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ON THE SPECTRAL DENSITY OF A CLASS OF CHAOTIC TIME SERIES

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ON THE SPECTRAL DENSITY OF A CLASS OF CHAOTIC TIME SERIES

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Abstract: The purpose of this paper is to show explicitly the spectral density of the stationary stochastic process determined by the map F_α (α is a parameter in $(0,1)$), the random variable $\phi(x,y) = x$ and the invariant probability ν described below.

We first define the transformation $T_\alpha : [0,1] \to [0,1]$ given by

$$T_\alpha(x) = \begin{cases} \frac{x}{\alpha}, & \text{if} \quad 0 \le x < \alpha \\ \frac{\alpha(x-\alpha)}{1-\alpha}, & \text{if} \quad \alpha \le x \le 1, \end{cases}$$

where $\alpha \in (0,1)$ is a constant. The map T_α describes a model for a particle (or the probability of a certain kind of element in a given population) that moves around, in discrete time, in the interval $[0,1]$.

The map F_α is defined from $K = ([0,1] \times (0,\alpha)) \cup ([0,\alpha] \times [\alpha,1])$ to itself and it is given by $F_\alpha(x,y) = (T_\alpha(x), G_\alpha(x,y))$, for $(x,y) \in K$ where

$$G_\alpha(x,y) = \begin{cases} \alpha\, y & , \quad 0 \le x < \alpha \\ \alpha + \left(\frac{1-\alpha}{\alpha}\right) y & , \quad \alpha \le x \le 1. \end{cases}$$

The spectral density function of the stationary process with probability ν (invariant and absolutely continuous measure with respect to the Lebesgue measure)

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0, Y_0)) + \xi_t, \quad \text{for} \quad t \in \mathbf{Z},$$

where $(X_0, Y_0) \in \mathbf{R}^2$ and $\{\xi_t\}_{t\in\mathbf{Z}}$ is a white noise process, will be given explicitly (see Theorem 1 in Section 3) by

$$f_Z(\lambda) = f_X(\lambda)+\frac{\sigma_\xi^2}{2\pi} = \frac{1}{2\pi}[\gamma(e^{i\lambda})+\gamma(e^{-i\lambda})-\rho_X(0)]+\frac{\sigma_\xi^2}{2\pi}, \quad \text{for all} \quad \lambda \in (-\pi, \pi],$$

where γ is given by the last equality in (2.13) at Proposition 5.

We will show the consistency of the periodogram in this situation. We shall also estimate the parameter α based on a time series.

Keywords: Spectral density; chaotic time series; dynamical system; periodogram.

1. INTRODUCTION

We shall present a complete spectral analysis of the stationary stochastic process

$$Z_t = X_t+\xi_t = \phi(F_\alpha^t(x_0, y_0))+\xi_t = \phi(F_\alpha(X_{t-1}, Y_{t-1})+\xi_t, \quad \text{for} \quad t \in \mathbf{Z}, \qquad (1.1)$$

where $\phi(x, y) = x$ is a random variable, ξ_t is a white noise process, and F_α is a transformation defined below.

The map F_α is defined from $K = ([0,1] \times (0, \alpha)) \cup ([0, \alpha] \times [\alpha, 1])$ to itself and it is given by $F_\alpha(x, y) = (T_\alpha(x), G_\alpha(x, y))$ where the transformation $T_\alpha : [0,1] \to [0,1]$ is defined by

$$T_\alpha(x) = \begin{cases} \frac{x}{\alpha}, & \text{if} \quad 0 \le x < \alpha \\ \frac{\alpha(x-\alpha)}{1-\alpha}, & \text{if} \quad \alpha \le x \le 1, \end{cases} \qquad (1.2)$$

with $\alpha \in (0,1)$ as a constant, and

$$G_\alpha(x, y) = \begin{cases} \alpha y & , \quad 0 \le x < \alpha \\ \alpha + \left(\frac{1-\alpha}{\alpha}\right) y & , \quad \alpha \le x \le 1. \end{cases} \qquad (1.3)$$

The graph of the map T_α is shown in Figure 1. The action of the difeomorphism F_α is presented in Figure 2. The transformation F_α is a modification of the well known Baker transformation.

The map T_α describes a model for a particle that moves around in the interval [0,1]. If the particle is at position x, then after a unit of time it jumps to $T_\alpha(x)$ and so on. According to the model considered here suppose the spatial position of the particle is $T_\alpha^t(x) = X_t$, $t \in \mathbf{N}$, in the interval [0,1]. If the particle X_t is in the interval $[0, \alpha)$, it has a uniformly spread possibility to jump to any point X_{t+1} in [0,1]. However, if it is in the interval $[\alpha, 1)$ it has a uniformly spread possibility to jump to any point X_{t+1} in the interval $[0, \alpha)$.

We are primarily interested in the map T_α, but for defining the spectral density we need a bijective map. Therefore, we have to consider F_α, *the natural extension of* T_α (Bogomolny and Carioli (1995)).

The difeomorphism F_α leaves invariant (see definition in Lopes and Lopes (1995)) an ergodic probability ν on $K \subset \mathbf{R}^2$, absolutely continuous with respect to the Lebesgue measure, that will be described in Section 3.

Choosing a point (x_0, y_0) at random, according to the Lebesgue measure (or according to ν), the spectral properties of the process Z_t will be analyzed.

ore precisely, we shall present explicitly the analytic expression of the spectral density function of such stochastic process (see Section 3).

We refer the reader to Lopes and Lopes (1995) for general definitions and more detailed explanations for the context of the class of problems we consider here.

In Section 2, we present the basic results for a map T_α that are used in Section 3 for obtaining results for the map F_α. In the appendix, we show the consistency of the periodogram in the model considered here.

The main result of this paper, the expression for the spectral density of F_α, is presented in Theorem 1 in Section 3.

The explicit expression of the spectral density function (as obtained here) of a stochastic process allows one to analyze the efficiency of a given numerical method for estimating the spectrum, based on the closeness of the estimation obtained from the method compared to the true spectral density function.

We also estimate the parameter α at the end of Section 2.

We refer the reader to Lopes and Lopes (1995) whenever definitions are used on this paper.

