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ESTIMATION OF PARAMETERS IN ARFIMA PROCESSES

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Abstract

It is known that, in the presence of short memory components, the estimation of the fractional parameter d in an Autoregressive Fractionally Integrated Moving Average, ARFIMA(p, d, q), process leads to some difficulties (Smith et al. (1997)). In this paper, we continue the efforts made by Smith et al. (1997) by conducting a simulation study to evaluate the convergence properties of the iterative estimation procedure suggested by Hosking (1981). In this context we consider some semiparametric approaches and a parametric method proposed by Whittle (1953). We also investigate the method proposed by Robinson (1995a) and a modification using the smoothed periodogram function.

AMS Subject Classifications: 62M10, 62M15, 60G18.

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1. Introduction

The autoregressive fractionally integrated moving average, ARFIMA(p, d, q), process has widely been used in different fields such as astronomy, hydrology, mathematics and computer science, to represent a time series with long memory property (see Beran (1994)). Recently a wide range of estimators for the fractional parameter d have appeared in the time series literature (see for instance, Hassler (1993), Reisen (1994), Chen et al. (1994), Robinson (1995a,b), Taqqu et al. (1995), Taqqu and Teverovsky (1996), Bisaglia and Guégan (1998), Hurvich et al. (1998), Hurvich and Deo (1999), and Velasco (1999)). These estimators can be categorized into two groups - parametric and semiparametric methods. Within the first group the methods proposed by Fox and Taqqu (1986) and Sowell (1992), which involve the likelihood function, are the most common. In the latter, the most popular, usually referred to as the GPH method, was proposed by Geweke and Porter-Hudak (1983); more recently, a modified form of this, was given by Robinson (1995a). When dealing with the ARFIMA (p, d, q) model, all the parameters, including the autoregressive and moving average ones in addition to the differencing parameter, have to be estimated. These parameters can be simultaneously estimated in the *parametric* approach. In the *semiparametric* methods, the parameters are estimated in two steps: only d is estimated in the first step and the autoregressive and moving average parameters are estimated in the second step.

Since Gaussian parametric estimates for long memory range dependent time series models have rigorously been justified by Fox and Taqqu (1986), Giraitis and Surgailis (1990), Sowell (1992), Dahlhaus (1989) and others, they provide an attractive alternative to the semiparametric methods. However, the Gaussian parametric methods require a great deal of computation and appropriate software is not yet widely available, while the least squares methods (semiparametric procedures) are easy to implement.

The main goal of this paper is to compare the performance of estimating all the parameters of an ARFIMA process based on the algorithm by Hosking (1981) with that of the parametric Whittle estimator (Fox and Taqqu (1986)). For this analysis we consider several estimators of d which are summarized in Section 2. Section 3 describes the algorithm to estimate the parameters. Section 4 presents the results of a simulation study and Section 5 gives a summary and some concluding remarks.

2. The ARFIMA(p,d,q) model

We summarize now some results for the ARFIMA(p, d, q) model with emphasis on the estimation of the differencing parameter d. Consider the simple ARFIMA(p, d, q) model of the form

$$\Phi(B)(1 - B)^d X_t = \Theta(B)\epsilon_t, \quad \text{for} \quad d \in (-0.5, 0.5), \tag{2.1}$$

where $\{\epsilon_t\}$ is a white noise process with $E(\epsilon_t) = 0$ and variance σ_ϵ^2 and B is the back-shift operator such that $BX_t = X_{t-1}$.

The polynomials $\Phi(B) = 1-\phi_1 B-\cdots-\phi_p B^p$ and $\Theta(B) = 1-\theta_1 B-\cdots-\theta_q B^q$ have orders p and q respectively with all their roots outside the unit circle. In this paper we assume that $\{X_t\}$ is a linear process without a deterministic term. We now define $U_t = (1-B)^d X_t$, so that $\{U_t\}$ is an ARMA(p,q) process. The process defined in (2.1) is stationary and invertible (see Hosking (1981)) and its spectral density function, $f_X(w)$, is given by

$$f_X(w) = f_U(w)(2\sin(w/2))^{-2d}, \quad w \in [-\pi, \pi], \tag{2.2}$$

where $f_U(w)$ is the spectral density function of the process $\{U_t\}$.

2.1. Estimation of d

Now we consider five alternative estimators of the parameter d. Four of them are semiparametric and are based on regression equations constructed from the logarithm of the expression in (2.2) The other one is a parametric method proposed by Fox and Taqqu (1986). The methods are summarized as follows:

Periodogram Estimator ($\hat{d}_p$)

The first one denoted by $\hat{d}_p$, and was proposed by Geweke and Porter-Hudak (1983) who used the periodogram function $I(w)$ as an estimate of the spectral density function in expression (2.2). The number of observations in the regression equation is a function $g(n)$ of the sample size n where $g(n) = n^\alpha, 0 < \alpha < 1$.

Smoothed Periodogram Estimator ($\hat{d}_{sp}$)

The second estimator, denoted by $\hat{d}_{sp}$ in the sequel, was suggested by Reisen (1994). This regression estimator is obtained by replacing the spectral density function in the expression (2.2) by the smoothed periodogram function with the Parzen lag window. In this method, $g(n)$ is chosen as above and the truncation point in the Parzen lag window is $m = n^{\beta}, 0 < \beta < 1$. The appropriate choice of α and β were investigated by Geweke and Porter-Hudak (1983) and Reisen (1994), respectively.

