Universidade Federal do Rio Grande do Sul
Instituto de Matemática
Cadernos de Matemática e Estatística
Série A: Trabalho de Pesquisa

WAVE FRONT PROPAGATION FOR A CAUCHY PROBLEM WITH A FAST COMPONENT

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Série A, nº 30, OUT/92

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Abstract

First we consider a nonlinear Cauchy problem depending on a small parameter $\varepsilon > 0$. The parcial differencial equation describes a "slow" diffusion (coefficient of order ε) in x-direction, $x \in \mathbf{R}$, and a "fast" motion in y-direction which is a homogeneous Markov process in a compact subset of $\mathbf{R}^r$ and whose generator has infinitesimal characteristics of order $\frac{1}{\varepsilon}$. Secondly, we generalize the above problem by considering as the "slow" motion a locally infinitely divisible process in $\mathbf{R}$. The Feynman-Kac formula provides a representation for the generalized solution of both problems. We use a stochastic approach (Freidlin's approach) to study the wave front propagation as $\varepsilon \downarrow 0$.

Keywords: slow motion, fast motion, wave front propagation, Markov process, locally infinitely divisible process, Large Deviation Principle, action functional.

1. Introduction

In this paper we study some generalizations of the following mixed problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \dfrac{1}{2\varepsilon}\dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \dfrac{\varepsilon a(x,y)}{2}\dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon} f(x,y,u^\varepsilon), \\ x\in \mathbf{R},\ y\in(-b;b),\ t>0 \\ u^\varepsilon(0,x,y) = g(x,y) \\ \dfrac{\partial u^\varepsilon(t,x,y)}{\partial y}\Big|_{y=\pm b} = 0 \end{cases}$$

This problem was studied in [2] using Freidlin's stochastic approach. We analyzed the wave front propagation as $\varepsilon \downarrow 0$ for the solution $u^\varepsilon(t,x,y)$ of (1.1). Using the Feynman-Kac formula and large deviations theory, we defined a function $V(t,x)$, $t>0$, $x\in\mathbf{R}$ such that, under suitable conditions (Condition (N) formulated by Freidlin [4]),

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = \begin{cases} 0 & \text{if } V(t,x)<0, \quad |y|\le b \\ 1 & \text{if } V(t,x)>0, \quad |y|\le b. \end{cases}$$

Clearly, the set $\{(t,x,y) : V(t,x)=0, |y|\le b\}$ determines the position of the wave front, as $\varepsilon \downarrow 0$. and $G_t = \{(x,y) : V(t,x)>0, |y|\le b\}$ represents the excited region at time t. Observe that in problem (1.1) the motion in y-direction is described by a Wiener process in $[-b;b]$ with instantaneous reflection at the end points of the interval. Its diffusion coefficient is of order $\frac{1}{\varepsilon}$ and for this reason it is called "fast motion". The motion in x-direction is a diffusion with coefficient $\frac{\varepsilon a(x,y)}{2}$ (of order ε) and is called "slow motion".

It is desirable, for instance, that the results obtained in [2] remain valid in the case of a weakly coupled system of equations of the type in (1.1). In this case, the fast motion is a process $(Y_{\frac{t}{\varepsilon}}, \nu_{\frac{t}{\varepsilon}})$ where Y_t is a Wiener process in $[-b;b]$ with instantaneous reflection at the end points of the interval and ν_t is a Markov chain in the phase space $\{1,\cdots,n\}$ with infinitesimal characteristics specified by a matrix $(q_{ij})_{i,j=1,\cdots,n}$, $q_{ij}\ge 0$ if $i\ne j$, $q_{ii} = -\sum_{j=1}^n q_{ij}$.

In this paper we include a wider class of problems by extending problem (1.1) in two directions. First we consider a Markov process in a compact subset of $\mathbf{R}^r$ as the fast motion (in y-direction). Secondly, we generalize the slow motion by considering processes belonging to the class of locally infinitely divisible processes in $\mathbf{R}$.

