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Dynamical Solutions
of
Linear Matrix Differential Equations

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Abstract

We discuss the m th-order linear differential equation with matrix coefficients in terms of a particular matrix solution that enjoys properties similar to the exponential of first order equations. A new formula for the exponential matrix is established with dynamical solutions related to the generalized Lucas polynomials.

It is enough to verify that such functions are solutions of (1) which satisfy the initial values (3). This was done in [13]. Here we shall prove it by using the relationships (9). For $j = 0$ we have

$$C'_0(t) = D(t)A_m = D^{(m)}(t) - \sum_{j=1}^{m-1} D^{(m-j)}(t)A_j, \quad ,$$

and (10) follows by integrating both sides between 0 and t . Let us assume that (10) is valid for $j-1 < m-1$. Then

$$C_j^{(j+1)}(t) = C_{j-1}^{(j)}(t) + D^{(j)}(t)A_{m-j}, \quad ,$$

can be written as

$$C_j^{(j+1)}(t) = D^{(m-1)}(t) - \sum_{i=1}^{m-j-1} D^{(m-i-1)}(t)A_i$$

We then integrate both sides, j -times between 0 and t , in order to obtain (10).

Remark

When the coefficients A_j commute among themselves, then they also do with $D(t)$ and its derivatives. This means that they can appear as right factors in (10), which is the case with certain higher-order evolution equations [2]. We should observe that their natural left position could have certain implicancies when thinking in numerical spatial discretizations for higher-order distributed linear systems.

Power of the Companion Matrix

We shall now proceed to derive a formula for the powers of the companion matrix A defined in (5). It is convenient to write

$$A^k = [A_{ij}^{(k)}] ,$$

where the components are square matrices. The matrix exponential e^{At} can be expressed operationally as

$$e^{At} = [I \quad d/dt \quad \dots \quad d^{m-1}/dt^{m-1}]' [C_0(t) \quad C_1(t) \quad \dots \quad C_{m-1}(t)]$$

Since the k th-derivative of e^{At} at $t=0$ equals to A^k it follows that

$$A_{ij}^{(k)} = C_{j-1}^{(k+i-1)}(0)$$

Consequently, we obtain from the recurrence lemma that

$$A^k = \begin{bmatrix} j-1 \\ \sum_{s=0}^{j-1} D_{k+i-j+s-1} A_{m-s} \end{bmatrix} \quad (11)$$

is a $m \times n$ matrix with $n \times n$ components where n is the order of the coefficients.

3. The Second-Order Damped Equation

We shall consider now the matrix equation

$$u''(t) = Bu'(t) + Au(t) , \quad (12)$$

which has the general solution

$$u(t) = C(t)u(0) + D(t)u'(0)$$

The complementary basis matrix solution $C(t)$ is given by

$$C(t) = D'(t) - D(t)B ,$$

as follows from (10). We claim that $C(t)$ is not a left solution of (12) unless the matrices A and B commute. In fact, to have $C'' = C'B + CA$ is equivalent to

$$D''' - D''B = (D'' - D'B)B + (D' - DB)A$$

or

$$D(t)AB = D(t)BA ,$$

because $D(t)$ and its derivatives are left solutions of (12). It is clear that the last equality holds only when A and B commute.

Let us now examine how analogous is (12) with a second order scalar equation. When the coefficients commute

$$AB = BA ,$$

it is easy to see that $D(t)$ is of the same form as for the case, that is

$$D(t) = e^{(B/2)t} \sinh \sqrt{\Delta} t / \sqrt{\Delta} , \quad \Delta = (B^2 + 4A)/4 \quad (13)$$

where this expression should be understood as the product of two power series or more appropriately as a matrix function of two commuting variables [15].

A necessary condition for the function $D(t)$ defined by (13) to be a solution of (12) is that commutes with $B^2 + 4A$.

This follows by differentiating (13) and evaluating $D'' - BD' - AD$. We claim that (13) can not be the dynamical solution of (12) unless A and B commute. In fact, if such $D(t)$ were the dynamical solution of (12), then $G(t) = (B^2 + 4A)D(t)$ will be also a solution. Then

$$(B^2 + 4A)D_{k+2} = B(B^2 + 4A)D_{k+1} + A(B^2 + 4A)D_k ; k=0,1,2,\dots$$

reduces to

$$(AB - BA)D_{k+1} = (AB^2 - B^2A)D_k$$

because of (7). The conclusion follows since for $k=0$ we must have $AB - BA = 0$. Therefore, the dynamical solution $D(t)$ a linear matrix differential equation could be considered as a mathematical object of its own and whose study should be pursued.