2. THE AUTOCORRELATION FUNCTION

Before considering the transformation F_α we will need to consider the

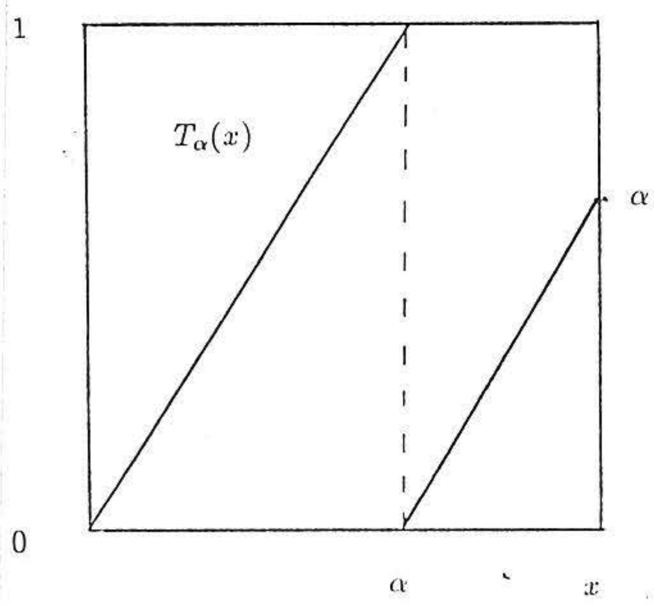

FIGURE 1

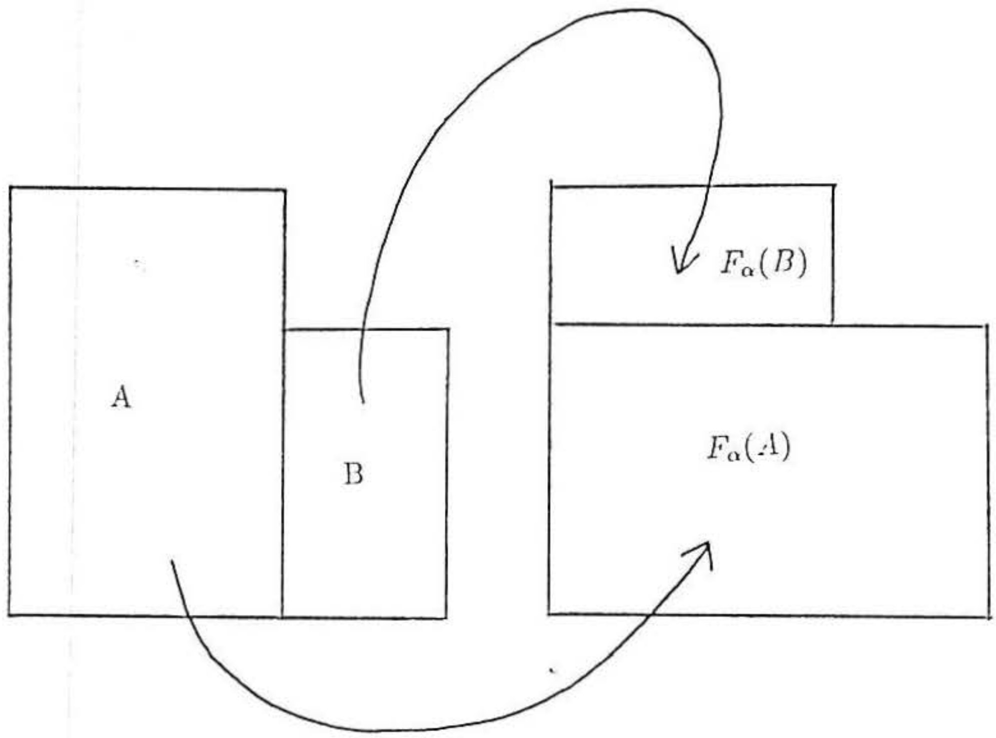

FIGURE 2

transformation T_α.

Let the transformation $T_\alpha : [0,1] \to [0,1]$ be given as in (1.2), where $\alpha \in (0,1)$ is a constant. The derivative of $T_\alpha(x)$ at x is $a = 1/\alpha$ if $0 \le x < \alpha$ and $b = \alpha/1-\alpha$ if $\alpha \le x \le 1$.

One observes that $\underline{a}$ is always greater than 1, however $b \le 1 \Leftrightarrow \alpha \le 1/2$ and $b > 1 \Leftrightarrow \alpha > 1/2$. The transformation T_α is an expansive map (see Robinson (1995)) when $\alpha > 1/2$. It is easy to show that when $\alpha < 1/2$, T_α^2 is an expansive map.

We will be interested here in finding the invariant measure μ absolutely continuous with respect to the Lebesgue measure (see Parry and Pollicott (1990); Robinson (1995); Ruelle (1978)) and also in analyzing the autocorrelation function associated with the stationary stochastic process (T_α^t, μ).

First we shall find the invariant measure for the transformation T_α. The transformation T_α has an invariant absolutely continuous measure μ with respect to the Lebesgue measure, if and only if, the Ruelle-Perron-Frobenius equation is satisfied (see Parry and Pollicott (1990); Ruelle (1978)), that is, if there exists a density function $g(x)$ such that

$$g(y) = \sum_{x:g(x)=y} g(x)\frac{1}{T'_\alpha(x)}$$

and $d\mu = g(x)dx$. The above equation implies that if $0 \le y < \alpha$ then

$$g(y) = g(x_1)\frac{1}{T'_\alpha(x_1)} + g(x_2)\frac{1}{T'_\alpha(x_2)},$$

for $0 \le x_1 < \alpha < x_2 \le 1$, and if $\alpha \le y \le 1$ then

$$g(y) = g(x_1)\frac{1}{T'_\alpha(x_1)}, \quad \text{for } 0 \le x_1 < \alpha.$$

Therefore, the Ruelle-Perron-Frobenius equation in this situation is given by

$$g(y) = \alpha\, g(x_1) + \Big(\frac{1-\alpha}{\alpha}\Big) g(x_2), \;\; 0 \le x_1 < \alpha < x_2 \le 1 \text{ and } 0 \le y < \alpha$$

and

$$g(y) = \alpha\, g(x_1), \quad 0 \le x_1 < \alpha \text{ and } \alpha \le y \le 1. \tag{2.2}$$

One considers the following density function

$$g(x) = \begin{cases} \frac{1}{\alpha(2-\alpha)}, & \text{if} \quad 0 \le x < \alpha \\ \frac{1}{2-\alpha}, & \text{if} \quad \alpha \le x \le 1. \end{cases} \tag{2.3}$$

For the above guess of the density function, it is easy to see that equations in (2.2) hold.

Therefore, the Ruelle-Perron-Frobenius equations are satisfied and the density function $g(x)$ given in expression (2.3) defines an invariant measure μ such that, for any Borel set A, $\mu(A) = \int_A g(x)dx$.

Consider in the sequel the following notation

$$c = \frac{1}{\alpha(2-\alpha)} \quad \text{and} \quad d = \frac{1}{2-\alpha}. \tag{2.4}$$

When T is expansive the measure μ is an ergodic one (see Parry and Pollicott (1990)). Hence, the measure μ given by the expression (2.3) is an ergodic measure (applying the last statement for T_α or T_α^2).

In an analogous way as in Lopes and Lopes (1995), consider $F_\alpha : K \to K$ (K will be defined later), *the natural extension of* T_α (see Bogomolny and Carioli (1995)). We shall give the explicit expression for the spectral density function of the stationary stochastic process $F_\alpha^t = X_t$, the random variable $\phi(x, y) = x$ and a measure ν that will be defined later.

The reason to consider $F_\alpha(x, y)$ and not $T_\alpha(x)$ in our reasoning is because F_α is a bijective map while T_α is not.

