

## GALERKIN LEAST-SQUARES APPROXIMATIONS FOR FLOWS OF CASSON FLUIDS THROUGH AN EXPANSION

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**Abstract.** Among non-newtonian fluid models, purely viscous constitutive equations play an important role in industrial applications regardless their lack of accuracy in non-viscometric flows. In this work we are concerned with the flow of viscoplastic shear-thinning fluids in complex geometries. Viscoplastic fluids are those that behave as a solid when submitted to low stresses and flow when submitted to stresses higher than a yield stress value. Usually, they also present shear-thinning behavior. Fluids such as molten chocolate, xanthan gum solutions, blood, wastewater sludges, muds, and polymer solutions present viscoplastic shear-thinning features. In order to give numerical approximations for viscoplastic shear-thinning flows we first describe a mechanical model based on continuum mechanics conservation laws of mass and momentum. The description of material behavior is such as to respect certain principles of objectivity and generality in continuum mechanics. The constitutive equation of Casson models viscoplasticity and shear-thinning. The numerical approximation of the equations is performed by a finite element method. To prevent the model from pathologies known for the classic Galerkin method, we chose to employ a stabilized method based on a Galerkin least-squares (GLS) scheme, which is designed to circumvent Babuška-Brezzi condition and deal with the asymmetry of the advective operator. We present approximations for the flow through a planar 4:1 sudden expansion. We investigate the influence of Reynolds and Casson numbers on the flow dynamics.

**Keywords.** Continuum mechanics, Casson, Galerkin least-squares, finite elements, viscoplastic fluids

### 1. Introduction

In the study of the flow of non-newtonian liquids, one has to choose among a gamut of constitutive equations available a model with the skill of describing features of interest of the fluid's behavior under specific flow conditions. We agree that, in most cases, there is no appropriate fluid classification, but rather some features of a fluid non-linear behavior under specific flow conditions that are the most relevant in order to predict variables of interest. For example, the experimentally observed phenomena of shear-thinning (pseudoplasticity) or shear-thickening is fitted by purely viscous constitutive equations for viscometric flows that do not account for effects such as normal stresses differences, time dependence or memory (Bird et al., 1987). Some of these models also describe a yield stress, i.e., a level of shear stress under which the material would not deform, and beyond which the material would have a fluid-like behavior (Papanastasiou, 1987). In fact, the yield stress is a model for the behavior of some structured liquids that present a dramatic change in mechanical properties inside a small range of stress, as pointed out by Barnes (1999) and references therein. These fluids are called viscoplastic.

In many industrial applications, shear-thinning and viscoplasticity represent the most relevant non-newtonian features when predicting flow variables of engineering interest. As examples of materials exhibiting these features we have molten chocolate (Chevalley, 1991), xanthan gum solutions (Casas et al., 2000), ketchup (Macosko, 1994), blood (Cokelet et al., 1969). In this work, we employ the Casson constitutive equation (Casson, 1959) to model that behavior, and we refer to those materials as Casson fluids. The Casson model is claimed to fit rheological data better than general viscoplastic models for many materials, and is the preferred rheological model for blood and chocolate (Joye, 2003).

The aim of this work is to generate numerical approximations for the flow of Casson fluids through a planar expansion. This geometry adds a geometric non-linearity to the flow problem, and is useful to study the stability of the numerical method and the features of the flow dynamics.

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The flows of viscoplastic materials inside ducts is known to form a plug flow region due to the low shear stress far from the walls (Bird et al., 1983). The flow of viscoplastic fluids in planar expansions have been investigated by authors such as Scott et al. (1988), who used the classical Galerkin method to obtain results for Bingham and Casson fluids. Vradis & Ötügen (1997) used a finite-difference scheme to study the flow of a Bingham material through a 1:2 sudden expansion, observing the growth of the reattachment length with the increase of Reynolds numbers and the decrease of recirculation with the increase of yield stress. The same authors in Hammad et al. (2001) studied the problem using a Hershel-Bulkley model and concluded that the flow field was strongly dependent on the yield stress, but weakly dependent on the power-law index of the model. Pak et al. (1990) observed experimentally that the reattachment length for purely viscous fluids was almost the same as that for newtonian fluids. Abdali et al. (1992) approximated via finite element method the contraction and exit flows from axisymmetric and plane ducts using the Bingham model and showing the yielded and unyielded regions. Pham & Mitsoulis (1994) studied entry and exit flows of Casson fluids using Papanastasiou modified equation (Papanastasiou, 1987) for the Casson model and the finite element method to approximate the equations. They showed results depicting zones of yielded and unyielded material and proposed correlations for the correction factor of pressure loss. Reis Junior & Naccache (2003) studied viscoplastic flow in contractions using the Carreau equation with a very high low-shear-rate viscosity and using a finite volume method. Neofytou & Drikakis (2003) investigated the flow of three fluid models (Casson, Quemada and power-law) in a sudden expansion channel. They did not make use of the symmetry conditions and determined instabilities and asymmetry for characteristic flow numbers higher than some critical values.

The numerical difficulty to deal with the von Mises criteria (Bird et al., 1983) has been dealt by modifications of the constitutive equation: some authors use the bi-viscosity model (Gartling & Phan-Thien, 1984; Naccache & Mendes, 1997) and others the Papanastasiou model (Papanastasiou, 1987), this latter which we used in this work.