Robinson Estimator ($\hat{d}_{pr}$)

The third one is the periodogram estimator with mild modifications suggested by Robinson (1995a), denoted hereafter by $\hat{d}_{pr}$. This estimator is a modified form of the log-periodogram which regresses $\{\ln I(w_i)\}$ on $\ln(2\sin(w_i/2))^2$, for $i = l, l+1, \cdots, g(n)$, where l is the lower truncation point which tends to infinity more slowly than $g(n)$. Now $g(n)$ takes a different form given by

$$g(n) = \begin{cases} A(d,\tau)n^{\frac{2\tau}{2\tau+1}}, & 0 \leq d \leq 0.25 \\ A(d,\tau)n^{\frac{\tau}{\tau+1-2d}}, & 0.25 < d \leq 0.5 \end{cases}$$

where τ and $A(d,t)$ need to be chosen appropiately. Robinson (1995a) derived some asymptotic results for $\hat{d}_{pr}$, when $d \in (-0.5, 0.5)$, and shows that this estimator is asymptotically less efficient than a Gaussian maximum likelihood estimator of d. As mentioned by the author this estimator is not attractive for practical purposes since it depends on the unknown parameters. This problem could be turned around by replacing the unknown parameter d in the $g(n)$ function by using a preliminary estimate, for instance, from the GPH method. The appropriate choice of the optimal $g(n)$ have been the subject of many papers such as, Hurvich et al. (1998) and Hurvich and Deo (1999). The optimal $g(n)$ in the sense of minimum mean squared error is given by

$g(n)(opt) = C\,n^{\frac{4}{5}}$, where C is a constant. They propose an estimator of C also based on a log-periodogram regression and derive its consistency and an asymptotical confidence interval for d when the number of frequencies used in the regression model is deterministic and proportional to $n^{\frac{4}{5}}$.

Robinson's estimator based on the smoothed periodogram ($\hat{d}_{spr}$)

We suggest, without any mathematical proof, the use of the smoothed periodogram function, with the Parzen lag window, to replace the periodogram in the Robinson's estimator. The truncation point is the same as the one chosen for $\hat{d}_{sp}$ and the number of observations in the regression equation is also the same as the one chosen for $\hat{d}_{pr}$.

Whittle estimator ($\hat{d}_W$)

The fifth estimator is a parametric procedure due to Whittle (1953) with modifications suggested by Fox and Taqqu (1986) and will be denoted hereafter by $\hat{d}_W$. The estimator $\hat{d}_W$ is based on the periodogram and it involves the function

$$Q(\zeta) = \int_{-\pi}^{\pi} \frac{I(w)}{f_X(w,\zeta)} d\zeta, \qquad (2.3)$$

where $f_X(w,\zeta)$ is the known spectral density function at frequency w and ζ denotes the vector of unknown parameters. The Whittle estimator is the value of ζ which minimizes the function $Q(\cdot)$. For the ARFIMA (p,d,q) process the vector ζ contains the parameter d and also all the unknown autoregressive and moving average parameters. For more details see Fox and Taqqu (1986), Dahlhaus (1989) and Beran (1994). For computational purposes the estimator $\hat{d}_W$ is obtained by using the discret form of $Q(\cdot)$, as in Dahlhaus (1989, page 1753), that is,

$$\mathcal{L}_n(\zeta) = \frac{1}{2n} \sum_{j=1}^{n-1} \left\{ \ln f_X(w_j, \zeta) + \frac{I(w_j)}{f_X(w_j, \zeta)} \right\}. \tag{2.4}$$

Dahlhaus (1989) and Yajima (1985) have shown that the maximum likelihood estimator of d is strongly consistent, asymptotically normally distributed and asymptotically efficient in the Fisher sense.

3. Identification and Estimation of an ARFIMA(p,d,q) Model

For the use of the regression techniques several steps are necessary to obtain an ARFIMA model for a set of time series data and these are given below (see Hosking (1981) and Brockwell and Davis (1991)).

Let $\{X_t\}$ be the process as defined in (2.1). Then $U_t = (1-B)^d X_t$ is an ARMA(p,q) process and $Y_t = \frac{\Phi(B)}{\Theta(B)} X_t$ is an ARFIMA$(0,d,0)$ process.

Model Building Steps:

1. Estimate d in the ARIMA(p,d,q) model; denote the estimate by $\hat{d}$.

2. Calculate $\hat{U}_t = (1-B)^{\hat{d}} X_t$.

3. Using Box-Jenkins modelling procedure (see Box and Jenkins (1976)) (or the AIC criterion, Akaike (1973)) identify and estimate ϕ and θ parameters in the ARMA(p,q) process $\phi(B)\hat{U}_t = \theta(B)\epsilon_t$.

4. Calculate $\hat{Y}_t = \frac{\hat{\phi}(B)}{\hat{\theta}(B)} X_t$.

5. Estimate d in the ARFIMA$(0, d, 0)$ model $(1-B)^{\hat{d}}\hat{Y}_t = \epsilon_t$. The value of $\hat{d}$ obtained in this step is now the new estimate of d.

6. Repeat steps 2 to 5, until the estimates of the parameters d, ϕ and θ converge.

In this algorithm, to estimate d we use the regression methods described in Section 2. It should be noted that usually only one iteration with Steps 1-3 is used to obtain a model (see, for instance, Brockwell and Davis (1991)). Related to Step 3, it has widely been discussed that the bias in the estimator of d can lead to the problem of identifying the short-memory parameters. This issue has been investigated by Schmidt and Tschernig (1993), Crato and Ray (1996) and, recently, by Smith et al. (1997) and Reisen and Lopes (1999). In this paper we assume that the true model is known and only the parameters need to be estimated.

4. Simulation Study

Now we investigate, by simulation experiments, the convergence of the iterative method of model estimation shown in Section 3. In this study, observations from the ARFIMA(p, d, q) process are generated using the method described in Hosking (1984) where the random variables ϵ_t are assumed to be identically and independently normally distributed as $N(0, 1.0)$ obtained from the subroutine RNNOR in the IMSL - Library. For the estimators $\hat{d}_p$ and $\hat{d}_{sp}$, we use $g(n) = n^{0.5}$ and $m = n^{0.9}$ (the truncation point in the Parzen lag window), as suggested in Geweke and Porter-Hudak (1983) and

Reisen (1994), respectively. In the case of Robinson's estimator we use $l = 2$, $\tau = 0.5$ and $A(d, \tau) = 1.0$. The respective numbers of observations involved in the regression equations are given in the tables. Three models are considered: ARFIMA$(0, d, 0)$, ARFIMA$(1, d, 0)$ and ARFIMA$(0, d, 1)$. ARFIMA $(0, d, 0)$ model is included here to verify the finite sample behaviour and also the performance of the smoothed periodogram function in the Robinson's method.