In the first case we keep the slow motion as in (1.1) but the fast motion is described by a family of processes $(Y_t^\varepsilon; \bar{P}_y^\varepsilon)$ where $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$ and $(Y_t; \bar{P}_y)$ is a homogeneous Markov family in the phase space $(D, \mathcal{B}(D))$ where $D \subset \mathbf{R}^r$ is compact and $\mathcal{B}(D)$ is the σ-field generated by the Borel subsets of D in the topology inherited from the Euclidean norm in $\mathbf{R}^r$. We denote by $\mathcal{A}^1$ and $\mathcal{A}^{1,\varepsilon}$ respectively the infinitesimal generator corresponding to the processes Y_t and Y_t^ε.

Now problem (1.1) becomes

$$(1.2)\qquad \begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \mathcal{A}^{1,\varepsilon} u^\varepsilon(t,x,y) + \dfrac{\varepsilon a(x,y)}{2}\dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon} f(x,y,u^\varepsilon), \\ x \in \mathbf{R},\ y \in D,\ t > 0 \\ u^\varepsilon(0,x,y) = g(x,y). \end{cases}$$

The boundary conditions are specified according to the infinitesimal generator $\mathcal{A}^{1,\varepsilon}$.

We say that a function $f(u)$ belongs to the class $\mathcal{F}_1$ (see [4]) if f is differentiable in u, $f(0) = f(1) = 0$, $f(u) > 0$ in $(0;1)$, $f(u) < 0$ if $u \notin [0;1]$, $f'(0) = \sup_{u \geq 0} \frac{f(u)}{u}$. We assume that for each x, y, $f(x,y,u) \in \mathcal{F}_1$. Put $\frac{f(x,y,u)}{u} = c(x,y,u)$ and $c(x,y) \equiv c(x,y,0) = \sup_{u \geq 0} c(x,y,u)$. Assume that $c(x,y,u)$ is Lipschitz continuous in x and u, continuous in y, and $0 < \underline{c} \leq c(x,y) \leq \bar{c}$.

The initial function $g(x)$ is bounded, nonnegative, continuous or with discontinuities of first kind, and $[G_0] = [(G_0)]$ where $G_0 = \text{supp}\, g \neq \mathbf{R}$. The set $[A]$ is the clousure of A and (A) is its interior. We also assume that $a(x,y)$ is Lipschitz continuous in x and $0 < \underline{a} \leq a(x,y) \leq \bar{a}$.

Let $\widetilde{X}_t^\varepsilon$ represent the slow motion (in x-direction). Then $\widetilde{X}_t^\varepsilon$ satisfies the stochastic differential equation

$$d\widetilde{X}_t^\varepsilon = \sqrt{\varepsilon a(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon)}\, dW_t, \qquad \widetilde{X}_0^\varepsilon = x,$$

where W_t is a Wiener process in $\mathbf{R}$, starting at zero, adapted to some increasing family of σ-fields and independent of Y_t. One can verify (see [4]) that the Markov process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$ is associated with the operator

$$L^\varepsilon = A^{1,\varepsilon} + \frac{\varepsilon a(x,y)}{2}\frac{\partial^2}{\partial x^2}.$$

It is known (see Freidlin [4]) that if $c(x,y,u)$ is Lipschitz continuous in x and u then the generalized Feynman-Kac formula

$$u^\varepsilon(t,x,y) = \widetilde{E}_{xy} g(\widetilde{X}^\varepsilon_t, Y^\varepsilon_t) \exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\widetilde{X}^\varepsilon_s, Y^\varepsilon_s, u^\varepsilon(t-s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\} \tag{1.3}$$

has a unique solution $u^\varepsilon(t,x,y)$. Besides, one can prove that a solution of (1.2) satisfies (1.3). In this sense we say that problem (1.2) has a unique generalized solution $u^\varepsilon(t,x,y)$. Since $c(x,y) = \sup_{u\geq 0} c(x,y,u)$ we have

$$u^\varepsilon(t,x,y) \leq \widetilde{E}_{xy} g(\widetilde{X}^\varepsilon_t, Y^\varepsilon_t) \exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\widetilde{X}^\varepsilon_s, Y^\varepsilon_s\right) ds\right\}.$$

As in the case of problem (1.1) (see [2]) the asymptotic behavior of $u^\varepsilon(t,x,y)$ as $\varepsilon \downarrow 0$ is related with probabilities of large deviations for the family of processes

$$\left(\widetilde{X}^\varepsilon_t, \int_0^t c(\widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\, ds\right). \tag{1.4}$$

We shall obtain the action functional for (1.4) using the same approach as in [2], section 2. In that approach the action functional for processes of the type $\int_0^t b(\varphi_s, Y^\varepsilon_s)\, ds$ where φ is a continuous function on $[0;T]$ into $\mathbf{R}$ plays an important role.