The following linear differential operators

$$Lu = u'' - Bu' - Au ; L^*w = w'' + w'B - wA$$

satisfy

$$\int_0^t w Lu ds = \int_0^t (L^*w)u ds + B(u,w) \Big|_0^t$$

where

$$B(u,w) = wu' - w'u - wBu$$

for column vectors u and row vectors w . Let $D^*(t)$ be the dynamical solution of the matrix equation $v'' = -Bv' + Av$. It will satisfy

$$w'' = -w'B + wA \quad (14)$$

Let $C^*(t)$ be the complementary basis solution of $D^*(t)$. Then

$$B(D(t), D^*(t)) = 0$$

implies

$$D^*(t)D'(t) = (D^{*'}(t) + D^*(t)B)D(t)$$

or simply

$$D^*(t)D'(t) = C^*(t)D(t)$$

We shall refer to (14) as the adjoint equation of (12). This equation is the same that we would get by considering the adjoint equation of $z' = Az$ where A is the companion matrix associated with (12).

Since $D(t)$ and $D'(t)$ does not vanish simultaneously, nor with $C(t)$, it follows that $D(t)$ and $D^*(t)$ vanish simultaneously.

4. Series Representation of Dynamic Solutions

The question of having an explicit series representation for the dynamical solution amounts to solve the matrix difference equation (7) or to take the appropriate projection of e^{At} when using the companion matrix A . Since the powers of this matrix have been obtained in terms of the power series coefficients of $D(t)$, we shall discuss the solution of the difference equation.

For simplicity, we restrict ourselves to the case

$m = 2$.

The arguments and formulae could be easily extended to higher-order equations. We consider

$$D_{k+2} = A_1 D_{k+1} + A_2 D_k \quad (15)$$

$$D_1 = I, \quad D_0 = 0,$$

where A_1, A_2 are arbitrary square matrices of the same order.

After writing out the first few terms D_k that satisfy (15), it is observed that a register of the involved lower indices for the coefficients, rather than exponents, will be a convenient way to keep track of a formation law. We claim

$$D_{2k} = \sum_{i=1}^k \sum_{\substack{s_1 \dots s_{2k-i} \\ s_1 + \dots + s_{2k-i} = 2k-1}} A_{s_1} \dots A_{s_{2k-i}},$$

$$D_{2k+1} = \sum_{i=1}^k \sum_{\substack{s_1 \dots s_{2k+1-i} \\ s_1 + \dots + s_{2k+1-i} = 2k}} A_{s_1} \dots A_{s_{2k+1-i}},$$
(16)

where the s_i 's can take only the values 1, 2 .

The above expression hold obviously when $k = 1$. Let assume that are valid for certain positive integer k . Then splitting the somatory for $s_1 = 1$ and $s_1 = 2$, it will turn out that

$$D_{2(k+1)} = A_1 \sum_{i=1}^k \sum_{\substack{t_1 \dots t_{2k+1-i} \\ t_1 + \dots + t_{2k+1-i} = 2k}} A_{t_1} \dots A_{t_{2k+1-i}} + A_2 \sum_{i=1}^k \sum_{\substack{t_1 \dots t_{2k-i} \\ t_1 + \dots + t_{2k-i} = 2k-1}} A_{t_1} \dots A_{t_{2k-i}}$$

$$= A_1 D_{2k+1} + A_2 D_{2k},$$

and similarly for $D_{2(k+1)+1}$.

Therefore, the dynamical solution of a second-order matrix differential equation is given by

$$D(t) = \sum_{j=1}^{\infty} \sum_{i=1}^{(j)} \left[A_{S_1} \dots A_{S_{j-i}} \right] t^{j-i}/(j-i)! \quad (17)$$

$$s_1 + \dots + s_{j-i} = j-1$$

where

$$(j) = \begin{cases} j/2 & , \quad j \text{ even} \\ (j+1)/2 & , \quad j \text{ odd} \end{cases}$$

The above formula furnishes a non-trivial example of a matrix function of two non-commuting variables. This kind of functions were considered in the work of Lappo-Danilevskii [6]. Certainly, this power series is not amenable for drawing easy analytical or computational conclusions. Therefore, we shall develop in what follows a more operational approach based in the theory of matrix functions due to Runckel-Pittelkow [14] and Schwerdtfeger [15].

