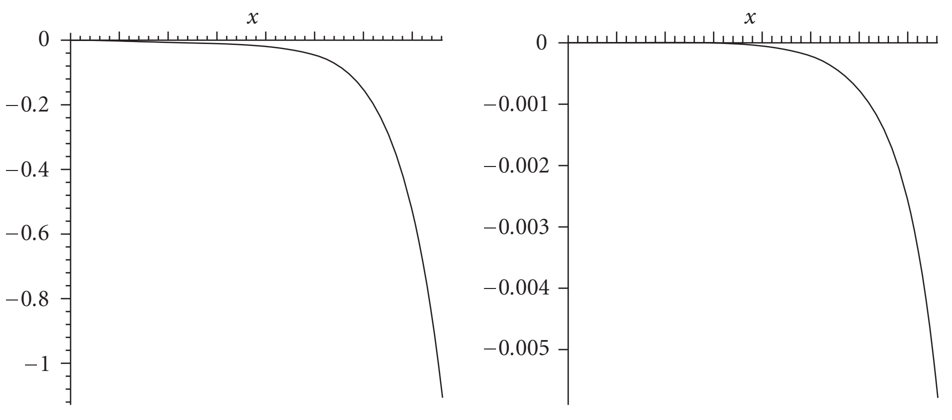
We can observe the effect of varying the parameters values in each segment of the beam on the modes shapes. The second and third modes are quite different from those of the uniform beam. This means that a beam with different sections some how influences



(a) Real part

(b) Imaginary part

Figure 6.9. Second mode to triple-span beam.



(a) Real part

(b) Imaginary part

Figure 6.10. Third mode to triple-span beam.

more the modes than external devices such as lumped mass, lumped stiffness, and lumped damping.

7. Conclusion

We have considered the study of the eigenanalysis of a triple-span Euler-Bernoulli beam subject to internal and external damping and to intermediate devices by keeping the original second-order Newtonian formulation. We also employed a matrix formulation that allows to observe the influence of the boundary and intermediate continuity conditions of the beam. Also, the values of a solution basis of the fourth-order differential equation for each segment. By choosing the elements of the basis in each segment as a convenient translation of the elements of a fundamental basis for the first segment, computations are reduced. This fundamental later is generated by a specific initial-value solution and its first three derivatives. The matrix method avoids the use of the first-order state formulation or to rely on the Euler basis of a differential equation of fourth order. It also allows to envision how conditions will influence a chosen basis.

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