In section 2 of this paper we formulate sufficient conditions on $(Y_t; \bar{P}_y)$ in order that families of processes of the type $\int_0^t b(\varphi_s, Y^\varepsilon_s)\, ds$ obey a Large Deviation Principle. The main tool here is the theory of semigroups of linear operators. We suggest Pazy [12] and Kato [10] as references.

In section 3 we establish a Large Deviation Principle for (1.4) under the conditions formulated in section 2. Our goal in section 3 is just to make clear that the theory developed in [2] can be applied when we consider Markov processes of a general type as the fast motion.

In section 4 we consider a weakly coupled system of equations of the type in (1.1) and also a problem with fast motion being a nondegenerated diffusion process in a compact subset of $\mathbf{R}^r$. We describe explicitely the wave front as $\varepsilon \downarrow 0$ in both examples, following the same approach as in [2].

In the remaining sections we deal with the slow motion which will be described by a locally infinitely divisible process. We are not interested in the most general concept but only on those processes with values in $\mathbf{R}$. The next definition was taken from Wentzell [15].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t : t \geq 0\}$ be a nondecreasing family of σ-fields with $\mathcal{F}_t \subset \mathcal{F}$, for all $t \geq 0$. Let $\mathcal{B}(\mathbf{R})$ be the σ-field generated by the Borel subsets of $\mathbf{R}$. A strong Markov process $(\xi_t; P_{tx})$ on $(\Omega, \mathcal{F}, P)$ with values in $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is called *locally infinitely divisible* if its sample functions are right-continuous with left-hand limits with probability one and whose infinitesimal generator is given by

$$\begin{aligned}\mathcal{A}_t f(x) =& b(x,t)\frac{df(x)}{dx} + \frac{1}{2}a(x,t)\frac{d^2 f(x)}{dx^2} + \\ &+ \int_{\mathbf{R}} \left[f(x+u) - f(x) - u\frac{df(x)}{dx} \right] \Pi_{x,t}(du)\end{aligned}$$

where $x \in \mathbf{R}$, $t \geq 0$, $\Pi_{x,t}(\cdot)$ is a nonrandom measure on $\mathcal{B}(\mathbf{R})$, measurable in x and t, $\Pi_{x,t}(\{0\}) = 0$, and

$$\int_{\mathbf{R}} u^2 \Pi_{x,t}(du) < \infty, \quad \text{for all } x,\, t.$$

The functions $b(x,t)$ and $a(x,t)$ are measurable with $0 < \underline{a} \leq a(x,t) \leq \bar{a}$, $\underline{b} \leq b(x,t) \leq \bar{b}$ and $f(x)$ is measurable, bounded, and twice-continuouslly differentiable. The question about large deviations for this class of processes is considered by Wentzell [15].

The main goal in sections 5 and 6 is to analyze the wave front propagation of the solution of a Cauchy problem of the type

$$(1.5) \qquad \begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \mathcal{A}^{1,\varepsilon} u^\varepsilon(t,x,y) + \mathcal{A}^{2,\varepsilon} u^\varepsilon(t,x,y) + \dfrac{1}{\varepsilon} f(x,y,u^\varepsilon) \\ \text{for } t > 0,\ x \in \mathbf{R},\ y \in (D),\ D \subset \mathbf{R} \\ u^\varepsilon(0,x,y) = g(x,y) \end{cases}$$

The operator $\mathcal{A}^{1,\varepsilon}$ is the infinitesimal generator of the process $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$ where $(Y_t; \bar{P}_y)$ satisfies suitable conditions formulated in section 2, and $\mathcal{A}^{2,\varepsilon}$ is the infinitesimal generator of the slow motion which is described by a locally infinitely divisible process with frequent small jumps (see [6] or [15]).