Let $f(z)$ be analytic in $|z| < K$ where $K > M$ with $M = \max |\lambda_i|$ and λ_i the different eigenvalues of a square matrix S of order N . Let $c(z) = \sum_{j=0}^N c_{n-j} z^j$ be the characteristic polynomial of S with $c_0 = 1$, and denote by d_s the coefficients of the Laurent expansion for $1/c(z)$ around the origin with $|z| > M$. Then

$$f(S) = \sum_{j=0}^{N-1} \phi_j(f) S^j \quad (18)$$

where

$$\begin{aligned}\phi_j(f) &= \sum_{s=j+1}^N c_{n-s} \sum_{k=N+j-s}^{\infty} f^{(k)}(0)/k! \\ &= \sum_{s=j}^{N-1} c_{s-j} \sum_{k=j}^{\infty} d_{k+n-s} f^{(k)}(0)/k! \\ &= [f^{(j)}(0)/j! - \sum_{s=0}^j c_{N-s} \sum_{k=N+j-s}^{\infty} f^{(k)}(0)/k!]\end{aligned}$$

Moreover, the d_s can be computed recursively by

$$d_s = - \sum_{k=1}^N c_k d_{s-k} \quad \text{for } s \geq N$$

using $d_s = 0$ for $s < N$ and $d_N = 1$. See [12] for details.

Any of these formulae could be used with $\exp(St)$ where S is the companion matrix associated with (12) and then take the appropriate projection, that is $[I \ 0] S^j \begin{bmatrix} 0 \\ I \end{bmatrix} = D_j$. We thus have

Theorem 1

The solution of the matrix difference equation $D_{k+2} = BD_{k+1} + AD_k$, $D_0 = 0$, $D_1 = I$ is given by

$$D_k = \sum_{j=0}^{N-1} \phi_j(k) D_j, \quad N = 2n \quad (19)$$

where

$$\begin{aligned}\phi_j(k) &= \sum_{s=j+1}^N c_{N-s} d_{k+s-j} = \\ &= \delta_{jk} - \sum_{s=0}^j c_{N-s} d_{k+s-j}\end{aligned}$$

Proof

From (18) it follows that $S^k = \sum_{j=0}^{N-1} j^{(k)} S^j$ with those given functions j and the conclusion is clear.

Therefore, the dynamical solution of a second order damped equation can be given by any of the following formulas:

$$D(t) = \sum_{r=m-1}^{N-1} D_r t^r \sum_{s=r+1}^N C_{N-s} \sum_{v=N+r-s}^{\infty} d_{v+s-r} \frac{t^v}{v!} \quad (20)$$

$$D(t) = \sum_{r=m-1}^{N-1} D_r \left[\frac{t^r}{r!} - \sum_{j=0}^r C_{N-s} \sum_{v=N+r-s}^{\infty} d_{v+s-r} \frac{t^v}{v!} \right] \quad (21)$$

5. The Lucas Polynomials and Dynamical Solutions

The relation between a higher order linear equation and the exponential of the corresponding companion matrix is a well known matter. We wish to discuss the converse, that is, how the exponential of an arbitrary square matrix relates with certain higher order linear equation. We shall see that there is a relationship that could be attractive from a numerical point of view.

Given a square matrix A of order N , we consider the scalar differential equation

$$u^{(N)}(t) = \sum_{j=1}^N a_j u^{(N-j)}(t), \quad (22)$$

associated with the characteristic polynomial

$$P(\lambda) = \det |\lambda I - A| = \lambda^N - \sum_{j=1}^N a_j \lambda^{N-j},$$

of the given matrix A . From the Cayley-Hamilton theorem we know that any power of A can be written as

$$A^k = \sum_{j=1}^N \alpha_j(k) A^{N-j} ; \quad k = 1, 2, \dots$$

for certain scalars $\alpha_j(k)$. Consequently

$$e^{At} = \sum_{j=1}^N \beta_{N-j}(t) A^{N-j}$$

where the coefficients

$$\beta_{N-j}(t) = \sum_{k=0}^{\infty} \alpha_j(k) \frac{t^k}{k!}$$

are to be identified. Bakarat and Baumann [1] established some time ago that the values $\alpha_j(k)$ could be expressed in terms of the generalized Lucas polynomials

$$d_k = d_k(a_1, \dots, a_N) ,$$

obtained by solving the N th-order scalar linear difference equation

$$d_k = \sum_{j=1}^N a_j d_{k-j} , \quad k \geq N \quad (23)$$

$$d_{N-1} = 1 , \quad d_0 = d_1 = \dots = d_{N-2} = 0$$

Although no compact formula was given for the α_j 's ; later on, Lavoie [9] showed their relation with the Bell polynomials, while Bruschi and Ricci [3] exhibited a generating function for the generalized Lucas polynomials d_k .