We analyze numerically the flow via the finite element approximation for the mixed variational problem. The stabilized method Galerkin least-squares (GLS) (Hughes et al., 1986) is employed in order to avoid undesirable pathologies (spurious oscillations, locking) to which the classical Galerkin formulation would be susceptible in mixed formulations. The Galerkin method in fluids suffers from two major difficulties. First, the need to satisfy Babuška-Brezzi condition (Ciarlet, 1978) in order to achieve a compatible combination of velocity and pressure subspaces. Further, the inherent instability of central difference schemes in approximating advective dominated equations (Brooks & Hughes, 1982). The GLS method gives stability to the original Galerkin formulation by adding terms evaluated elementwise that are obtained by minimizing the square of the equation residual (Hughes et al., 1989). The magnitude of these terms is controlled by stability parameters that are designed to optimize stability and convergence (Franca & Frey, 1992). The GLS method has the ability to circumvent Babuška-Brezzi condition and to generate stable approximations for highly advective flows preserving good accuracy properties. This is achieved by adding residual-based terms to the classic Galerkin formulation, retaining its weighted residual structure and not damaging its consistency. The GLS formulation we employ is the one proposed by Franca & Frey (1992) for the Navier-Stokes problem, which its linearized steady counterpart reduces to the GLS scheme proposed by Hughes et al. (1989) in the case of zero pressure.

In this work we use the notation put briefly as follows. The problems are defined on a bounded domain  $\Omega \subset \mathcal{R}^{nsd=2,3}$  with a polygonal or polyhedral boundary  $\Gamma$

$$\begin{cases} \Gamma = \bar{\Gamma}_g \cup \bar{\Gamma}_h, \\ \Gamma_g \cap \Gamma_h = \emptyset, \quad \Gamma_g = 0 \end{cases} \quad (1)$$

where  $\Gamma_g$  is the region on which essential (Dirichlet) conditions are imposed and  $\Gamma_h$  is the region is subjected to natural (Neumann) boundary conditions.

A partition  $C_h$  of  $\bar{\Omega}$  into elements consisting of triangles, tetrahedrons, convex quadrilaterals or hexahedrons is performed in the usual way: no overlapping is allowed between any two elements, the union of all element domains  $\Omega_K$  reproduces  $\bar{\Omega}$  and a combination of triangles and quadrilaterals for the two-dimensional case can be accommodated. Quasiuniformity is not assumed,

$$\begin{cases} \bar{\Omega} = \bigcup_{K \in C_h} \bar{\Omega}_K \\ \Omega_{K_1} \cap \Omega_{K_2} = \emptyset \quad \forall K_1, K_2 \in C_h \end{cases} \quad (2)$$

For polynomial spaces, we adopt:

$$R_m(K) = \begin{cases} P_m(K), & \text{if } K \text{ is a triangle or tetrahedron,} \\ Q_m(K), & \text{if } K \text{ is a quadrilateral or hexaedron} \end{cases} \quad (3)$$

where for each integer  $m \geq 0$ ,  $P_m$  and  $Q_m$  have the usual meaning (Ciarlet, 1978). The function spaces employed are:

$$\begin{aligned} L^2(\Omega) &= \{ q \mid \int_{\Omega} q^2 d\Omega < \infty \} \\ L_0^2(\Omega) &= \{ q \in L^2(\Omega) \mid \int_{\Omega} q d\Omega = 0 \} \\ H^1(\Omega) &= \{ v \in L^2(\Omega) \mid \partial v / \partial x_i \in L^2(\Omega) \}, \quad i = 1, nsd \\ H_0^1(\Omega) &= \{ v \in L^2(\Omega) \mid \partial v / \partial x_i \in L^2(\Omega) \mid v = 0 \text{ on } \Gamma_g \}, \quad i = 1, nsd \end{aligned} \quad (4)$$

## 2. Mechanical model

The mechanical model presented herein concerns a material body  $B$  for which flow is defined by the triple velocity, mass density and stress tensor ( $\mathbf{u}$ ,  $\rho$ ,  $\mathbf{T}$ ) and the associated force system of contact and body forces ( $\mathbf{t}(\mathbf{n})$ ,  $\mathbf{f}$ ). We write the following equations for a region  $\Omega$  of frontier  $\Gamma$  of a body  $B$ .

### 2.1 Principle of mass conservation

The mass of a body does not change with time. Mathematically,

$$\frac{d}{dt} \int_{\Omega} \rho d\Omega = 0 \quad (5)$$

Applying Reynolds Transport Theorem (Truesdell & Toupin, 1960), we obtain the differential mass conservation equation:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \quad (6)$$

where  $\mathbf{u}$  denotes the velocity field. For incompressible fluids, the motion is called isochoric,  $\rho$  is a constant and the weak form of the continuity equation is given by:

$$\int_{\Omega} q \text{div} \mathbf{u} = 0 \quad (7)$$

### 2.3 Cauchy's theorem

Gurtin (1981): This theorem concerns the form of the contact forces. It asserts the linearity of  $\mathbf{t}(\mathbf{n})$  in  $\mathbf{n}$ ,  $\mathbf{n}$  is the unit vector outward to any surface. It asserts that a necessary and sufficient condition that the momentum balance laws be satisfied is that exists a spatial tensor field  $\mathbf{T}$  (stress tensor) such that for each unit vector  $\mathbf{n}$ :