We shall study two cases. In section 5 the slow motion is a locally infinitely divisible process whose infinitesimal generator is

$$\mathcal{A}^{2,\varepsilon} f(x) = b(y)\frac{df(x)}{dx} + \frac{\varepsilon a(y)}{2}\frac{d^2 f(x)}{dx^2} + \tag{1.6}$$
$$+ \frac{1}{\varepsilon}\int\limits_{\mathbf{R}} \left[f(x+\varepsilon\beta) - f(x) - \varepsilon\beta\frac{d(f(x))}{dx} \right] \Pi(d\beta)$$

where $\Pi(\cdot)$ is a nonrandom measure with $\Pi(\{0\}) = 0$ and $\int\limits_{\mathbf{R}} \beta^2 \Pi(d\beta) < \infty$. Notice that the coefficients $b(y)$ and $a(y)$ depend only on the fast variable y.

In section 6 the slow motion is a locally infinitely divisible process in $\mathbf{R}$, independent of the fast motion, whose infinitesimal generator is

$$\mathcal{A}^{2,\varepsilon} f(x) = b(x)\frac{df(x)}{dx} + \frac{\varepsilon a(x)}{2}\frac{d^2 f(x)}{dx^2} + \tag{1.7}$$
$$+ \frac{1}{\varepsilon}\int\limits_{\mathbf{R}} \left[f(x+\varepsilon\beta) - f(x) - \varepsilon\beta\frac{d(f(x))}{dx} \right] \Pi_x(d\beta)$$

where $\Pi_x(\cdot)$ is a nonrandom measure with $\Pi_x(\{0\}) = 0$ and $\int\limits_{\mathbf{R}} \beta^2 \Pi_x(d\beta) < \infty$ for all $x \in \mathbf{R}$. Observe that in this case the infinitesimal characteristics of the slow motion depend only on x.

In both cases we shall use the action functional for the family of random processes (1.4) to describe the wave front for the generalized solution of (1.5).

Acknowledgement

The author is gratefull to M. I. Freidlin for the formulation of the problem discussed in [2] as well as its generalizations considered in this paper. His comments and suggestions were very important to the completion of this work.

2. A Large Deviation Principle - sufficient conditions

In this section the fast motion is a homogeneous Markov family $(Y_t;\bar{P}_y)$ in the phase space $(D,\mathcal{B}(D))$ where $D\subset \mathbf{R}^r$ is compact and $\mathcal{B}(D)$ is the σ-field of the Borel subsets of D in the topology inherited from the Euclidean norm in $\mathbf{R}^r$.

The semigroup $\{T_t\}_{t\geq 0}$ associated with $(Y_t;\bar{P}_y)$ is

$$(2.1)\qquad T_t h(y)=\int_D h(z)\bar{P}(t,y,dz)=\bar{E}_y h(Y_t),$$

h being a bounded and measurable function, $\bar{P}(t,y,\cdot)$ the transition function of the process Y_t, and $\bar{E}_y$ the corresponding conditional expectation. The infinitesimal generator is denoted by $\mathcal{A}^1$.

Let us introduce some notation:

B : space of bounded, $\mathcal{B}(D)$-measurable numerical functions on D.

C_D : space of continuous numerical functions on D.

B_0 : subspace of B of "strong continuity" of $\{T_t\}_{t\geq 0}$.

$C_{[0,T]}(\mathbf{R}^m)$: space of continuous functions on $[0,T]$ into $\mathbf{R}^m$.

$\rho_{0T}((\varphi^1,\cdots,\varphi^m),(\psi^1,\cdots,\psi^m))=\sum_{i=1}^m\|\varphi^i-\psi^i\|$ where $\|\cdot\|$ is the supremum norm in $C_{[0,T]}(\mathbf{R})$.