A Monte Carlo study analyzing the behaviour of the finite sample efficiency of the maximum likelihood estimators using an approximate frequency-domain (Fox and Taqqu (1986)) and the exat time-domain (Sowell (1992)) approaches may be found in Cheung and Diebold (1994).

In the Whittle method, the parameters of the process are estimated simultaneously by the use of the subroutine BCONF in the IMSL - Library. In the case of the semiparametric methods, the autoregressive and moving average parameters are estimated by using the subroutine NSLE in the IMSL - Library, after the time series has been differentiated by the estimate of d. As mentioned in the previous section, in our simulation, we assume that the true model is known and only the parameters need to be estimated. The results for all estimation procedures are based on the same 500 replications.

ARFIMA$(0, d, 0)$:

INSERT TABLE 4.1 ABOUT HERE

Table 4.1, gives the mean value $E(\hat{d})$, the standard deviation (sd, in parenthesis), the bias $(\hat{d})$, the mean squared error (mse), and the values of $g(n)$ (the upper limit of the frequencies involved in the semiparametric approaches). As expected, the Whittle's method for estimating d is more accurate than the other methods. Nevertheless, the other methods give good results as well. The results get better when the sample size increases. For the Robinson methods, the choice of the number of frequencies is crucial for estimating d. For $d = 0.2$, $\hat{d}_{pr}$ and $\hat{d}_{spr}$ have bigger mean squared errors compared to the other methods. In this case, the regression is built from $l = 2, \cdots, g(n)$, that is, less observations are used to obtain $\hat{d}_{pr}$ and $\hat{d}_{spr}$. For d =0.3 and 0.45, both estimators improve with smaller bias and mean squared error. For $d = 0.45$ they are very competitive to the Whittle's estimator. $\hat{d}_{spr}$ dominates $\hat{d}_{pr}$ and $\hat{d}_{sp}$ outperform $\hat{d}_p$ in terms of mean squared error.

ARFIMA (p, d, q) MODELS:

These models contain short memory components and the estimation of all parameters is the goal. Thus, the long memory parameter d is estimated taking into account the additional uncertainty due to the contemporary estimation of the autoregressive or moving average parameters.

Following the procedure described in Section 3, for each d, ϕ and θ we generate a time series of size $n = 300$, estimate the fractional parameter d and then obtain $\hat{U}_t = (1-B)^{\hat{d}} X_t$ (see Step 2 in Section 3) from which the autoregressive or the moving average coefficient estimate is obtained as in Step 3. Then we obtain $\hat{Y}_t = (\hat{\phi}(B)/\hat{\theta}(B))X_t$ which is an ARFIMA$(0, d, 0)$ process and use it to estimate d. Steps 2-5 are repeated until the values of $(\hat{d}, \hat{\phi}, \hat{\theta})$ do not change much from one iteration to the next. In each iteration d is estimated using $\hat{d}_p$, $\hat{d}_{sp}$, $\hat{d}_{pr}$ and $\hat{d}_{spr}$. This procedure is repeated 500 times.

In each replication, the maximum number of iteration is fixed at 20. An extensive simulation study was performed considering different values of d, ϕ and θ with $p = q = 0, 1$. However, we only present some of them here since the pattern is the same for the other cases.

The results are shown in Tables 4.2 to 4.15. The first part of the tables gives the results corresponding to the first iteration. These are the average of $\hat{d}$, $(E(\hat{d}))$, bias, *sd*, mean squared error (mse), the average of the coefficient estimate ($E(\hat{\phi})$ or $E(\hat{\theta})$), bias in the coefficient estimate and the *sd* of the coefficient obtained from the first iteration over the 500 replications. The second part of the tables gives the value of l_i, the maximum iteration to obtain the convergence, and the corresponding estimation results as in part one. Note that, in the second part of the table, there are no results for the Whittle's method.

From the results we can discuss the following issues:

i. The number of iterations (l_i) needed to obtain convergence for the estimates.

ii. The impact of the values of d, ϕ, θ for convergence. The convergence of the parameter estimates to the true values.

iii. The behaviour of the estimators $\hat{d}_p$, $\hat{d}_{sp}$, $\hat{d}_{pr}$, $\hat{d}_{spr}$ and $\hat{d}_W$.

iv. The comparison between parametric and semiparametric methods.

ARFIMA $(1, d, 0)$:

TABLES 4.2 to 4.9 ABOUT HERE

Tables 4.2 to 4.9 present the results corresponding to $d = 0.2,\ 0.45$ and $\phi = -0.6, -0.2, 0.2, 0.6$. We summarize the findings as follows:

i. The number of iterations to stabilize the estimates increases with ϕ and d, and its value is larger when ϕ is positive. In most of the cases considered here, the estimates of d and ϕ obtained in the first iteration (steps 1-3) are very good. The iterative method has not shown a substantial improvement in the estimation of the parameters. The computational effort involved in the procedure is not simple and the problem of order identification when the time series is differenciated many times must be considered. In certain cases there were difficulties to achieve convergence of the parameters, especially for those closer to the non-stationary boundary in the Robinson's method. Thus, we feel that only one iteration (steps 1-3) is needed in the model building algorithm described in Section 3. We also computed the averages of the standard deviations calculated from the estimates in the 20 iterations in each replication (the results are not presented here). These values are very small and they indicate that the changes in the values of the estimates from iteration to iteration are very small. This confirms our earlier assertion that estimates from the first iteration would be sufficient for practical purposes.