$$i) \quad \mathbf{t}(\mathbf{n}) = \mathbf{T} \mathbf{n} \quad (8)$$

ii)  $\mathbf{T}$  is symmetric;

iii)  $\mathbf{T}$  satisfies the equation:

$$\text{div} \mathbf{T} + \mathbf{f} = \rho \dot{\mathbf{u}} \quad (9)$$

### 2.4 Principle of power expended

It is a consequence of the laws of momentum balance (Gurtin, 1981). It asserts that, for any material volume  $\Omega_t \subset B_t$  (where  $B_t$  represents the configuration of a material body  $B$  in the time  $t$ ) and  $V(\Omega_t)$  the space of virtual velocities associated to  $\Omega_t$ , where  $V(\Omega_t) = \{ \mathbf{v}: \Omega_t \rightarrow \mathcal{R}^{nsd} \mid \nabla \mathbf{v} \text{ is defined in } \Omega_t \}$ :

i) The power expended on  $\Omega_t$  by body forces ( $\mathbf{f}$ ) and surface forces ( $\mathbf{t}(\mathbf{n})$ ) is equal to the stress power plus the rate of change of kinetic energy:

$$P_i(\Omega_t, \mathbf{v}) + P_a(\Omega_t, \mathbf{v}) = P_e(\Omega_t, \mathbf{v}), \quad \forall \mathbf{v} \in V(\Omega_t) \quad (10)$$

where the stress power is

$$P_i(\Omega_t, \mathbf{v}) = \int_{\Omega_t} \mathbf{T} \cdot \nabla \mathbf{v} \, d\Omega_t \quad (11)$$

the kinetic energy power is

$$P_a(\Omega_t, \mathbf{v}) = \int_{\Omega_t} \rho \dot{\mathbf{v}} \cdot \mathbf{v} \, d\Omega_t \quad (12)$$

and the external forces power is

$$P_e(\Omega_t, \mathbf{v}) = \int_{\Omega_t} \rho \mathbf{f} \cdot \mathbf{v} \, d\Omega_t + \int_{\Gamma_t} \mathbf{t}(\mathbf{n}) \cdot \mathbf{v} \, d\Gamma_t \quad (13)$$

The following notation is used:  $\mathbf{T}$  the stress tensor;  $\mathbf{D}$  is called the rate of strain, and denotes the symmetric part of the relative velocity tensor,  $\nabla \mathbf{v}$ .

ii) For every rigid displacement ( $\mathbf{v}(\mathbf{x}, t) = \mathbf{W}(t)(\mathbf{x} - \mathbf{x}_0) + \mathbf{c}(t)$ ), with  $\mathbf{W} = -\mathbf{W}^T$ , it follows that  $P_i(\Omega_t, \mathbf{v}) = 0$ .

The alternative form of the PPE to enunciate the axioms of conservation in continuum mechanics has the advantage of giving a variational formulation for mechanical, allowing a natural and intuitive introduction of variational methods, in particular the Finite Element Method, which was used in this study and is exploited afterwards.

## 2.5 Material behavior

Although Cauchy's Theorem describes the form of contact forces, stated for any continuous body, the way in which materials deform or flow when submitted any dynamic condition is not stated by this theorem. Besides, the behavior of continuous bodies submitted to arbitrary conditions differ drastically, due to the material dependent relation between contact forces within the body (accounted by the stress tensor,  $\mathbf{T}$ ) upon its motion and deformation. This relation is described mathematically by the called rheological or material constitutive equations. The constitutive equations are constructed in order to obey certain rules that assure their physical meaning and generality. These rules are summarized in three principles, according to Astarita & Marrucci (1974) and Slattery (1999):

i) Principle of determinism: The stress in a body is determined by the history of the motion that the body has previously undergone.

ii) Principle of local action: The contact forces upon a material point  $P$  are determined uniquely by its closest neighborhood.

iii) Principle of frame indifference: By the Stokes' postulate of isotropy of continuous bodies (White, 1974), the description of material behavior is invariant under changes of frame of reference.

iv) Principle of fading memory: the influence on past deformations on the present stress is weaker for the distant past than for the recent past (Astarita & Marrucci, 1974), an important statement for materials with memory.

v) Satisfaction of the second law of thermodynamics. This requirement an important feature for complex constitutive equations. It is satisfied when the viscous dissipation term in the energy equation remains always positive (Astarita & Marrucci, 1974; Slattery, 1999).