Our goal in this section is to formulate sufficient conditions in order that processes satisfying equation

$$(2.2)\qquad \dot{\xi}^\varepsilon_t=b(\xi^\varepsilon_t;Y_{\frac{t}{\varepsilon}}),\quad \xi^\varepsilon_0=x\in\mathbf{R}^m,\ t\geq 0,$$

obey a Large Deviation Principle. The function $b(x,y)=(b^1(x,y),\cdots,b^m(x,y))$, $x\in\mathbf{R}^m$, $y\in\mathbf{R}^r$ is assumed to be bounded and Lipschitz continuous in both variables.

Let us introduce the following conditions:

(i) for any $x,\ \beta\in\mathbf{R}^m$,

$$(2.3)\qquad \lim_{T\to\infty}\frac{1}{T}\log\bar{E}_y\exp\left\{\int_0^T(\beta,b(x,Y_s))\,ds\right\}=H(x,\beta)$$

exists uniformly in $y\in\mathbf{R}^r$,

(ii) $H(x,\beta)$ is differentiable in β.

Under conditions (i)-(ii), Freidlin proved (see Theorem 7.4.1 and Lemma 7.4.3 in [6]) that the normalized action functional on the space $(C_{[0,T]}(\mathbf{R}^m), \rho_{0T})$ for the family of processes $\{\xi_t^\epsilon\}$, solution of (2.1), is given by

$$S_{0T}(\varphi) = \begin{cases} \int_0^T L(\varphi_s;\dot{\varphi}_s)\,ds, & \text{if } \varphi \text{ is a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}^m) \end{cases} \tag{2.4}$$

with normalizing coefficient $\frac{1}{\epsilon}$. The function $L(x,\alpha)$ is the Legendre transform of $H(x,\beta)$ with respect to the variable β, i.e.,

$$L(x,\alpha) = \sup_{\beta\in\mathbf{R}^m} \{\langle\alpha,\beta\rangle - H(x,\beta)\}, \quad \text{for } \alpha \in \mathbf{R}^m$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbf{R}^m$. Therefore, we shall formulate conditions on $(Y_t;\bar{P}_y)$ in order that conditions (i) and (ii) be fulfilled.

It is well known (see [14] or [3]) that in the case of a compact phase space D, if $C_D \subseteq B_0$ then the transition function of the process Y_t is uniquely determined by the semigroup $\{T_t\}_{t\geq 0}$ acting on C_D. Besides, the infinitesimal generator $\mathcal{A}^1$ of $\{T_t\}_{t\geq 0}$ uniquely determines the transition function and hence the finite-dimensional distributions of the Markov family.

In our case $C_D \subseteq B$ but C_D is not contained in B_0 necessarily. However, if $(Y_t;\bar{P}_y)$ is uniformly stochastically continuous then $C_D \subseteq B_0$ (see Problem 10.9 in [14]). Moreover, with probability one, the paths are continuous on the right and have limit on the left at each point.

It is also known (see Chapter 9 in [14]) that all the Feller-Markov families with paths continuous on the right exhibit the strong Markov property with respect to the family of σ-fields $\mathcal{F}_{\leq t+} = \bigcap_{s>t} \mathcal{F}_{\leq s}$, $t \in [0,+\infty)$ where $\mathcal{F}_{\leq t}$ is the smallest σ-field such that Y_s, $0 \leq s \leq t$ are measurable.

Relying on the above facts we can enunciate the following result:

Proposition 2.1. *If $(Y_t;\bar{P}_y)$ is a homogeneous Feller-Markov family, uniformly stochastically continuous, then*

(a) $\{T_t\}_{t\geq 0}$ can be regarded as acting on C_D; further, it is a strongly continuous semigroup in the sense that $\|T_t f - f\| \to 0$ as $t \downarrow 0$ for all $f \in C_D$.

(b) $(Y_t; \bar{P}_y)$ has paths continuous on the right with left-hand limits at all points, with probability one.

(c) $(Y_t; \bar{P}_y)$ is a strong Markov family with respect to the σ-fields $\mathcal{F}_{\leq t+}$.