Consider now the stationary stochastic process given by (1.1), where $\{\xi_t\}_{t\in\mathbf{Z}}$ is a noise process. For simplicity of the exposition we suppose $\xi_t \sim N(0, \sigma_\xi^2)$, for any $t \in \mathbf{Z}$, that is, a Gaussian white noise process. We assume that $\{(X_t, Y_t)\}_{t\in\mathbf{Z}}$ and $\{\xi_t\}_{t\in\mathbf{Z}}$ are uncorrelated processes.

Define the *autocorrelation function of order* k of the process $\{X_t\}_{t\in\mathbf{Z}}$ by

$$\rho_X(k) = \frac{Cov(X_t, X_{t+k})}{\sqrt{Var(X_t)Var(X_{t+k})}} =$$

$$= \frac{E[X_t\phi(F_\alpha^k(X_t, Y_t))] - E(X_t)E[\phi(F_\alpha^k(X_t, Y_t))]}{\sqrt{Var(X_t)Var[\phi(F_\alpha^k(X_t, Y_t))]}} =$$

$$= \frac{E(xT_\alpha^k(x)) - E(x^2)}{Var(x)}. \tag{2.6}$$

Our goal is to derive the spectral density function of the process $\{X_t\}_{t\in\mathbf{Z}}$

$$f_X(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \exp -ik\lambda\rho_X(k), \quad \text{for any} \quad \lambda \in (-\pi,\pi].$$

Hence, one needs to derive the autocorrelation function, $\rho_X(k)$, defined above.

By abuse of the notation, we shall denote $\phi(x) = x$ and $\phi(x,y) = x$ by the same letter ϕ.

In an analogous way as in Lopes and Lopes (1995; Section 2), we will show that for positive k the autocorrelation function of order k of the dynamical systems $(F_\alpha(x,y), \phi(x,y), \nu)$ and $(T_\alpha(x), \phi(x), \mu)$ are the same. For negative values of k the autocorrelation function of order k of F_α is equal to the corresponding autocorrelation function of positive lag k of (T_α, ϕ, μ). These properties will be described in Section 2.

There is no meaning for the autocorrelation function of T_α at negative lag k because T_α is not an invertible map.

First one needs three technical propositions involving the transformation T_α.

The following proposition gives a characterization of the k-th iterated of the transformation $T_\alpha(x)$ by a recursive formula.

PROPOSITION 1: *The k-th iterated of the transformation $T_\alpha(x)$ given by the expression (1.2) is defined by*

$$T_\alpha^k(x) = \begin{cases} T_\alpha^{k-1}\left(\frac{x}{\alpha}\right), & \text{if} \quad 0 \le x < \alpha \\ \\ T_\alpha^{k-2}\left(\frac{x-\alpha}{1-\alpha}\right), & \text{if} \quad \alpha \le x \le 1, \end{cases} \tag{2.7}$$

for any integer $k \le 2$.

PROOF: The proof is given by an induction in k. First one wants to show that the expression (2.7) holds for $k = 2$ knowing that

$$T_\alpha^2(x) = \begin{cases} \frac{x}{\alpha^2}, & \text{if} \quad 0 \le x < \alpha^2 \\ \frac{x-\alpha^2}{1-\alpha}, & \text{if} \quad \alpha^2 \le x < \alpha \\ \frac{x-\alpha}{1-\alpha}, & \text{if} \quad \alpha \le x \le 1. \end{cases}$$

Since $T_\alpha^0 \equiv Id$, by using the recurrence formula when $k = 2$ one has

$$T_\alpha^2(x) = \begin{cases} T_\alpha^1\left(\frac{x}{\alpha}\right), & \text{if} \quad 0 \le x < \alpha \\ \frac{x-\alpha}{1-\alpha}, & \text{if} \quad \alpha \le x \le 1. \end{cases}$$

We know that

$$T_\alpha^1\left(\frac{x}{\alpha}\right) = \frac{\frac{x}{\alpha}}{\alpha} = \frac{x}{\alpha^2}, \quad \text{if} \quad 0 \le \frac{x}{\alpha} < \alpha,$$

that is,

$$T_\alpha^1\left(\frac{x}{\alpha}\right) = \frac{x}{\alpha^2}, \quad \text{if} \quad 0 \le x < \alpha^2$$

or

$$T_\alpha^1\left(\frac{x}{\alpha}\right) = \frac{\alpha\left(\frac{x}{\alpha} - \alpha\right)}{1-\alpha} = \frac{x-\alpha^2}{1-\alpha}, \quad \text{if} \quad \alpha \le \frac{x}{\alpha} \le 1,$$

that is,

$$T_\alpha^1\left(\frac{x}{\alpha}\right) = \frac{x-\alpha^2}{1-\alpha}, \quad \text{if} \quad \alpha^2 \le x \le \alpha.$$

Therefore, we have

$$T_\alpha^2(x) = \begin{cases} \frac{x}{\alpha^2}, & \text{if} \quad 0 \le x < \alpha^2 \\ \frac{x-\alpha^2}{1-\alpha}, & \text{if} \quad \alpha^2 \le x < \alpha \\ \frac{x-\alpha}{1-\alpha}, & \text{if} \quad \alpha \le x \le 1. \end{cases}$$

Hence, the recurrence formula holds for $k = 2$. Suppose now that (2.7) holds for k. One wants to show it also holds for $k+1$. Suppose $0 \le x < \alpha$. Then,

$$T_\alpha^{k+1}(x) = T_\alpha^k(T_\alpha(x)) = T_\alpha^k\Big(\frac{x}{\alpha}\Big).$$

If $\alpha \le x \le 1$ then

$$\begin{aligned} T_\alpha^{k+1}(x) &= T_\alpha^k(T_\alpha(x)) = T_\alpha^k\Big(\frac{\alpha(x-\alpha)}{1-\alpha}\Big) = \\ &= T_\alpha^{k-1}\Big(T_\alpha\Big(\frac{\alpha(x-\alpha)}{1-\alpha}\Big)\Big) = T_\alpha^{k-1}\Big(\frac{x-\alpha}{1-\alpha}\Big), \end{aligned}$$

since $\alpha(x-\alpha)/(1-\alpha) \le \alpha$ whenever $x \in [0,1]$. Therefore,

$$T_\alpha^{k+1}(x) = \begin{cases} T_\alpha^k\Big(\frac{x}{\alpha}\Big), & \text{if} \quad 0 \le x < \alpha \\ T_\alpha^{k-1}\Big(\frac{x-\alpha}{1-\alpha}\Big), & \text{if} \quad \alpha \le x \le 1 \end{cases}$$

and the proposition is proved.

PROPOSITION 2: *The integral*

$$A(k) = \int_0^1 T_\alpha^k(x)dx$$

satisfies the recursive equation

$$A(k) = \alpha A(k-1) + (1-\alpha)A(k-2), \tag{2.8}$$

for any integer $k \ge 2$, *with initial values* $A(0) = 1/2$ *and* $A(1) = (2-\alpha)\alpha/2$.