Being so, the dependence of  $\mathbf{T}$  upon the body's motion and deformation might give a function of the type

$$\mathbf{T} = \mathbf{T}(\mathbf{u}, \nabla \mathbf{u}) \quad (14)$$

Every tensor may be expressed uniquely as the sum of a symmetric tensor and a skew-symmetric tensor (Gurtin, 1981). For the relative velocity tensor ( $\nabla \mathbf{u}$ ), it is defined as the sum of the symmetric tensor,  $\mathbf{D}$ , the rate of strain; and the skew-symmetric tensor,  $\mathbf{W}$ , the vorticity. The rate of strain,  $\mathbf{D}$ , describes the motions of volumetric dilatation and distortions. As it is a symmetric tensor, it is isotropic, or frame indifferent. The tensor,  $\mathbf{W}$ , is not frame indifferent, it describes the motions of pure translation and rigid body rotation, which do not cause deformation. The velocity field,  $\mathbf{u}$ , is not also a frame indifferent field. Therefore, a constitutive equation for  $\mathbf{T}$  must be written in terms of  $\mathbf{D}$ , and besides, this equation must also be an isotropic function, i.e., must be a frame indifferent tensor function (Gurtin, 1981). The simplest constitutive equation that obeys those rules is a linear relation between  $\mathbf{T}$  and  $\mathbf{D}$ , which is written based on the representation theorem for isotropic linear tensor functions, which guarantees its isotropy (Gurtin, 1981):

$$\mathbf{T}(\mathbf{D}) = \psi(\text{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D} \quad (15)$$

where  $\psi$  is a function of  $\text{tr}\mathbf{D}$ , the first invariant of  $\mathbf{D}$ ,  $I_D$  ( $I_D = \text{tr}\mathbf{D}$ ). By the definition of  $\mathbf{D}$  as the symmetric part of  $\nabla \mathbf{u}$ , it is easy to find that  $\text{tr}\mathbf{D} = \text{div}\mathbf{u}$ . When the fluid is at rest ( $\mathbf{D} = \mathbf{0}$ ), it develops a uniform field of hydrostatic stress, identical to the thermodynamic pressure,  $P$ . In this case,

$$\mathbf{T} = -P\mathbf{I} \quad (16)$$

In this form, one can write  $\psi$  as a linear function of  $I_D$ :

$$\psi = -P + \lambda \operatorname{div} \mathbf{u} \quad (17)$$

Since  $\psi$  is associated only with volume expansion, it is usually called coefficient of bulk viscosity (White, 1974). The physical significance of  $\psi$  is connected with the mechanism of dissipation when the volume of the fluid element is changed at a finite rate as well as with the relation between the total stress tensor and the thermodynamic pressure (Schlichting, 1968). In view, of Eqs. (15) and (17), the deformation law, which was first given by Stokes, in 1945 (White, 1974), is written as:

$$\mathbf{T} = (-P + \lambda \operatorname{tr} \mathbf{D}) \mathbf{I} + 2\mu \mathbf{D} \quad (18)$$

Equation (18) brings some consequences that were the basis for the establishment of the Stokes' hypothesis. First, is the definition of the mean pressure,  $p$ , as the mean of the normal components of  $\mathbf{T}$ . After Eq.(18),  $p$  is defined as

$$p \equiv -\frac{1}{3} \operatorname{tr} \mathbf{T} = P - (\lambda + \frac{2}{3} \mu) \operatorname{tr} \mathbf{D} \quad (19)$$

which means that the pressure for a viscous deforming material is not equal to the thermodynamic pressure. For incompressible materials, as  $\operatorname{tr} \mathbf{D} = \operatorname{div} \mathbf{u} = 0$ , this difference is not a concern. For compressible materials, usually the Stokes hypothesis is employed. It assumes

$$\lambda + \frac{2}{3} \mu = 0 \quad (20)$$

and the problem is overcome. Stokes hypothesis is supported by the monoatomic gas theory, but causes controversial among scientists because in cases where viscous normal stresses are important it already has been indicated its invalidity (Landau & Lifshitz 1959; White, 1974). Also after the concept of mean pressure, one can define a deviatoric stress field,  $\mathbf{S}$ , as

$$\mathbf{S} \equiv \mathbf{T} + p \mathbf{I} \quad (21)$$

where  $\mathbf{S}$  is the viscous part of the stress tensor, i.e., the components of stress that arise in motion and which produces dissipation by friction.

Being so, the previous assumptions allow the linear constitutive equation in Eq.(15) to be written for an incompressible material in the form

$$\mathbf{T} = -p \mathbf{I} + 2\mu \mathbf{D} \quad (22)$$

which is the *newtonian fluid model*, or the constitutive equation for a *newtonian fluid*. It describes the linear behavior shown by many common fluids (water, oils, milk, honey) of stress upon deformation.

The concept of generalized newtonian fluid was created to explain the non-newtonian behavior similar to the linear newtonian behavior of some fluids. This concept is useful in describing a large scope of fluids in isochoric motion (incompressible cases) in pure shear flows. The constitutive equations known as generalized fluid models are those in which the relation between  $\mathbf{T}$  and  $\mathbf{D}$  is not linear, but may be represented as so, in the form:

$$\mathbf{T} = -p \mathbf{I} + 2\eta(\dot{\gamma}) \mathbf{D} \quad (23)$$

where  $\eta(\dot{\gamma})$  is the viscosity function, given by an empiric or a theoretical model, and the scalar  $\dot{\gamma}$  is defined as the magnitude of the rate of deformation tensor (Slattery, 1999):

$$\dot{\gamma} = (2 \Pi_D)^{1/2} = (2 \operatorname{tr} \mathbf{D}^2)^{1/2} \quad (24)$$

The stress magnitude is given by

$$\tau = (1/2 \operatorname{tr} \mathbf{S}^2)^{1/2} \quad (25)$$

Empiric models based on Eq.(23) can predict the behavior of viscometric flows, in which the material particle are subjected to a constant deformation history, without memory effects (Bird et al., 1987). Besides viscometric flows are

characterized by the absence of normal stresses, being known then as shear flows. As examples of shear flows there are the flow in a closed channel, Couette flow, cone-and-plate viscometer steady flow.