ii. The estimation of AR coefficients do impact the estimation of d and also the iterative procedure in section 3. When ϕ goes from -0.2 to -0.6 the biases in $\hat{d}$ seem to decrease slightly. However, when $\phi > 0$ biases of all the estimators of d increase with ϕ. When $|\phi| = .6$ the biases in all estimators of d are large (except for $\hat{d}_{sp}$), so are the biases in the estimators of ϕ but in the opposite direction. This indicates that the bias in $\hat{d}$ is being compensated by the bias in $\hat{\phi}$. When ϕ is negative the estimates of the parameters are typically better behaved than in the positive case. Also, the number of iterations needed to attain the convergence is smaller (compare, for instance, the cases ARFIMA$(1, 0.45, 0)$ when $\phi = -0.6$ and $\phi = 0.6$).

iii. $\hat{d}_{sp}$ has smaller mean squared error and, in general, also has smaller bias compared to the other regression estimators. When ϕ is negative, $\hat{d}_{sp}$ underestimates d while $\hat{d}_p$, $\hat{d}_{pr}$ and $\hat{d}_{spr}$ overestimates d most of the time except when, $d = 0.45$ where $\hat{d}_{pr}$ and $\hat{d}_{spr}$ underestimate the true value. $\hat{d}_{sp}$, and its corresponding $\hat{\phi}$, move more rapidly to true values compared to $\hat{d}_p$, and its corresponding $\hat{\phi}$. Also, as expected, s.d.$(\hat{d}_{sp}) <$ s.d.$(\hat{d}_p)$. It should also be noted that the simulated standard deviations are close to the asymptotic values. For instance, when $d = 0.45$ and $\phi = 0.2$ the simulated standard deviations for $\hat{d}_{sp}$ and $\hat{d}_p$ are 0.0725 and 0.1231, respectively, while the asymptotic values are 0.0876 and 0.2018, respectively. It is clear that the biases of $\hat{d}_{pr}$ and $\hat{d}_{spr}$ are more pronounced than those of the usual $\hat{d}_p$ and $\hat{d}_{sp}$ estimators. The first two methods involve more frequencies in the regression equation and this yields estimates with large bias and large mean squared error, specially when ϕ is positive and d is large. This may be caused by the fact that the AR component enlarges the value of the spectral density function. The results are different from the ones in the ARFIMA$(0, d, 0)$ model. $\hat{d}_{spr}$ has a smaller mean squared error compared with $\hat{d}_{pr}$ as expected since the spectral density function is estimated by the smoothed periodogram function.

iv. For large and positive ϕ, the semiparametric methods, especially the smoothed periodogram performs better than the Whittle's method which improves when ϕ is negative but not in all cases, only for the AR values not closer to the non-stationary boundary. We also note that, if the order of the process is not correct, this estimator gives inconsistent estimates for the parameters.

ARFIMA$(0, d, 1)$:

TABLES 4.10 to 4.15 HERE

Simulation results for the ARFIMA$(0, d, 1)$ process are given in Tables 4.10 to 4.15. We considered several values of θ. However, we present the results only for $d = 0.3$ and $\theta = -0.3, 0.3$ and for $d = 0.45$ and $\theta = -0.6, -0.3, 0.3, 0.6$ since the pattern is similar for other cases.

When θ goes from -.6 to -.3 the biases in the estimator of d decrease and when θ is positive the biases increase with θ but in the opposite direction. These results are opposite to the ones obtained in the ARFIMA$(1, d, 0)$ model. The estimator $\hat{d}_{sp}$ outperforms the other methods including the Whittle's estimator $\hat{d}_W$.

As in the ARFIMA$(1, d, 0)$ model, $\hat{d}_p$ and $\hat{d}_{sp}$ need only small number of iterations to achieve convergence with the latter requiring the smallest number of iterations. Results for the estimators $\hat{d}_{pr}$ and $\hat{d}_{spr}$ are not very good. If we consider only one iteration then, in general, the two regression estimators perform much better than $\hat{d}_{pr}$ and$\hat{d}_{spr}$.

We also encountered some convergence difficulties for the Robinson's estimator $\hat{d}_{pr}$ especially for positive and large values of θ. In most of the cases, the least squares estimation of the parameters failed to converge. Both $\hat{d}_{pr}$ and $\hat{d}_{spr}$ estimators, have very large sample variances. Extensive computational efforts were necessary to obtain 500 successfull replications with a maximum of 20 iterations in each.

5. Summary and Concluding Remarks

In this paper we considered a simulation study to evaluate the procedures for estimating the parameters of an ARFIMA process. We considered both parametric and semiparametric methods and also use the smoothed periodogram function in the modified regression estimator. The results indicate that the regression methods outperforms the parametric Whittle's method when AR or MA components are involved. Performance of the Robinson estimator usually is not as good as the other semiparametric methods; it has large bias, standard deviation, and mean squared error. The use of the smoothed periodogram in Robinson's method improves the estimates, however, the results are still not as good as the usual regression methods. The results also indicate that the estimates from the first iteration (steps 1-3) are sufficient for practical purposes.

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Table 4.1: Estimation of d: ARFIMA (0,d,0)