PROOF: From Proposition 1 one has, for any integer $k \ge 2$,

$$A(k) = \int_0^1 T_\alpha^k(x)dx = \int_0^\alpha T_\alpha^{k-1}\Big(\frac{x}{\alpha}\Big)dx + \int_\alpha^1 T_\alpha^{k-2}\Big(\frac{x-\alpha}{1-\alpha}\Big)dx.$$

By changing the variable x to $y = x/\alpha$ in the first and the variable x to $z = (x-\alpha)/(1-\alpha)$ in the second above integrals one has

$$A(k) = \int_0^1 T_\alpha^{k-1}(y)\alpha dy + \int_0^1 T_\alpha^{k-2}(z)(1-\alpha)dz = \alpha A(k-1) + (1-\alpha)A(k-2),$$

for any integer $k \geq 2$. So, the equation (2.8) holds. Now one observes that

$$A(0) = \int_0^1 T_\alpha^0(x)dx = \int_0^1 x dx = \frac{1}{2}$$

and

$$\begin{aligned} A(1) &= \int_0^1 T_\alpha^1(x)dx = \int_0^\alpha \frac{x}{\alpha}dx + \int_\alpha^1 \alpha\frac{(x-\alpha)}{1-\alpha}dx = \\ &= \frac{\alpha}{2} + \frac{\alpha}{1-\alpha}\Big(\frac{1}{2} - \alpha - \frac{\alpha^2}{2} + \alpha^2\Big) = \\ &= \frac{\alpha}{2} + \frac{\alpha}{1-\alpha}\Big(\frac{1}{2} - \alpha + \frac{\alpha^2}{2}\Big) = \frac{\alpha}{2} + \frac{\alpha}{2}(1-\alpha) = \frac{\alpha}{2}(2-\alpha). \end{aligned}$$

The proposition is proved.

PROPOSITION 3: *The integral*

$$B(k) = \int_0^1 xT_\alpha^k(x)dx$$

satisfies the recursive equation

$$B(k) = \alpha^2 B(k-1) + (1-\alpha)^2 B(k-2) + \alpha(1-\alpha)A(k-2), \tag{2.9}$$

for any integer $k \geq 2$, *with initial values* $B(0) = 1/3$ *and* $B(1) = (1+\alpha)(2-\alpha)\alpha/6$.

PROOF: From Proposition 1, for any integer $k \geq 2$,

$$B(k) = \int_0^1 xT_\alpha^k(x)dx = \int_0^\alpha xT_\alpha^{k-1}\Big(\frac{x}{\alpha}\Big)dx + \int_\alpha^1 xT_\alpha^{k-2}\Big(\frac{x-\alpha}{1-\alpha}\Big)dx.$$

By changing the variable x to $y = x/\alpha$ in the first and the variable x to $z = (x-\alpha)/(1-\alpha)$ in the second above integrals one has

$$\begin{aligned} B(k) &= \int_0^1 \alpha y T_\alpha^{k-1}(y)\alpha dy + \int_0^1 [z(1-\alpha)+\alpha]T_\alpha^{k-2}(z)(1-\alpha)dz = \\ &= \alpha^2 \int_0^1 yT_\alpha^{k-1}(y)dy + (1-\alpha)^2\int_0^1 zT_\alpha^{k-2}(z)dz + \\ &+ \alpha(1-\alpha)\int_0^1 T_\alpha^{k-2}(z)dz = \alpha^2 B(k-1) + (1-\alpha)^2 B(k-2) + \\ &+ \alpha(1-\alpha)A(k-2), \end{aligned}$$

for any integer $k \geq 2$. So, the equation (2.9) holds. The initial values are given by

$$B(0) = \int_0^1 xT_\alpha^0(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$$

and

$$\begin{aligned} B(1) &= \int_0^1 xT_\alpha^1(x)dx = \int_0^\alpha x\frac{x}{\alpha}dx + \int_\alpha^1 x\frac{\alpha(x-\alpha)}{1-\alpha}dx = \\ &= \frac{\alpha^2}{3} + \frac{\alpha}{1-\alpha}\Big(\frac{1}{3} - \frac{\alpha}{2} - \frac{\alpha^3}{3} + \frac{\alpha^3}{2}\Big) = \\ &= \frac{\alpha^2}{3} + \frac{\alpha}{6(1-\alpha)}(2 - 3\alpha + \alpha^3) = \\ &= \frac{\alpha^2}{3} - \frac{\alpha}{6}(\alpha^2 + \alpha - 2) = \frac{\alpha}{6}(2+\alpha-\alpha^2) = \frac{\alpha}{6}(1+\alpha)(2-\alpha). \end{aligned}$$

And the proposition is proved.

From Propositions 2 and 3 we shall derive the autocorrelation function $\rho_X(k)$, $k \geq 0$, of the process $\{X_t\}_{t\in\mathbf{N}}$.

PROPOSITION 4: *Let $\{X_t\}_{t\in\mathbf{N}}$ be the stationary stochastic process given in (2.5). The autocorrelation function of order k of the process $\{X_t\}_{t\in\mathbf{N}}$ defined in expression (2.6) is given by*

$$\rho_X(k) = \frac{E[X_t T_\alpha^k(X_t)] - \left(\frac{1+\alpha-\alpha^2}{2(2-\alpha)}\right)^2}{\frac{(\alpha^2-\alpha+1)(\alpha^2-5\alpha+5)}{12(2-\alpha)^2}}, \tag{2.10}$$

where $E[X_t T_\alpha^k(X_t)]$, *denoted by* $C(k)$, *is given by the three term relation*

$$C(k) = \alpha^2\, c\, B(k-1) + (1-\alpha)^2 d\, B(k-2) + \alpha(1-\alpha) d\, A(k-2), \qquad (2.11)$$

for any integer $k \geq 2$, *with* $A(k)$ *given by (2.8),* $B(k)$ *given by (2.9) and the constants* c *and* d *are defined in the expression (2.4). Moreover, the initial values* $C(0)$ *and* $C(1)$ *are given by*

$$C(0) = \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)} \quad \textit{and} \quad C(1) = \frac{\alpha(4-\alpha-\alpha^2)}{6(2-\alpha)}.$$

PROOF: From the stationarity of the process $\{X_t\}_{t\in\mathbf{N}}$, one observes that

$$E(X_t) = E(X_{t+k}) \equiv E(T_\alpha^k(X_t))$$

and

$$Var(X_t) = Var(X_{t+k}) \equiv Var(T_\alpha^k(X_t)).$$

The expected value of the process $\{X_t\}_{t\in\mathbf{N}}$ is given by

$$\begin{aligned} E(X_t) &= \int_0^1 x\,d\mu(x) = c\int_0^\alpha x\,dx + d\int_\alpha^1 x\,dx = \\ &= \frac{\alpha^2}{2\alpha(2-\alpha)} + \frac{1-\alpha^2}{2(2-\alpha)} = \frac{1+\alpha-\alpha^2}{2(2-\alpha)}. \end{aligned}$$