In this work we employ as the constitutive equation for  $\mathbf{T}$  a general newtonian model using the Casson equation (Casson, 1959) for the viscosity function. The Casson model fits the viscoplastic behavior of fluids by the stress-deformation relation below:

$$\begin{aligned} \tau^{1/2} &= \tau_0^{1/2} + (\eta_0 \dot{\gamma})^{1/2} & \text{for } |\tau| > \tau_0 \\ \dot{\gamma} &= 0 & \text{for } |\tau| \leq \tau_0 \end{aligned} \quad (26)$$

where  $\tau_0$  is the yield stress and  $\eta_0$  the plastic viscosity. To avoid the discontinuity inherent to the von Mises criterion for the magnitude of the shear stress, Papanastasiou (1987) proposed a modification for viscoplastic models that when applied to the Casson equation gives the following modified Casson viscosity function:

$$\eta^{1/2} = \eta_0^{1/2} + (\tau_0 / \dot{\gamma})^{1/2} \left[ 1 - \exp(-m \dot{\gamma}^{1/2}) \right] \quad (27)$$

This equation is valid both in yielded and unyielded regions, and the exponent  $m$  controls the smoothness of the function. For  $m \geq 100$ , Eq. (27) mimics the original Casson model (Pham & Mitsoulis, 1994), as shown in Fig.(1) in a shear stress versus shear rate graph, for  $\tau_0=3$   $\eta_0=1$ .

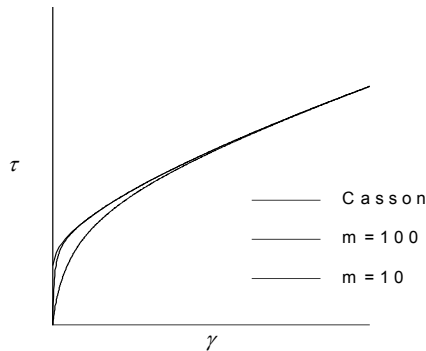


Figure 1: Papanastasiou (1987) approximation for the Casson model

Thus, a Casson flow may be characterized by two dimensionless parameters,  $Re_{Ca}$ , the Reynolds number, and  $Ca$ , the Casson number (Pham & Mitsoulis, 1994). These are defined as defined as:

$$Re_{Ca} = \frac{Lu_0 \rho}{\eta_0}; \quad Ca = \frac{\tau_0 L}{\eta_0 u_0} \quad (28)$$

where  $L$  is a characteristic length and  $u_0$  a characteristic velocity. The magnitude of Reynolds number gives the relative importance of the advection term over the diffusivity term, in the process of momentum transport.

We assume a constant viscosity  $\eta$ , in order to reformulate the term of stress power on the conservation of momentum equation, Eq.(10), and integrate it by parts as follows:

$$\begin{aligned} \int_{\Omega} \mathbf{T} \cdot \nabla \mathbf{v} d\Omega &= \int_{\Omega} (-p \mathbf{I} + 2\eta \mathbf{D}(\mathbf{u})) \cdot \nabla \mathbf{v} d\Omega \\ &= \int_{\Omega} -p \mathbf{I} \cdot \nabla \mathbf{v} d\Omega + \int_{\Omega} 2\eta \mathbf{D}(\mathbf{u}) \cdot \nabla \mathbf{v} d\Omega \\ &= \int_{\Omega} -p \operatorname{div} \mathbf{v} d\Omega + \int_{\Omega} 2\eta \mathbf{D}(\mathbf{u}) \cdot \mathbf{D}(\mathbf{v}) d\Omega \end{aligned} \quad (29)$$

Then we use the generalized newtonian assumption for  $\mathbf{T}$  and give a non-dimensional form for the mechanical model comprehending the mass and momentum balance:

$$\int_{\Omega} \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} d\Omega + \int_{\Omega} [\nabla \mathbf{u}] \cdot \mathbf{v} d\Omega + \int_{\Omega} 2 Re^{-1} \eta^*(\dot{\gamma}) \mathbf{D}(\mathbf{u}) \cdot \mathbf{D}(\mathbf{v}) d\Omega - \int_{\Omega} p \operatorname{div} \mathbf{v} d\Omega - \int_{\Omega} q \operatorname{div} \mathbf{u} d\Omega = \int_{\Omega} Fr^{-2} \mathbf{f} \cdot \mathbf{v} d\Omega + \int_{\Gamma_h} \mathbf{t}_h \cdot \mathbf{v} d\Gamma_h \quad (30)$$

where the variables have been normalized as follows, and the definition of Froude, Fr, number is given:

$$\mathbf{x} = \frac{\mathbf{x}}{L}; \quad \mathbf{u} = \frac{\mathbf{u}}{u_0}; \quad t = \frac{t}{L/u_0}; \quad p = \frac{p}{\rho/u_0^2}; \quad \mathbf{f} = \frac{\mathbf{f}}{f}; \quad \eta^*(\dot{\gamma}) = \frac{\eta(\dot{\gamma})}{\eta_0}; \quad \text{Fr} = \frac{u_0}{\sqrt{Lf}} \quad (31)$$

where  $f$  is a characteristic body force and  $\eta^*(\dot{\gamma})$  represents a dimensionless shear-stress dependent viscosity.