Theorem 2.1. *Let us assume that $\{\widetilde{T}_t\}_{t\geq 0}$ is a compact strongly continuous semigroup with $\|\widetilde{T}_t\| \leq Me^{wt}$ for some $M > 0$ and $w \geq 0$. Let $C^+ = \{f \in B_0 : f \geq 0\}$ and assume that $\{\widetilde{T}_t\}_{t\geq 0}$ is strongly positive with respect to C^+. Then the eigenvalue λ .of the infinitesimal generator $\widetilde{\mathcal{A}}$ of $\{\widetilde{T}_t\}_{t\geq 0}$ with the maximal real part is real and simple; the corresponding eigenvector ϕ is positive and $\|\phi\| = 1$.*

Proof: This result is well known in the theory of semigroups of linear operators and we omit its proof. The reader may consult Pazy **[12]**.

■

For each $h \in C_D$ let us introduce the operator $T_t^{(h)}$ on B defined by

$$T_t^{(h)} f(y) = \bar{E}_y f(Y_t) \exp\left\{ \int_0^t h(Y_s)\, ds \right\}, \quad f \in B. \tag{2.5}$$

Proposition 2.2.

(a) $\{T_t^{(h)}\}_{t\geq 0}$ *is a semigroup of bounded linear operators and* $\|T_t^{(h)}\| \leq e^{wt}$ *for some* $w \in \mathbf{R}$.

(b) $\{T_t^{(h)}\}_{t\geq 0}$ *is strongly continuous in the same space* B_0 *as* $\{T_t\}_{t\geq 0}$ *and only on this space. Moreover, its infinitesimal generator is given by*

$$\mathcal{A}^{(h)} f(y) = \mathcal{A}^1 f(y) + h(y) f(y), \qquad f \in \mathcal{D}_{\mathcal{A}^{(h)}}$$

and $\mathcal{D}_{\mathcal{A}^{(h)}} = \mathcal{D}_{\mathcal{A}^1}$ where $\mathcal{D}_{\mathcal{A}^1}$ is the domain of definition of $\mathcal{A}^1$.

Proof: The proof of this proposition can be found in Wentzell **[14]** and we omit it.

■

Theorem 2.2. *Let $(Y_t; \bar{P}_y)$ be a homogeneous Markov family in the phase space $(D, \mathcal{B}(D))$, $D \subset \mathbf{R}^r$ compact and $\mathcal{B}(D)$ the σ-field of the Borel subsets of D in the topology inherited from the Euclidean norm in $\mathbf{R}^r$. Assume that*

(L.1) $(Y_t; \bar{P}_y)$ *is a homogeneous Feller-Markov process.*

(L.2) $(Y_t; \bar{P}_y)$ *is uniformly stochastically continuous, i.e.,*

$$\forall \varepsilon > 0, \quad \bar{P}_y(|Y_s - Y_t| \geq \varepsilon) \to 0 \quad \text{as } t - s \to 0$$

uniformly in $y \in D$ and in $t, s \in [0, +\infty)$.

(L.3) The semigroup $\{T_t\}_{t\geq 0}$ in (2.1) is strongly positive with respect to the cone $\{f \in C_D : f \geq 0\}$.

(L.4) For each $h \in C_D$, the semigroup $\{T_t^{(h)}\}_{t\geq 0}$ in (2.5) satisfies the Feller-condition, i.e., $T^{(h)}C_D \subseteq C_D$.

(L.5) For each $h \in C_D$, $\{T_t^{(h)}\}_{t\geq 0}$ is a compact semigroup.

Then, for any $\beta \in \mathbf{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \log \bar{E}_y \exp\left\{\int_0^t \beta\, h(Y_s)\, ds\right\} = H(\beta) \tag{2.6}$$

exists uniformly in y. Moreover, $H(\beta)$ is differentiable and convex in β.

Proof: Conditions **(L.1)**-**(L.5)** imply that $\{T_t^{(\beta h)}\}_{t\geq 0}$ is a compact, strongly continuous semigroup acting on C_D and strongly positive with respect to $C^+ = \{f \in C_D : f \geq 0\}$ (this is a consequence of Proposition 2.1 and Proposition 2.2). Then, by Theorem 2.1 the maximal eigenvalue $\lambda(\beta)$ of $\mathcal{A}^{(\beta h)}$ is real and simple; the corresponding eigenvector ϕ is positive and $||\phi|| = 1$.