The second moment of the process $\{X_t\}_{t\in\mathbf{N}}$ is given by

$$\begin{aligned} E(X_t^2) &= \int_0^1 x^2 d\mu(x) = c\int_0^\alpha x^2 dx + d\int_\alpha^1 x^2 dx = \\ &= \frac{\alpha^2}{3(2-\alpha)} + \frac{1-\alpha^3}{3(2-\alpha)} = \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)}. \end{aligned}$$

Hence, the variance of the process $\{X_t\}_{t\in\mathbf{N}}$ is given by

$$\begin{aligned} Var(X_t) &= E(X_t^2) - [E(X_t)]^2 = \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)} - \left(\frac{1+\alpha-\alpha^2}{2(2-\alpha)}\right)^2 = \\ &= \frac{\alpha^4 - 6\alpha^3 + 11\alpha^2 - 10\alpha + 5}{12(2-\alpha)^2} = \frac{(\alpha^2-\alpha+1)(\alpha^2-5\alpha+5)}{12(2-\alpha)^2}. \end{aligned}$$

The autocorrelation function of order k of the process $\{X_t\}_{t\in\mathbf{N}}$ is defined in (2.6) where $E[X_t\, T_\alpha^k(X_t)]$ is given by

$$\begin{aligned} C(k) &\equiv E[X_t\, T_\alpha^k(X_t)] = \int_0^1 xT_\alpha^k(x)d\mu(x) = \\ &= c\int_0^\alpha xT_\alpha^k(x)dx + d\int_\alpha^1 xT_\alpha^k(x)dx = \\ &= c\int_0^\alpha xT_\alpha^{k-1}\Big(\frac{x}{\alpha}\Big)dx + d\int_\alpha^1 xT_\alpha^{k-2}\Big(\frac{x-\alpha}{1-\alpha}\Big)dx. \end{aligned}$$

The last above equality is due to Proposition 1. By changing the variable x to $y = x/\alpha$ in the first and variable x to $z = (x-\alpha)/(1-\alpha)$ in the second above integrals one has

$$\begin{aligned} C(k) &= c\int_0^1 \alpha y T_\alpha^{k-1}(y)\alpha dy + d\int_0^1((1-\alpha)z+\alpha)T_\alpha^{k-2}(z)(1-\alpha)dz = \\ &= c\alpha^2\int_0^1 yT_\alpha^{k-1}(y)dy + d(1-\alpha^2)\int_0^1 zT_\alpha^{k-2}(z)dz + \\ &+ d\alpha(1-\alpha)\int_0^1 T_\alpha^{k-2}(z)dz = \\ &= \alpha^2\, c\, B(k-1) + (1-\alpha)^2\, d\, B(k-2) + \alpha\,(1-\alpha)\, d\, A(k-2), \end{aligned}$$

for any integer $k \geq 2$, where $A(k)$ is given by (2.8), $B(k)$ is given by (2.9) and the constants c and d by (2.4). Hence, the expression (2.11) holds.

One observes that the initial values for $C(k)$ are given by

$$\begin{aligned} C(0) &= \int_0^1 xT_\alpha^0(x)d\mu(x) = \int_0^1 x^2 d\mu(x) = \\ &= c\int_0^\alpha x^2dx + d\int_\alpha^1 x^2dx = c\frac{\alpha^3}{3} + \frac{d}{3}(1-\alpha^3) = \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)} \end{aligned}$$

and

$$\begin{aligned} C(1) &= \int_0^1 xT_\alpha^1(x)d\mu(x) = c\int_0^\alpha xT_\alpha(x)dx + d\int_\alpha^1 xT_\alpha(x)dx = \\ &= c\int_0^\alpha x\frac{x}{\alpha}dx + d\int_\alpha^1 x\alpha\frac{(x-\alpha)}{1-\alpha}dx = \\ &= \frac{2\alpha-\alpha^3-\alpha^2+2\alpha}{6(2-\alpha)} = \frac{\alpha(4-\alpha-\alpha^2)}{6(2-\alpha)}, \end{aligned}$$

and the Proposition is proved.

From (2.10) we still need to know the quantity $C(k)$ in order to give the autocorrelation function $\rho_X(k)$ of the process $\{X_t\}_{t\in\mathbf{Z}}$. One can equivalently describe the quantities $A(k)$, $B(k)$ and $C(k)$ by the following power series

$$\varphi(z) = \sum_{k\geq 0} A(k)z^k, \quad \Psi(z) = \sum_{k\geq 0} B(k)z^k \quad \text{and} \quad \gamma(z) = \sum_{k\geq 0} C(k)z^k. \tag{2.12}$$

PROPOSITION 5: *The power series for $A(k)$, $B(k)$ and $C(k)$ as in expression (2.12) are given, respectively, by*

$$\begin{aligned}
\varphi(z) &= \frac{1+\alpha z(1-\alpha)}{2[(1-\alpha)z+1](1-z)}, \\
\psi(z) &= \frac{2-\alpha z(\alpha^2+\alpha-2)+6\alpha(1-\alpha)z^2\varphi(z)}{6[1-\alpha^2 z-(1-\alpha)^2 z^2]} \quad \textit{and} \\
\gamma(z) &= \frac{2\alpha^2(1-\alpha)+2+\alpha z(2-\alpha-\alpha^2)}{6(2-\alpha)} + \\
&+ \left[\frac{\alpha z+(1-\alpha)^2 z^2}{2-\alpha}\right]\psi(z) + \frac{\alpha(1-\alpha)z^2}{2-\alpha}\varphi(z).
\end{aligned} \tag{2.13}$$

PROOF: From Proposition 2 one has the recursive formula for $A(k)$, for any integer $k \geq 2$, and the two initial values. Hence,

$$\begin{aligned}
\varphi(z) &= A(0)+A(1)z+\sum_{k\geq 2}[\alpha A(k-1)+(1-\alpha)A(k-2)]z^k = \\
&= A(0)+A(1)z+\alpha\sum_{k\geq 2}A(k-1)z^k+(1-\alpha)\sum_{k\geq 2}A(k-2)z^k = \\
&= \frac{1}{2}+\frac{\alpha}{2}(2-\alpha)z+\alpha z[\varphi(z)-A(0)]+(1-\alpha)z^2\varphi(z) = \\
&= \frac{1}{2}+\frac{\alpha}{2}z(2-\alpha)+\alpha z\varphi(z)-\frac{\alpha z}{2}+(1-\alpha)z^2\varphi(z).
\end{aligned}$$

Therefore,

$$\varphi(z) = \frac{1+\alpha z-\alpha^2 z}{2[1-\alpha z-(1-\alpha)z^2]} = \frac{1+\alpha z(1-\alpha)}{2[(1-\alpha)z+1](1-z)}.$$

And the first equality in (2.13) holds.