n	d	Whittle $\hat{d}_W$	Smoothed Periodogram $\hat{d}_{sp}$	Periodogram $\hat{d}_p$	Robinson Sm. Perio. $\hat{d}_{spr}$	Robinson Periodogram $\hat{d}_{pr}$
	0.2					
	$E(\hat{d})$	0.1983	0.1396	0.2110	0.2153	0.2252
	Sd	(0.0749)	(0.1915)	(0.2470)	(0.2862)	(0.4289)
	bias $(\hat{d})$	0.0017	0.0604	-0.0110	-0.0153	-0.0252
	mse	0.0056	0.0402	0.0610	0.0819	0.1841
	$g(n)$				12	12
150	0.3					
	$E(\hat{d})$	0.3073	0.2361	0.3248	0.3245	0.3263
	Sd	(0.0719)	(0.1957)	(0.2612)	(0.2299)	(0.0321)
	bias $(\hat{d})$	-0.0073	0.0639	-0.0248	-0.0299	-0.0263
	mse	0.0025	0.0423	0.0687	0.0533	0.1035
	$g(n)$				16	16
	0.45					
	$E(\hat{d})$	0.4768	0.3724	0.4500	0.4653	0.4615
	Sd	(0.0379)	(0.1879)	(0.2275)	(0.0828)	(0.1108)
	bias	-0.0268	0.0776	0.0	-0.0153	-0.0115
	mse	0.0021	0.0412	0.0516	0.0071	0.0124
	$g(n)$				65	65
	0.2					
	$E(\hat{d})$	0.2033	0.1562	0.2018	0.2175	0.2075
	Sd	(0.0494)	(0.1501)	(0.1970)	(0.2160)	(0.3088)
	bias	-0.0033	0.0438	-0.0018	-0.0160	-0.0075
	mse	0.0024	0.0244	0.0387	0.0468	0.0952
	$g(n)$				17	17
300	0.3					
	$E(\hat{d})$	0.3006	0.2491	0.3010	0.3113	0.3036
	Sd	(0.0478)	(0.1499)	(0.1871)	(0.1734)	(0.2481)
	bias	-0.0006	0.0509	-0.0010	-0.0113	-0.0036
	mse	0.0022	0.0250	0.0349	0.0301	0.0614
	$g(n)$				23	23
	0.45					
	$E(\hat{d})$	0.4721	0.4020	0.4594	0.4593	0.4556
	Sd	(0.0351)	(0.1631)	(0.2040)	(0.0646)	(0.0835)
	bias	-0.0221	0.0480	-0.0094	0.0093	-0.0056
	mse	0.0017	0.0218	0.0416	0.0043	0.007
	$g(n)$				115	115

Table 4.2: Estimation for d = 0.2: ARFIMA(1,d,0), ϕ = -0.6

	$d = 0.2$			ϕ = -0.6		
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 17$	Whittle
I	$E(\hat{d}_i)$	0.2440	0.1886	0.2413	0.2340	0.1883
	bias($\hat{d}_i$)	-0.0440	**0.0114**	-0.0413	-0.0340	0.0117
	sd($\hat{d}_i$)	0.1274	0.0723	0.2172	0.1092	0.0586
	mse($\hat{d}_i$)	0.0181	0.0054	0.0488	0.0131	**0.0036**
	$E(\hat{\phi}_i)$	-0.6020	-0.5818	-0.5656	-0.5995	-0.5958
	bias($\hat{\phi}_i$)	**0.0020**	-0.0182	-0.0344	-0.0005	-0.0042
	sd($\hat{\phi}_i$)	0.0804	0.0666	0.1973	0.0772	0.0640
II	l_i	3	3	6	3	–
	$E(\hat{d}_i^*)$	0.2485	0.1934	0.2450	0.2407	–
	bias($\hat{d}_i^*$)	-0.0485	**0.0066**	-0.0450	-0.0407	–
	sd($\hat{d}_i^*$)	0.1262	0.0726	0.2269	0.1098	–
	$E(\hat{\phi}_i^*)$	-0.6044	-0.5843	-0.5688	-0.6027	–
	bias($\hat{\phi}_i^*$)	**0.0044**	-0.0157	-0.0312	0.0027	–
	sd($\hat{\phi}_i^*$)	0.0804	0.0660	0.2028	0.0764	–

Table 4.3: Estimation for d = 0.2: ARFIMA(1,d,0), ϕ = -0.2

	$d = 0.2$		ϕ = -0.2			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Peridogram	Robinson Smoothed $g(n) = 17$	Whittle
I	$E(\hat{d}_i)$	0.2507	0.1950	0.2511	0.2450	0.1902
	bias$(\hat{d}_i)$	-0.0507	**0.0050**	-0.0511	-0.0450	0.0098
	sd$(\hat{d}_i)$	0.1269	0.0734	0.2094	0.1103	0.0734
	mse$(\hat{d}_i)$	0.0186	**0.0054**	0.0464	0.0142	0.0055
	$E(\hat{\phi}_i)$	-0.2245	-0.1875	-0.1935	-0.2232	-0.1913
	bias$(\hat{\phi}_i)$	0.0245	-0.0125	**-0.0065**	0.0232	-0.0087
	sd$(\hat{\phi}_i)$	0.1233	0.0917	0.2342	0.1156	0.0911
II	l_i	2	2	7	2	–
	$E(\hat{d}_i^*)$	0.2534	0.1977	0.2386	0.2490	–
	bias$(\hat{d}_i^*)$	- 0.0534	**0.0023**	-0.0386	-0.0490	–
	sd$(\hat{d}_i^*)$	0.1290	0.0743	0.2695	0.1119	–
	$E(\hat{\phi}_i^*)$	-0.2262	-0.1898	-0.1856	-0.2261	–
	bias$(\hat{\phi}_i^*)$	0.0262	**-0.0102**	-0.0144	0.0261	–
	sd$(\hat{\phi}_i^*)$	0.1252	0.0924	0.2692	0.1170	–

Table 4.4: **Estimation for $d = 0.2$: ARFIMA(1,d,0), $\phi = 0.2$**

	$d = 0.2$		$\phi = 0.2$			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 17$	Whittle
I	$E(\hat{d}_i)$	0.2568	0.1942	0.2610	0.2428	0.1762
	bias$(\hat{d}_i)$	-0.0568	**0.0058**	-0.0610	-0.0428	0.0238
	sd$(\hat{d}_i)$	0.1268	0.0683	0.1984	0.1034	0.1295
	mse$(\hat{d}_i)$	0.0193	**0.0047**	0.0430	0.0125	0.0173
	$E(\hat{\phi}_i)$	0.1530	0.2093	0.1633	0.1623	0.2177
	bias$(\hat{\phi}_i)$	0.0470	**-0.0093**	0.0367	0.0377	-0.0177
	sd$(\hat{\phi}_i)$	0.1384	0.0941	0.2126	0.1206	0.1394
II	l_i	6	3	6	6	–
	$E(\hat{d}_i^*)$	0.2496	0.1854	0.2103	0.2329	–
	bias$(\hat{d}_i^*)$	-0.0496	0.0146	**-0.0103**	-0.0329	–
	sd$(\hat{d}_i^*)$	0.1340	0.0729	0.3221	0.1131	–
	$E(\hat{\phi}_i^*)$	0.1615	0.2194	0.1965	0.1739	–
	bias$(\hat{\phi}_i^*)$	0.0385	-0.0194	**0.0035**	0.0261	–
	sd$(\hat{\phi}_i^*)$	0.1484	0.1017	0.2759	0.1354	–