For us, it is quite clear that these polynomials are just the derivative values at zero of the dynamical solution $d(t)$ of the scalar equation (22). Thus any formula for the α_j is has to involve the scalar dynamical solution $d(t)$. In fact, if we let $d_k = 0$ for negative integers k and set $a_0 = 1$, we can write Bakarat and Baumann's formulae in the compact form

$$\alpha_j(k) = \sum_{i=0}^{N-j} a_{j+i} d_{k-i-1} = \sum_{i=j}^N a_i d_{k+j-i-1} = \sum_{i=0}^{j-1} b_i d_{k+j-i-1}$$

where $b_0 = a_0$, $b_i = -a_i$. Thus

$$\beta_{N-j}(t) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{j-1} b_i d_{k+j-i-1} \right) t^k / k!,$$

which is nothing else but the complementary basis solutions $c_{N-j}(t)$ of equation (23) as it follows from our relation (10). We have

Theorem 2

For any square matrix A of order N having the characteristic po

ynomial $P(\lambda) = \sum_{j=0}^N b_j \lambda^{N-j}$, $b_0 = 1$, we have

$$e^{At} = \sum_{j=1}^N c_{N-j}(t) A^{N-j} = \sum_{j=1}^N \left(\sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) \right) A^{N-j}, \quad (24)$$

where $d(t)$ is the dynamical solution of the scalar equation

$$\sum_{j=0}^N b_j u^{(N-j)}(t) = 0 \quad (25)$$

and $c_j(t)$ is the solution of the above equation with initial

values $c_j^{(k)}(0) = \delta_{jk}$ for $k = 0, 1, \dots, N-1$.

Corollary

The dynamical solution of the m th-order equation

$$u^{(m)}(t) = \sum_{j=1}^m A_j u^{(m-j)}(t), \quad A_j \text{ } n \times n$$

is given by

$$D(t) = \sum_{j=1}^{mn} \sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) D_{mn-j} \quad (26)$$

where $d(t)$ is the scalar dynamical solution of the equation associated with the characteristic polynomial

$$P(\lambda) = \det[\lambda^m - \sum_{j=1}^m A_j \lambda^{m-j}] = \sum_{k=0}^{mn} b_k \lambda^{mn-k} \quad (27)$$

Proof.

It is immediate from (24) with A being the block companion matrix of the coefficients A_j .

Remarks

1. From (26) we conclude that the solution of the matrix difference equation

$$D_{k+m} = \sum_{j=1}^m A_j D_{k+m-j}, \quad D_{m-1} = I, \quad D_0 = D_1 = \dots = D_{m-2} = 0$$

is given by

$$D_k = \sum_{j=1}^N \sum_{i=0}^{j-1} b_i d^{(k+j-i-1)}(0) D_{N-j}, \quad N = mn \quad (28)$$

This solution could be considered as an extension of the Cayley-Hamilton theorem for the case of two or more matrix variables.

2. We observe from (27) that the numerical stability of the matrices D_k can be easily related with that of the coefficients $d^{(k+j-i-1)}(0)$, i.e., to determine when the characteristic polynomial (27) has its roots within the unit circle. [6], [8], [4].

6. Numerical Aspects

The formula we have derived for the exponential matrix could be considered from a numerical point of view. We first assume that the coefficient of the characteristic polynomial of the given matrix A are known. This task could be done with the Faddeva-Frame-Leverrier algorithm [10]

$$k b_k = -\text{trace } A h_{k-1}(A).$$

$$h_k(A) = A h_{k-1}(A) + b_k I$$

where $h_0(A) = I$ and $h_N(A) = 0$. Then, we have the alternative of using single or multistep o.d.e's solves for computing the coefficients $c_j(t)$ or numerical inversion of the Laplace transform

or use matrix methods as follows. Let A denote the companion matrix associated with the coefficients b_k . The calculation of the values $c_j(t)$ could be done by computing e^{At} as it follows from section 2.

Since the companion matrix A is sparse, we could compute e^{At} through scaling and squaring, that is

$$e^A = (e^{A/m})^m,$$

where $e^{A/m}$ could be computed by either Taylor or Padé approximation. The value for m can be determined by splitting arguments with the Lie-Trotter formula

$$e^{B+C} = \lim_{m \rightarrow \infty} (e^{B/m} e^{C/m})^m,$$

as it was suggested by M. Gunzburger and D. Gottlieb [10].

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