To prove the second equality in (2.13) one considers the recursive formula for $B(k)$, for any integer $k \geq 2$, with two initial values given in Proposition 3. Hence,

$$\begin{aligned}
\psi(z) &= \sum_{k\geq 0} B(k)z^k = B(0) + B(1)z + \sum_{k\geq 2}[\alpha^2 B(k-1) + \\
&+ (1-\alpha)^2 B(k-2) + \alpha(1-\alpha)A(k-2)]z^k = B(0) + B(1)z + \\
&+ \alpha^2 \sum_{k\geq 2} B(k-1)z^k + (1-\alpha)^2 \sum_{k\geq 2} B(k-2)z^k + \\
&+ \alpha(1-\alpha)\sum_{k\geq 2} A(k-2)z^k = \frac{1}{3} + \frac{\alpha}{6}z(1+\alpha)(2-\alpha) + \\
&+ \alpha^2 z\psi(z) - \frac{\alpha^2 z}{3} + (1-\alpha)^2 z^2\psi(z) + \alpha(1-\alpha)z^2\varphi(z).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi(z) &= \frac{2 + \alpha(1+\alpha)(2-\alpha)z - 2\alpha^2 z + 6\alpha(1-\alpha)z^2\varphi(z)}{6[1 - \alpha^2 z - (1-\alpha)^2 z^2]} = \\
&= \frac{2 - \alpha z(\alpha^2 + \alpha - 2) + 6\alpha(1-\alpha)z^2\varphi(z)}{6[1 - \alpha^2 z - (1-\alpha)^2 z^2]}.
\end{aligned}$$

And the second equality in (2.13) holds. To prove the third equality in (2.13) one considers the recursive formula for $C(k)$, for any integer $k \geq 2$, with two initial values given in Proposition 4. Hence,

$$\begin{aligned}
\gamma(z) &= \sum_{k\geq 0} C(k)z^k = C(0) + C(1)z + \sum_{k\geq 2}\alpha^2 c\, B(k-1)z^k + \\
&+ \sum_{k\geq 2}(1-\alpha)^2 d\; B(k-2) + \alpha(1-\alpha)d\; A(k-2)z^k = \\
&= C(0) + C(1)z + \alpha^2 c\sum_{k\geq 2} B(k-1)z^k + (1-\alpha)^2 d\sum_{k\geq 2} B(k-2)z^k + \\
&+ \alpha(1-\alpha)d\sum_{k\geq 2} A(k-2)z^k = \\
&= \frac{1+\alpha^2-\alpha^3}{3(2-\alpha)} + \frac{\alpha(4-\alpha-\alpha^2)z}{6(2-\alpha)} + \alpha^2 c\, z[\psi(z) - B(0)] +
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (1-\alpha)^2 d\, z^2\psi(z) + \alpha(1-\alpha)d\, z^2\varphi(z) = \\
&= \quad \frac{2(1+\alpha^2-\alpha^3)+\alpha(4-\alpha-\alpha^2)z}{6(2-\alpha)} + \alpha^2 c\, z\psi(z) - \frac{1}{3}\alpha^2 c\, z + \\
&+ \quad (1-\alpha)^2 d\, z^2\psi(z) + \alpha(1-\alpha)d\, z^2\varphi(z) = \\
&= \quad \frac{2(1+\alpha^2-\alpha^3)+\alpha z(2-\alpha-\alpha^2)}{6(2-\alpha)} + \left[\frac{\alpha z + (1-\alpha)^2 z^2}{2-\alpha}\right]\psi(z) + \\
&+ \quad \frac{\alpha(1-\alpha)z^2}{2-\alpha}\varphi(z).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\gamma(z) &= \frac{2\alpha^2(1-\alpha)+2+\alpha z(2-\alpha-\alpha^2)}{6(2-\alpha)} + \\
&+ \frac{1}{2-\alpha}\Big[(\alpha z + (1-\alpha)^2 z^2)\psi(z) + \alpha(1-\alpha)z^2\varphi(z)\Big].
\end{aligned}
$$

Hence, the proposition is proved.

Note that the estimation of α for the stationary stochastic process (1.1) can be obtained from Proposition 4. This follows from the fact that

$$
\frac{\alpha(4-\alpha-\alpha^2)}{6(2-\alpha)} = C(1) = \int_0^1 x T_\alpha^0(x)d\mu(x).
$$

Considering a times series $\{Z_t\}_{t=1}^N$ and using the Birkhoff's Ergodic Theorem one can estimate α by solving the equation

$$
C(\hat{1}) = \frac{1}{N}\sum_{t=1}^{N-1} Z_t Z_{t+1} = \frac{\hat{\alpha}(4-\hat{\alpha}-\hat{\alpha}^2)}{6(2-\hat{\alpha})},
$$

in the variable $\hat{\alpha}$.

3. THE SPECTRAL DENSITY FUNCTION

In this section we shall use the results for T_α obtained in the previous section for the map F_α.

Now we define the natural extension F_α of the transformation T_α. Consider the transformation $F_\alpha : K \to K$, where $K = ([0,1] \times (0,\alpha)) \cup ([0,\alpha] \times [\alpha, 1])$, given by $F_\alpha(x,y) = (T_\alpha(x), G_\alpha(x,y))$ where $G_\alpha(x,y)$ is defined by (1.2).

One observes that F_α is a homeomorphism of K and F_α^n is of the form

$$F_\alpha^n(x,y) = (T_\alpha^n(x), G_{\alpha,n}(x,y)),$$

that is, the action of F_α in the first variable is just the action of T_α.

Now we shall define the F_α-invariant measure ν on K, absolutely continuous with respect to the Lebesgue measure $dxdy$.

For sets of the form $A_1 \times A_2$, where $A_1 \subset (0,\alpha)$ and $A_2 \subset (\alpha, 1)$ or $A_1 \subset (\alpha, 1)$ and $A_2 \subset (0,\alpha)$, we define $\nu(A_1 \times A_2) = (2-\alpha)\,\mu(A_1)\,\mu(A_2)$.

For sets of the form $A_1 \times A_2$, where $A_1 \subset (0,\alpha)$ and $A_2 \subset (0,\alpha)$, we define $\nu(A_1 \times A_2) = (2-\alpha)\,\alpha\,\mu(A_1)\,\mu(A_2)$.

It is not difficult to see that ν is invariant for F_α and is absolutely continuous with respect to the Lebesgue measure. The measure ν satisfies $\nu(A \times (0,1)) = \mu(A)$, when $A \subset (0,\alpha)$ and $\nu(A \times (0,\alpha)) = \mu(A)$, when $A \subset (\alpha, 1)$.

THEOREM 1: *The spectral density function of the process*

$$Z_t = X_t + \xi_t = \phi(F_\alpha^t(X_0, Y_0)) + \xi_t, \quad \textit{for } t \in \mathbf{Z},$$

is given by

$$f_Z(\lambda) = \frac{1}{2\pi}[\gamma(e^{i\lambda}) + \gamma(e^{-i\lambda}) - \rho_X(0)] + \frac{\sigma_\xi^2}{2\pi}, \tag{2.14}$$

for any $\lambda \in (-\pi, \pi]$, where $\gamma(z)$ is given by the third equality in expression (2.13) of Proposition 5. The point (X_0, Y_0) is chosen randomly according to the measure ν (or according to the Lebesgue measure).