### 3. Finite Element Approximation

The mechanical modeling in Eq.(30) was approximated by a stabilized finite element method, namely GLS (Hughes et al., 1986) The approximation is performed over the partition  $C^h$  of the closed domain  $\bar{\Omega}$ . The usual spaces of functions (Ciarlet, 1978) are employed to define the finite element subspaces to approximate the velocity field ( $\mathbf{V}^h$ ), the pressure field ( $P^h$ ) (Franca & Frey, 1992):

$$\begin{aligned} \mathbf{V}^h &= \{ \mathbf{v} \in H_0^1(\Omega)^{nsd} \mid \mathbf{v}|_K \in R_k(K)^{nsd}, K \in C^h \} \\ P^h &= \{ p \in C^0(\Omega) \cap L_0^2(\Omega) \mid p|_K \in R_l(K), K \in C^h \} \\ \mathbf{V}_h^g &= \{ \mathbf{v}(\cdot, t) \in H^1(\Omega)^{nsd}, t \in [0, t_\infty] \mid \mathbf{v}|_K \in R_k(K)^{nsd}, K \in C^h, \mathbf{v}(\cdot, t) = \mathbf{u}_g \text{ on } \Gamma_g \} \end{aligned} \quad (32)$$

The functions  $\mathbf{u}$  and  $p$  are approximated by functions  $\mathbf{u}^h$  and  $p^h$ , in which  $h$  is referred to the association of these functions with a the partition or mesh  $C_h$ , parameterized by a characteristic length  $h$ . The finite element approximation for the model presented previously, using a GLS stabilized method (Hughes et al., 1986; Franca & Frey, 1992), is:  
Find the pair  $(\mathbf{u}^h, p^h) \in \mathbf{V}_h \times P_h$  such as:

$$B(\mathbf{u}^h, p^h; \mathbf{v}, q) = F(\mathbf{v}, q), \quad (\mathbf{v}, q) \in \mathbf{V}_h \times P_h \quad (33)$$

where

$$\begin{aligned} B(\mathbf{u}, p; \mathbf{v}, q) &= \int_{\Omega} \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} \, d\Omega + \int_{\Omega} [\nabla \mathbf{u}] \mathbf{u} \cdot \mathbf{v} \, d\Omega + \int_{\Omega} 2 \text{Re}^{-1} \eta^*(\dot{\gamma}) \mathbf{D}(\mathbf{u}) \cdot \mathbf{D}(\mathbf{v}) \, d\Omega - \int_{\Omega} p \, \text{div} \, \mathbf{v} \, d\Omega - \int_{\Omega} q \, \text{div} \, \mathbf{u} \, d\Omega + \\ &\sum_{K \in C_h} \int_{\Omega_K} \left( \frac{\partial \mathbf{u}}{\partial t} + [\nabla \mathbf{u}] \mathbf{u} + \nabla p - 2 \text{Re}^{-1} \text{div}(\eta^*(\dot{\gamma}) \mathbf{D}(\mathbf{u})) \right) \cdot \left( \tau(\text{Re}_K)([\nabla \mathbf{v}] \mathbf{u} - 2 \text{Re}^{-1} \text{div}(\eta^*(\dot{\gamma}) \mathbf{D}(\mathbf{v})) - \nabla q) \right) \, d\Omega_K \\ F(\mathbf{v}, q) &= \int_{\Omega} \text{Fr}^{-2} \mathbf{f} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma} \mathbf{t} \cdot \mathbf{v} \, d\Gamma + \sum_{K \in C_h} \int_{\Omega_K} \text{Fr}^{-2} \mathbf{f} \cdot (\tau(\text{Re}_K)([\nabla \mathbf{v}] \mathbf{u} - 2 \text{Re}^{-1} \text{div}(\eta^*(\dot{\gamma}) \mathbf{D}(\mathbf{v})) - \nabla q)) \, d\Omega_K \end{aligned} \quad (34)$$

where the terms before the sums are original terms in Galerkin formulation, and the terms within the sums correspond to added stabilized terms, that are evaluated elementwise. The stability parameter for the velocity-pressure terms is gives as in Franca & Frey (1992):

$$\tau(\text{Re}_K) = \frac{h_K}{2|\mathbf{u}|_p} \xi(\text{Re}_K); \quad (35)$$

where

$$\begin{aligned} \xi(\text{Re}_K) &= \begin{cases} \text{Re}_K, & 0 \leq \text{Re}_K < 1 \\ 1, & \text{Re}_K \geq 1 \end{cases}, \\ \text{Re}_K &= \frac{m_k |\mathbf{u}|_p h_K}{4\eta(\dot{\gamma})/\rho}, \\ m_k &= \min \{ 1/3, 2C_k \}. \end{aligned} \quad (36)$$

The  $p$  norm is defined as in Franca & Frey (1992), and constants  $C_k$  derives from inverse estimate (Ciarlet, 1978). As indicated in Franca et al. (1992),  $C_k = \infty$  for a bilinear element and  $C_k = 1/24$  for a biquadratic one.