Table 4.5: **Estimation for $d = 0.2$: ARFIMA(1,d,0), $\phi = 0.6$**

	$d = 0.2$		$\phi = 0.6$			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 17$	Whittle
I	$E(\hat{d}_i)$	0.3374	0.2650	0.3888	0.3476	0.5159
	bias$(\hat{d}_i)$	-0.1374	**-0.0650**	-0.1888	-0.1476	-0.3159
	sd$(\hat{d}_i)$	0.0945	0.0517	0.1718	0.0887	0.3609
	mse$(\hat{d}_i)$	0.0278	**0.0069**	0.0651	0.0296	0.2298
	$E(\hat{\phi}_i)$	0.4468	0.5167	0.3986	0.4357	0.3080
	bias$(\hat{\phi}_i)$	0.1532	**0.0833**	0.2014	0.1643	0.2920
	sd$(\hat{\phi}_i)$	0.0996	0.0595	0.1646	0.0897	0.1838
II	l_i	15	15	16	16	–
	$E(\hat{d}_i^*)$	0.2914	0.1981	0.2214	0.2696	–
	bias$(\hat{d}_i^*)$	-0.0914	**0.0019**	-0.0214	-0.0696	–
	sd$(\hat{d}_i^*)$	0.1234	0.0742	0.4381	0.1957	–
	$E(\hat{\phi}_i^*)$	0.4907	0.5811	0.4858	0.4986	–
	bias$(\hat{\phi}_i^*)$	0.1093	**0.0189**	0.1142	0.1014	–
	sd$(\hat{\phi}_i^*)$	0.1271	0.0850	0.2676	0.1414	–

Table 4.6: Estimation for d = 0.45: ARFIMA(1,d,0), ϕ = -0.6

	$d = 0.45$		ϕ = -0.6			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.5184	0.4457	0.2357	0.2411	0.5273
	bias$(\hat{d}_i)$	-0.0684	**0.0043**	0.2144	0.2089	-0.0773
	sd$(\hat{d}_i)$	0.1254	0.0759	0.0765	0.0560	0.0734
	mse$(\hat{d}_i)$	0.0204	**0.0058**	0.0518	0.0469	0.1134
	$E(\hat{\phi}_i)$	-0.6124	-0.5833	-0.3717	-0.3908	-0.7098
	bias$(\hat{\phi}_i)$	**0.0124**	-0.0167	-0.2283	-0.2092	0.1098
	sd$(\hat{\phi}_i)$	0.0905	0.0698	0.1515	0.1147	0.1695
II	l_i	3	3	10	10	–
	$E(\hat{d}_i^*)$	0.5241	0.4505	0.4282	0.4308	–
	bias$(\hat{d}_i^*)$	-0.0741	**-0.0005**	0.0219	-0.0031	–
	sd$(\hat{d}_i^*)$	0.1241	0.0764	0.2109	0.1346	–
	$E(\hat{\phi}_i^*)$	-0.6154	-0.5861	-0.5457	-0.5809	–
	bias$(\hat{\phi}_i^*)$	0.0154	**-0.0139**	-0.0543	-0.0191	–
	sd$(\hat{\phi}_i^*)$	0.0882	0.0690	0.2903	0.1871	–

Table 4.7: **Estimation for d = 0.45: ARFIMA(1,d,0), ϕ = -0.2**

	$d = 0.45$		ϕ = -0.2			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.5123	0.4449	0.3616	0.3679	0.5230
	bias($\hat{d}_i$)	-0.0623	**0.0051**	0.0884	0.0821	-0.0730
	sd($\hat{d}_i$)	0.1296	0.0739	0.0747	0.0563	0.0800
	mse($\hat{d}_i$)	0.0206	**0.0055**	0.0134	0.0099	0.0117
	$E(\hat{\phi}_i)$	-0.2234	-0.1773	-0.0848	-0.0981	-0.2568
	bias($\hat{\phi}_i$)	0.0234	**-0.0227**	-0.1152	-0.1019	0.0568
	sd($\hat{\phi}_i$)	0.1389	0.0981	0.0993	0.0711	0.0815
II	l_i	5	3	9	4	–
	$E(\hat{d}^*_i)$	0.5154	0.4475	0.3308	0.4459	–
	bias($\hat{d}^*_i$)	-0.0654	**0.0025**	0.1192	0.0041	–
	sd($\hat{d}^*_i$)	0.1310	0.0750	0.3850	0.1333	–
	$E(\hat{\phi}^*_i)$	-0.2254	-0.1796	-0.0446	-0.1695	–
	bias($\hat{\phi}^*_i$)	0.0254	**-0.0204**	-0.1554	-0.0305	–
	sd($\hat{\phi}^*_i$)	0.1410	0.0997	0.4273	0.1960	–