PROOF: The integral of ν with respect to any function H that depends only on the x variable is such that

$$\int H(x)d\nu(x,y) = \int H(x)d\mu(x). \tag{2.15}$$

One observes that $\phi(x,y) = x$ is a random variable and $F_\alpha : K \to K$ defines a stationary stochastic process $X_t = \phi(F_\alpha^t(X_0, Y_0))$ with respect to the invariant probability ν defined above.

From the expression (2.15) and for any positive $t \in \mathbf{N}$,

$$\int \phi(F_\alpha^t(x,y))\phi(x,y)d\nu(x,y) = \int \phi(T_\alpha^t(x))\phi(x)d\mu(x).$$

For any positive $t \in \mathbf{N}$ (that is, when $-t$ is negative)

$$\int \phi(F_\alpha^{-t}(x,y))\phi(x,y)d\nu(x,y) = \int \phi(x,y)\phi(F_\alpha^t(x,y))d\nu(x,y)$$

because ν is invariant for F_α. Therefore, from (2.15) and for any positivé $t \in \mathbf{N}$

$$\int \phi(F_\alpha^{-t}(x,y))\phi(x,y)d\nu(x,y) = \int \phi(T_\alpha^t(x))\phi(x)d\mu(x). \tag{2.16}$$

The conclusion is that the autocorrelation coefficients $C(t) = C(-t)$, $t \in \mathbf{N}$ of the stochastic process given by the random variable $\phi(x,y) = x$, the transformation F_α and the probability ν can be obtained from the autocorrelation coefficients obtained previously for the stochastic process given by the random variable $\phi(x) = x$, the transformation T_α and the probability μ.

The spectral density function of the process $\{X_t\}_{t\in\mathbf{Z}}$ is given by

$$f_X(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\exp(-ik\lambda)\rho_X(k) \geq 0,$$

for all $\lambda \in (-\pi, \pi]$ (see Brockwell and Davis (1987)). Therefore, the spectral density function of the process (1.1) is (see (2.16)

$$f_Z(\lambda) = f_X(\lambda) + \frac{\sigma_\xi^2}{2\pi} = \frac{1}{2\pi}[\gamma(e^{i\lambda}) + \gamma(e^{-i\lambda}) - \rho_X(0)] + \frac{\sigma_\xi^2}{2\pi}, \quad \text{for all} \quad \lambda \in (-\pi, \pi],$$

where γ is explicitly given by the expression (2.13) in Proposition 5.

From Parry and Pollicott (1990) it is known that $\rho_X(k)$ decays exponentially to zero, hence $f_X(\lambda)$ is an analytic function for any $\lambda \in (-\pi, \pi]$.

We can also analyze alternatively $f_Z(u)$, $u \in (-1, 1)$ when $u = \frac{\lambda}{\pi}$. We shall use this notation in the appendix. If $\rho_X(k) = \rho_X(-k)$ (as in the present case), one just needs to consider $f_Z(u)$, $u \in (0, 1)$, because $f_Z(u) = f_Z(-u)$.

APPENDIX. THE CONSISTENCY OF THE PERIODOGRAM

We analyze in this section the periodogram for (ϕ, T_α) (or for (ϕ, F_α)). Our purpose here is to show how to obtain an approximation of the spectral measure $f_X(u)$ from a time series data $X_t = T_\alpha^t(X_0)$, for $t \in \{1, \cdots, N\}$, where X_0 is chosen at random according to the measure μ (or according to the Lebesgue measure). We can alternatively estimate

$$\sum_{h=-\infty}^{\infty} E(X_0 X_h) \exp(-2\pi i h u),$$

with $X_h = \phi(F_\alpha^h(x, y))$ and from this result estimate the spectral measure $f_X(u)$. By abuse of the notation we shall also call the above expression as the spectral measure.

Note that as the random variable $\phi(x, y)$ depends only on x (for positive t, $\phi(F_\alpha^t(x, y)) = T_\alpha^t(x)$ independently of y) we shall consider the periodogram for T_α instead of F_α.

In fact, the proof presented here works for any expansive map T, any Holder random variable ϕ and the ergodic absolutely continuous invariant probability μ for T. We leave to the reader the extension of the reasoning below to such case.

Consider the transformation $T_\alpha : [0, 1] \to [0, 1]$, where $\alpha \in (0, 1)$, given by (1.2). The map T_α (or T_α^2) is an expanding one.

We shall assume, for the sake of simplicity, that ϕ is the random variable $\phi(x) = x$ and $d\mu(x) = g(x)dx$ is the unique ergodic and absolutely continuous invariant probability for T_α.

The goal here is to sketch the proof of the smoothed periodogram's consistency in the above case. One denotes $X_t = (\phi \circ T_\alpha^t)(X_0) = T_\alpha^t(X_0) = \phi(F_\alpha^t(X_0, Y_0))$, and $\{X_t\}_{t=1}^N$ a time series of N observations where X_0 is an initial point chosen randomly according to μ. From the Birkhoff's Ergodic

Theorem (μ is ergodic for T_α), for each subinterval $\Delta_j = (a_j, b_j) \subset [0,1]$ and for μ-almost every $x_0 \in [0,1]$

$$\mu(\Delta_j) = \int_{\Delta_j} g(x)dx = \lim_{N\to\infty} \frac{1}{N}(\#\{t \mid 1 \le t \le N, T_\alpha^t(x_0) \in \Delta_j\}).$$

If $|b_j - a_j| = \epsilon$ is small and N is large enough, then

$$A_N(\epsilon) = \frac{1}{N}(\#\{t \mid 1 \le t \le N, T_\alpha^t(X_0) \in \Delta_j\}) \approx g(c_j)\Delta_j = B_N(\epsilon), \tag{2.17}$$

for some $c_j = c_j(N) \in \Delta_j$.

The expression $A_N(\epsilon) \approx B_N(\epsilon)$ means that the quotient $A_N(\epsilon)/B_N(\epsilon)$ goes to one when N goes to infinity and ϵ goes to zero.

Consider the discrete Fourier transform of the spatial position of the data obtained as the sampled time series $X_t = T_\alpha^t(X_0)$, for $1 \le t \le N$,

$$f(k) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} X_t \exp(-i\omega_k t),$$

where $\omega_k = 2\pi k N^{-1}, k = 1, 2, \cdots, N$, are the so-called *the Fourier frequencies* of the time series X_t, $1 \le t \le N$. The periodogram value $I(\omega_k)$ at the frequency ω_k, for

$$k \in \left\{ j \in \mathbf{Z}; 0 < \omega_j = \frac{2\pi j}{N} \le 2\pi \right\},$$

is defined in terms of the discrete Fourier transform $f(k)$ of a sample X_t, for $1 \le t \le N$, by

$$\begin{aligned} I(\omega_k) &= f(k)\overline{f(k)} = \frac{1}{N} \sum_{t=1}^{N} X_t \exp(-i\omega_k t) \sum_{s=1}^{N} X_s \exp(i\omega_k s) = \\ &= \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} X_t X_s \exp(-i(t-s)\omega_k), \end{aligned}$$

where $\overline{z}$ denotes the complex conjugate of z.