The functions  $\mathbf{u}^h, p^h$  are substituted by finite element expansions, resulting in an algebraic system of equations such as:

$$\begin{aligned}
 [\mathbf{M} + \mathbf{M}_\tau^y] \mathbf{a} + \mathbf{N}(\mathbf{u}) + \mathbf{N}_\tau^y(\mathbf{u}) + [\mathbf{K} + \mathbf{K}_\tau^y] \mathbf{u} + [\mathbf{G} + \mathbf{G}_\tau^y] \mathbf{p} &= \mathbf{F} + \mathbf{F}_\tau \\
 [\mathbf{M}_\tau^q] \mathbf{a} + \mathbf{N}_\tau^y(\mathbf{u}) + [\mathbf{G}^T + \mathbf{K}_\tau^q] \mathbf{u} + [\mathbf{G}_\tau^q] \mathbf{p} &= \mathbf{E}_\tau
 \end{aligned}
 \tag{37}$$

where  $\mathbf{u}$ ,  $p$  are the degrees of freedom for  $\mathbf{u}^h$ ,  $p^h$ , respectively,  $\mathbf{a}$  is the vector of degrees of freedom for the transient term  $\partial \mathbf{u}^h / \partial t$ . The matrices  $[\mathbf{M}]$ ,  $[\mathbf{K}]$ ,  $[\mathbf{G}]$  arise from the transient, viscous and pressure terms, respectively.  $\mathbf{N}(\mathbf{u})$  arise from the advective transport terms. The matrices on the right-hand-side are the font terms, and all the other matrices correspond to the stabilized terms. The time integration is performed after Franca & Frey (1992), where the incremental matrices are simmetrized as in Tezduyar et al. (1992) The viscosity function is updated at each time step using the values for the components of  $\mathbf{D}(\mathbf{u})$  from the last iteration.

#### 4. Numerical results

We performed numerical tests for the GLS approximation of Eq.(33). The problem is the flow over a 4:1 planar expansion. As the geometry is symmetric, we perform tests and show only the upper half geometry, employing a symmetry boundary condition. A plane velocity profile equal to  $u_0$  is prescribed at the flow inlet, at  $x_1=0$ , and this is taken as the characteristic flow velocity. At the walls non-slippery is prescribed, i.e., all velocities vanish at the walls. At the outlet, the flow is assumed completely developed and free traction is imposed. To have an initial condition, we solve the flow for the equivalent Stokes problem and use that result. We march in time according to the algorithm proposed by Franca & Frey (1992) and assume that steady-state is achieved when the residuals for all variables have values no higher than  $10^{-5}$ . We tested that formulation in the sudden expansion problem for  $Ca=3, 5, 10$  and for  $Re_{Ca}=1, 10$ . The results shown here were performed using a mesh consisting of 3260  $Q_1/Q_1$  finite elements (3433 nodes), which are non-conforming in a Babuška-Brezzi condition sense. All computations were performed using the resources of the Laboratory of Applied and Computational Fluid Mechanics (LAMAC-UFRGS), which are two Pentium III 1.0 GHz. The GLS method was able to deal and yield stable results.

The finite element mesh is shown in Fig. 2. In regions near the walls and near the expansion the mesh is refined to capture the highest gradients and flow details.

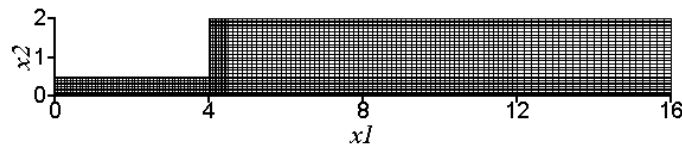


Figure 2: Finite element mesh

The flows of viscoplastic materials inside ducts is known to form a plug flow region due to the low shear stress far from the walls (Bird et al., 1983) In Fig. 3 we show the grow of the plug flow region with the increase of  $Ca$  for  $Re_{Ca}=1$ . For comparison, for a newtonian fluid would give a parabolic profile with  $u_1/u_0=1.5$ .

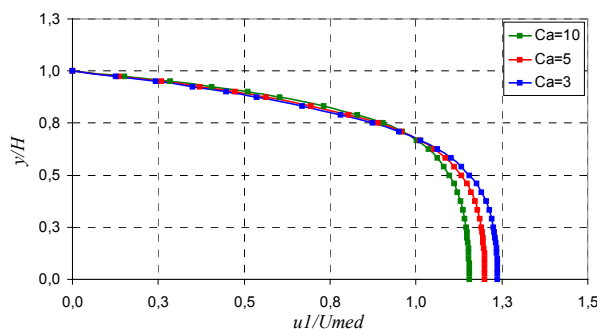


Figure 3: Velocity profiles

The local pressure drop at the upper wall after the expansion is variable for the same  $Re_{Ca}$  and different  $Ca$ , as shown in the graphs of Fig. 4. It is possible to notice that the higher  $Ca$  causes higher viscosities, i.e., more energy losses by friction. After the expansion there is a sudden velocity decrease, causing a positive pressure gradient, but right after that the flow is stabilized and the pressure drop is again linear with position.