Table 4.8: **Estimation for $d = 0.45$: ARFIMA(1,d,0), $\phi = 0.2$**

	$d = 0.45$			$\phi = 0.2$		
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.5097	0.4491	0.5928	0.5958	0.6362
	bias$(\hat{d}_i)$	-0.0597	**0.0009**	-0.1428	-0.1458	-0.1862
	sd$(\hat{d}_i)$	0.1231	0.0725	0.0741	0.0579	0.1471
	mse$(\hat{d}_i)$	0.0187	**0.0052**	0.0259	0.0246	0.0562
	$E(\hat{\phi}_i)$	0.1552	0.2139	0.0642	0.0601	0.0376
	bias$(\hat{\phi}_i)$	0.0448	**-0.0139**	0.1358	0.1399	0.1624
	sd$(\hat{\phi}_i)$	0.1426	0.1005	0.0654	0.0503	0.1322
II	l_i	8	6	10	10	–
	$E(\hat{d}_i^*)$	0.5009	0.4393	0.3581	0.4118	–
	bias$(\hat{d}_i^*)$	-0.0510	**0.0107**	0.0919	0.0382	–
	sd$(\hat{d}_i^*)$	0.1318	0.0781	0.3690	0.2923	–
	$E(\hat{\phi}_i^*)$	0.1664	0.2266	0.3026	0.2500	–
	bias$(\hat{\phi}_i^*)$	0.0336	**-0.0266**	-0.1026	-0.0500	–
	sd$(\hat{\phi}_i^*)$	0.1573	0.1123	0.3627	0.3062	–

Table 4.9: Estimation for $d = 0.45$: ARFIMA(1,d,0), $\phi = 0.6$

	$d = 0.45$		$\phi = 0.6$			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.5968	0.5201	0.9257	0.9280	0.8140
	bias($\hat{d}_i$)	-0.1468	**-0.0701**	-0.4757	-0.4780	-0.3640
	sd($\hat{d}_i$)	0.0942	0.0520	0.0716	0.0520	0.1020
	mse($\hat{d}_i$)	0.0304	**0.0076**	0.2314	0.2312	0.1427
	$E(\hat{\phi}_i)$	0.4417	0.5178	0.1383	0.1353	0.2520
	bias($\hat{\phi}_i$)	0.1583	**0.0822**	0.4617	0.4647	0.3480
	sd($\hat{\phi}_i$)	0.0930	0.0553	0.0631	0.4800	0.1195
II	l_i	18	18	18	18	–
	$E(\hat{d}_i^*)$	0.5561	0.4520	0.5228	0.6013	–
	bias($\hat{d}_i^*$)	-0.1061	**-0.0020**	-0.0728	-0.1513	–
	sd($\hat{d}_i^*$)	0.1217	0.0729	0.3789	0.2700	–
	$E(\hat{\phi}_i^*)$	0.4817	0.5869	0.4857	0.4233	–
	bias($\hat{\phi}_i^*$)	0.1183	**0.0131**	0.1143	0.1767	–
	sd($\hat{\phi}_i^*$)	0.1201	0.0804	0.3249	0.2489	–

Table 4.10: **Estimation for d = 0.3: ARFIMA(0,d,1), θ = -0.3**

	$d = 0.3$		θ = -0.3			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 23$	Whittle
I	$E(\hat{d}_i)$	0.3458	0.2962	0.3528	0.3501	0.3153
	bias$(\hat{d}_i)$	-0.0458	**0.0038**	-0.0528	-0.0501	-0.0153
	sd$(\hat{d}_i)$	0.1315	0.0761	0.1967	0.1234	0.0059
	mse$(\hat{d}_i)$	0.0193	0.0058	0.0413	0.0177	**0.0037**
	$E(\hat{\theta}_i)$	-0.2624	-0.3046	-0.2519	-0.2571	-0.2897
	bias$(\hat{\theta}_i)$	-0.0376	**0.0046**	-0.0481	-0.0429	-0.0103
	sd$(\hat{\theta}_i)$	0.1270	0.0903	0.1844	0.1262	0.0763
II	l_i	3	3	15	3	–
	$E(\hat{d}_i^*)$	0.3423	0.2922	0.3466	0.3427	–
	bias$(\hat{d}_i^*)$	-0.0423	**0.0078**	-0.0466	-0.0427	–
	sd$(\hat{d}_i^*)$	0.1324	0.0767	0.2044	0.1258	–
	$E(\hat{\theta}_i^*)$	-0.2652	-0.3078	-0.2559	-0.2632	–
	bias$(\hat{\theta}_i^*)$	-0.0348	**0.0078**	-0.0441	-0.0368	–
	sd$(\hat{\theta}_i^*)$	0.1277	0.0907	0.1957	0.1282	–

Table 4.11: **Estimation for $d = 0.3$: ARFIMA(0,d,1), $\theta = 0.3$**

	$d = 0.3$		$\theta = 0.3$			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 23$	Whittle
I	$E(\hat{d}_i)$	0.3409	0.2788	0.2581	0.2804	0.3385
	bias($\hat{d}_i$)	-0.0408	**0.0212**	0.0419	0.0196	-0.0385
	sd($\hat{d}_i$)	0.0999	0.0686	0.1544	0.1005	0.1018
	mse($\hat{d}_i$)	0.0116	**0.0051**	0.0255	0.0104	0.0118
	$E(\hat{\theta}_i)$	0.3365	0.2703	0.2422	0.2704	0.3288
	bias($\hat{\theta}_i$)	-0.0365	**0.0297**	0.0578	0.0296	-0.0288
	sd($\hat{\theta}_i$)	0.1242	0.0898	0.1801	0.1168	0.1154
II	l_i	10	6	15	15	–
	$E(\hat{d}_i^*)$	0.3638	0.2924	0.2989	0.3186	–
	bias($\hat{d}_i^*$)	-0.0638	**0.0076**	0.0010	-0.0186	–
	sd($\hat{d}_i^*$)	0.1149	0.0755	0.1981	0.1337	–
	$E(\hat{\theta}_i^*)$	0.3607	0.2851	0.2826	0.3097	–
	bias($\hat{\theta}_i^*$)	-0.0607	0.0149	0.0174	**-0.0097**	–
	sd($\hat{\theta}_i^*$)	0.1410	0.0987	0.2171	0.1485	–