For each $h \in \mathbf{Z}$ consider t and s such that $t - s = h$. Then,

$$I(\omega_k) = \frac{1}{N} \left(\sum_{h=0}^{N-1} \sum_{s=1}^{N-h} X_s X_{s+h} \exp(-ih\omega_k) + \sum_{h=-1}^{1-N} \sum_{s=-h}^{N} X_s X_{s+h} \exp(-ih\omega_k) \right) =$$

$$= \frac{1}{N}\sum_{h=0}^{N-1}\sum_{s=1}^{N-h} X_s\phi(F_\alpha^h(X_s,Y_s))\exp(-ih\omega_k)+$$

$$+\frac{1}{N}\sum_{h=-1}^{1-N}\sum_{s=-h}^{N} X_s\phi(F_\alpha^h(X_s,Y_s))\exp(-ih\omega_k). \tag{2.18}$$

Now if we take Δ_j, $1 \leq j \leq v$, as a partition by intervals (of the same size) of the interval [0,1], with $|\Delta_j| = \epsilon = 1/v$ small, one observes from (2.17) that

$$\frac{\#[X_j \in \Delta_j]}{N} \approx \Delta_j\, g(c_j),$$

where $c_j \in \Delta_j$, $1 \leq j \leq v$.

We shall sum up $X_s = T_\alpha^s(X_0) = \phi(F_\alpha^s(X_0, Y_0))$ according to its position in each Δ_j. Hence,

$$\Delta_j\, g(c_j)N \approx \#\{s \mid 1 \leq s \leq N, X_s \in \Delta_j\}.$$

Then, from (2.18)

$$I(\omega_k) \approx \frac{1}{N}\sum_{|h|<N}\sum_{j=1}^{v} c_j\, \phi(F_\alpha^h(c_j, y_j))(\Delta_j\, g(c_j)N)\exp(-ih\omega_k) =$$

$$= \sum_{|h|<N}\sum_{j=1}^{v} c_j\, \phi(F_\alpha^h(c_j, y_j))\, g(c_j)\, \Delta_j\, \exp(-ih\omega_k). \tag{2.19}$$

We shall show that for any X_0 chosen at random, then $\sum_{k=1}^{N} I(\omega_k)\delta_{\omega_k}$ converges in the distribution sense to the spectral density function

$$\sum_{h\in\mathbf{Z}} E(X_0X_h)\exp(-2\pi ihu)$$

, where δ_{ω_k} is the Dirac delta function concentrated at the frequency ω_k, $1 \leq k \leq N$. Hence, we will show that for any test function $z(u)$, $u \in [0,1]$,

$$\int_0^1 z(u)d\left(\sum_{k=1}^{N} I(\omega_k)\delta_{\omega_k}\right)$$

converges to

$$\int_0^1 z(u) \left(\sum_{h \in \mathbf{Z}} E(X_0 X_h) \exp(-2\pi i h u) \right) du$$

when N goes to infinity.

By integrating the smoothed periodogram against a test function $z(u)$, $u \in [0,1]$, and using (2.19)

$$\lim_{N\to\infty} \lim_{v\to\infty} \sum_{k=1}^{N} I(\omega_k)\, z\left(\frac{k}{N}\right) =$$

$$= \lim_{N\to\infty} \lim_{v\to\infty} \sum_{k=1}^{N} \left(\sum_{|h|<N} \sum_{j=1}^{v} c_j F_\alpha^h(c_j, y_j) g(c_j) \Delta_j \exp(-ih\frac{2\pi k}{N}) \right) z\left(\frac{k}{N}\right) =$$

$$= \int_0^1 \left[\sum_{h \in \mathbf{Z}} \left(\int_0^1 x\, \phi(F_\alpha^h(x,y))\, g(x) dx \right) \exp(-2\pi i h u) \right] z(u) du =$$

$$= \int_0^1 \left(\sum_{h \in \mathbf{Z}} E(X_0 X_h) \exp(-2\pi i h u) \right) z(u) du. \qquad (2.20)$$

Therefore, the smoothed periodogram converges in distribution sense to the spectral density function.

The property considered above in (2.20) describes a method for obtaining a good approximation to the spectral density function. This method will be explained below.

Consider $z(u) = I_{[x-\epsilon, x+\epsilon]}(u)$ for a fixed x and a small fixed ϵ.

From the reasoning described before, for such $z(u)$, $(2\epsilon)^{-1} \sum_{k=1}^{N} I(\omega_k) z(k/N)$ is approximately equal to

$$\sum_{h=1-N}^{N-1} E(X_0 X_h) \exp(-2\pi i h u),$$

if N is large and ϵ small enough.

Considering now several $z_i(u) = I_{[x_i-\epsilon, x_i+\epsilon]}(u)$, where x_i are equally spaced,

$$[x_1 - \epsilon, x_1 + \epsilon) \cup [x_2 - \epsilon, x_2 + \epsilon) \cup .... \cup [x_n - \epsilon, x_n + \epsilon]$$

is a partition of $[0,1]$ and applying the same reasoning to each $z_i(u)$, we obtain the approximate shape of the graph of

$$\sum_{h=1-N}^{N-1} E(X_0 X_h) \exp(-2\pi i\, hu) \quad , \quad u \in [0,1],$$

as a function of u.

From the above expression, one can derive the approximate graph of the spectral density $f_X(u)$ or $f_Z(u)$.

The proceeding just described above is called *smoothing the data* (see Brockwell and Davis (1987)). For instance, if one takes a large sample $T_\alpha^t(x_0)$, for $1 \leq t \leq 10,000$, the periodogram is given by

$$I(\omega_k) = N^{-1} \sum_{t=1}^{N} X_t \exp(-i\omega_k t) \sum_{s=1}^{N} X_s \exp(i\omega_k s) =$$

$$= N^{-1} \sum_{t=1}^{N} \sum_{s=1}^{N} X_s X_t \exp(-i(t-s)\omega_k)$$

and one can plot this real function in the interval $(0, 2\pi]$ as a function of ω_k. This graph will show a sparse amount of data, but if one takes a partition of the interval in small intervals and takes means of this data in each small interval (also called *smoothing the data*), then the graph of a well defined spectral density

$$\sum_{h=-\infty}^{\infty} E(X_0 X_h) \exp(-2\pi i h u)$$

as described in this section will be obtained.

REFERENCES

BOGOMOLNY, E. and CARIOLI, M. (1995) Quantum Maps of Geodesic Flows on Surfaces of Constant Negative Curvature. To appear in *Phy sica D*.

BROCKWELL, P. J. and DAVIS, R. A. (1987) *Time Series: Theory and ethods*. New York: Springer-Verlag.

LOPES, A. and LOPES, S. (1995) Parametric Estimation and Spectral Anal ysis of Chaotic Time Series. Submitted.

PARRY, W. and POLLICOTT, M. (1990) Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics. *Asterisque*, Vol. 187-188.
ROBINSON, C. (1995) *Dynamical Systems*, New York: CRC Press.
RUELLE, D. (1978) *Thermodynamic Formalism*, assachusetts: Addison Wesley.

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