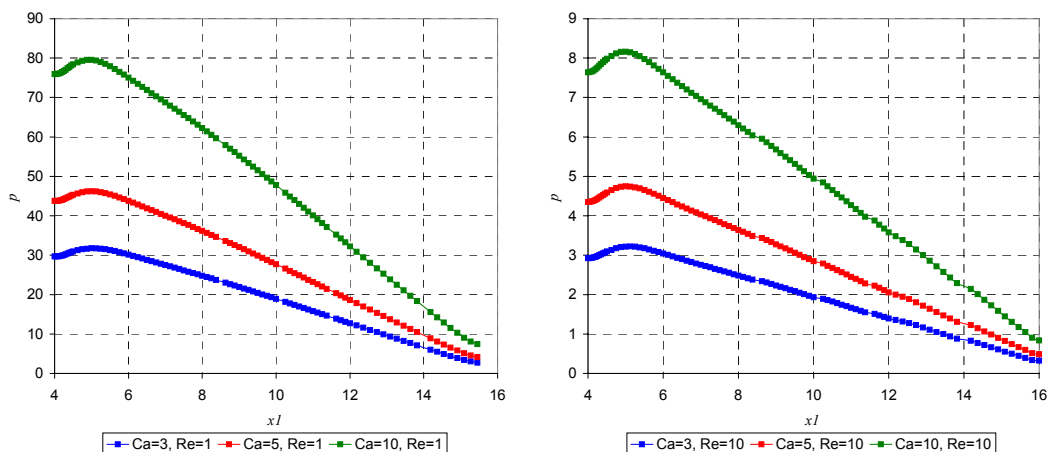


Figure 4: Pressure drop

The steady-state solution for viscosity function in the upper symmetric part of the geometry (detail near the expansion) is also depicted for: (a)  $Re_{Ca}=1$ ,  $Ca=3$ ; (b)  $Re_{Ca}=1$ ,  $Ca=5$ ; (c)  $Re_{Ca}=1$ ,  $Ca=10$ ; (d)  $Re_{Ca}=10$ ,  $Ca=3$ ; (e)  $Re_{Ca}=10$ ,  $Ca=5$ ; (f)  $Re_{Ca}=10$ ,  $Ca=10$ . It is possible to observe the regions of higher viscosity that are larger for larger  $Ca$ . This is a reason also for the formation of unyielded regions in the center of the channel.

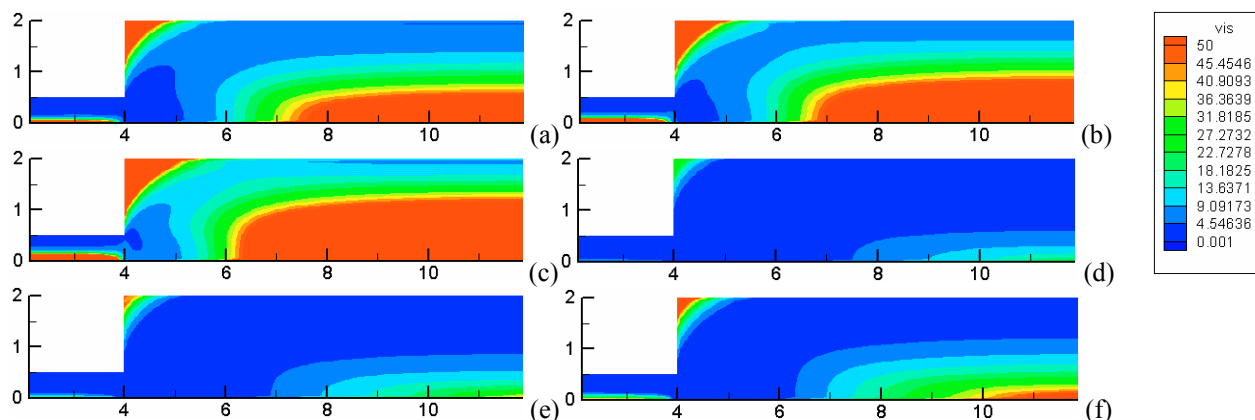


Figure 4: Viscosity function

As shown by Phan & Mitsoulis, the range of  $Ca$  tested in this work is not high enough to generate large regions of unyielded material. But for  $Ca=10$  and  $Re_{Ca}=1$ , we already find an unyielded region of a good magnitude, as shown in Fig. 5 in the complete developed flow. The criteria for yielding we used was to apply the von Mises criteria as in Eq. (26) directly when post-processing.

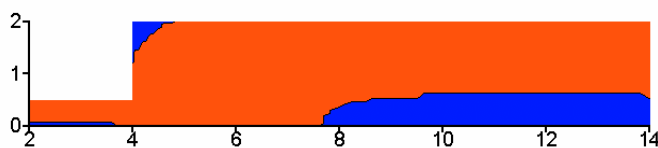


Figure 5: Unyielded material

We still need to perform numerical tests to evaluate the performance of our numerical code in other geometries, analyze the formation of yielded and unyielded regions, give correlations for the pressure drop, etc. But this is a work in progress and we expect to be producing more results in the sequel.

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