Table 4.12: Estimation for d = 0.45: ARFIMA(0,d,1), θ = -0.6

	$d = 0.45$			$\theta = $ -0.6		
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.5205	0.4519	0.6780	0.6838	0.7539
	bias($\hat{d}_i$)	-0.0705	**-0.0019**	-0.2281	-0.2338	-0.3039
	sd($\hat{d}_i$)	0.1325	0.0724	0.0739	0.0559	0.3625
	mse($\hat{d}_i$)	0.0225	**0.0052**	0.0575	0.0578	0.2234
	$E(\hat{\theta}_i)$	-0.5653	-0.6047	-0.4750	-0.4716	-0.2529
	bias($\hat{\theta}_i$)	-0.0347	**0.0047**	-0.1250	-0.1284	-0.3461
	sd($\hat{\theta}_i$)	0.0981	0.0625	0.0746	0.0703	0.4793
II	l_i	3	3	7	8	–
	$E(\hat{d}_i^*)$	0.5149	0.4456	0.4491	0.4555	–
	bias($\hat{d}_i^*$)	-0.0649	0.0044	**0.0009**	-0.0055	–
	sd($\hat{d}_i^*$)	0.1323	0.0730	0.0826	0.0627	–
	$E(\hat{\theta}_i^*)$	-0.5681	-0.6082	-0.6067	-0.6028	–
	bias($\hat{\theta}_i^*$)	-0.0319	0.0082	0.0067	**0.0028**	–
	sd($\hat{\theta}_i^*$)	0.0980	0.0626	0.0674	0.0608	–

Table 4.13: **Estimation for $d = 0.45$: ARFIMA(0,d,1), $\theta = -0.3$**

	$d = 0.45$			$\theta = -0.3$		
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.5081	0.4506	0.5996	0.6032	0.8608
	bias$(\hat{d}_i)$	-0.0581	**-0.0006**	-0.1496	-0.1532	-0.4108
	sd$(\hat{d}_i)$	0.1294	0.0761	0.0777	0.0561	0.3916
	mse$(\hat{d}_i)$	0.0201	**0.0058**	0.0284	0.0266	0.3218
	$E(\hat{\theta}_i)$	-0.2518	-0.3032	-0.1757	-0.1728	-0.1699
	bias$(\hat{\theta}_i)$	-0.0482	**0.0032**	-0.1243	-0.1272	-0.1301
	sd$(\hat{\theta}_i)$	0.1302	0.0868	0.0754	0.0641	0.4559
II	l_i	4	3	9	8	–
	$E(\hat{d}_i^*)$	0.5039	0.4465	0.4621	0.4668	–
	bias$(\hat{d}_i^*)$	-0.0539	**0.0035**	-0.0121	-0.0168	–
	sd$(\hat{d}_i^*)$	0.1318	0.0764	0.1079	0.0763	–
	$E(\hat{\theta}_i^*)$	-0.2555	-0.3068	-0.2954	0.2911	–
	bias$(\hat{\theta}_i^*)$	-0.0445	0.0068	**-0.0046**	-0.0089	–
	sd$(\hat{\theta}_i^*)$	0.1330	0.0876	0.1101	0.0915	–

Table 4.14: Estimation for $d = 0.45$: ARFIMA(0,d,1), $\theta = 0.3$

	$d = 0.45$		$\theta = 0.3$			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.4903	0.4360	0.2537	0.2595	0.9078
	bias($\hat{d}_i$)	-0.0403	**0.0140**	0.1963	0.1405	-0.4578
	sd($\hat{d}_i$)	0.1221	0.0683	0.0759	0.0559	0.1846
	mse($\hat{d}_i$)	0.0165	**0.0049**	0.0443	0.0394	0.2436
	$E(\hat{\theta}_i)$	0.3312	0.2724	0.0377	0.0474	0.7145
	bias($\hat{\theta}_i$)	-0.0312	**0.0276**	0.2623	0.2526	-0.4145
	sd($\hat{\theta}_i$)	0.1421	0.0905	0.1124	0.0851	0.1759
II	l_i	7	5	15	15	–
	$E(\hat{d}_i^*)$	0.5126	0.4490	0.5030	0.4805	–
	bias($\hat{d}_i^*$)	-0.0626	**0.0010**	-0.0530	-0.0305	–
	sd($\hat{d}_i^*$)	0.1346	0.0742	0.4124	0.3114	–
	$E(\hat{\theta}_i^*)$	0.3554	0.2872	0.2569	0.2565	–
	bias($\hat{\theta}_i^*$)	-0.0554	**0.0128**	0.0431	0.0435	–
	sd($\hat{\theta}_i^*$)	0.1564	0.0984	0.4817	0.3959	–

Table 4.15: Estimation for $d = 0.45$: ARFIMA(0,d,1), $\theta = 0.6$

	$d = 0.45$		$\theta = 0.6$			
	i	Periodogram	Smoothed $g(n) = 17$	Robinson Periodogram	Robinson Smoothed $g(n) = 115$	Whittle
I	$E(\hat{d}_i)$	0.4101	0.3761	-0.0144	-0.0094	0.6893
	bias$(\hat{d}_i)$	**0.0399**	0.0739	0.4644	0.4594	-0.2393
	sd$(\hat{d}_i)$	0.1016	0.0529	0.0742	0.0556	0.0977
	mse$(\hat{d}_i)$	0.0119	**0.0082**	0.2211	0.2141	0.0676
	$E(\hat{\theta}_i)$	0.5193	0.4889	-0.0976	-0.0883	0.7699
	bias$(\hat{\theta}_i)$	**0.0807**	0.1111	0.6976	0.6883	0.0656
	sd$(\hat{\theta}_i)$	0.1304	0.0771	0.1825	0.1610	0.0332
II	l_i	15	15	15	15	–
	$E(\hat{d}_i^*)$	0.4959	0.4337	0.0201	-0.0038	–
	bias$(\hat{d}_i^*)$	-0.0459	**0.0163**	0.4299	0.4538	–
	sd$(\hat{d}_i^*)$	0.1480	0.0753	0.4077	0.3382	–
	$E(\hat{\theta}_i^*)$	0.5991	0.5497	-0.1609	-0.1747	–
	bias$(\hat{\theta}_i^*)$	**0.0009**	0.0503	0.7609	0.7747	–
	sd$(\hat{\theta}_i^*)$	0.1620	0.1014	0.5857	0.5356	–