It is known (see Pazy **[12]**) that for a strongly continuous semigroup $\{\widetilde{T}_t\}_{t\geq 0}$,

$$B_\lambda(t)(\lambda I - \widetilde{\mathcal{A}})\, f = (e^{\lambda t} - \widetilde{T}_t)\, f, \quad \text{for } \lambda \in \mathbf{C},\, f \in \mathcal{D}_{\widetilde{\mathcal{A}}}$$

where $B_\lambda(t) = \int_0^t e^{\lambda(t-s)} \widetilde{T}_s f\, ds$ and $\mathbf{C}$ is the set of complex numbers. Using the above relation, one can see that $e^{\lambda(\beta)t}$ is an eigenvalue of $T_t^{(\beta h)}$ with the same eigenvector ϕ.

Since D is compact, there exists a constant $K > 0$ such that $0 < K \leq \phi(y) \leq 1$ for all $y \in D$. Also, $T_t^{(\beta h)}\phi(y) = e^{\lambda(\beta)t}\phi(y)$. Hence, $0 < K\phi(y) < \phi(y) \leq 1\,(y)$, for all $y \in D$ and then $0 < KT_t^{(\beta h)}1\,(y) < T_t^{(\beta h)}\phi(y) = e^{\lambda(\beta)\,t}\phi(y) \leq T_t^{(\beta h)}1\,(y)$ which implies that

$$\lim_{t\to\infty} \frac{1}{t} \log T_t^{(\beta h)} 1\,(y) = \lambda(\beta).$$

Notice that $E_y \exp\left\{\int_0^t \beta h(Y_s)\, ds\right\} = T_t^{(\beta h)} 1\,(y)$. Take $H(\beta) \equiv \lambda(\beta)$ and we get (2.6).

The function $H(\beta)$ is continuous and convex (see Lemma 7.4.1 in [6]). Besides, it is not difficult to verify that $T_t^{(\beta h)}$ is real-holomorphic in β near $\beta = 0$ ($T_t^{(\beta h)} f(y)$ has a Taylor expansion in β, near zero). Now, using the fact that $e^{\lambda(\beta) t}$ is an isolated eigenvalue of $T_t^{(\beta h)}$, we obtain from Theorem 1.8, Chapter VII in [10] that $\lambda(\beta)$ is differentiable.

■

Now we can conclude that if $(Y_t; \bar{P}_y)$ satisfies conditions **(L.1)-(L.5)** then the limit in (2.3) exists uniformly in $y \in \mathrm{R}^r$ and $H(x,\beta)$ is differentiable in β. Therefore, the family of processes $\{\xi_t^\varepsilon\}$, solution of (2.2), obeys a Large Deviation Principle with action functional given by (2.4). The function $L(x,\alpha)$ is the Legendre transform of the first eigenvalue $\lambda(x,\beta)$ with respect to β of the operator $\mathcal{A}^1 + \beta\, b(x,y)$.

Remark 2.1. In particular, if ξ_t^ε is defined by $\xi_t^\varepsilon = x + \int_0^t b(Y_{\frac{s}{\varepsilon}})\, ds$ or $\xi_t^\varepsilon = x + \int_0^t b(\psi_s, Y_{\frac{s}{\varepsilon}})\, ds$ where $\psi \in C_{[0,T]}(\mathrm{R}^m)$ is a fixed function, then the functional in (2.4) becomes respectively $S_{0T}(\varphi) = \int_0^T L(\dot\varphi_s)\, ds$ and $S_{0T}(\varphi) = \int_0^T L(\psi_s; \dot\varphi_s)\, ds$, φ a.c. The functions $L(\alpha)$ and $L(x,\alpha)$ are respectively the Legendre transform of the first eigenvalue of the operators $\mathcal{A}^1 + \beta\, b(y)$ and $\mathcal{A}^1 + \beta\, b(x,y)$.

Remark 2.2. If $(Y_t; \bar{P}_y)$ satisfies conditions **(L.1)-(L.5)** then the process Y_t has a unique invariant probability measure. The existence of an ergodic probability measure follows from the fact that Y_t is a homogeneous Feller-Markov family on a compact set (see Theorem 21, Chapter I in [13]). The uniqueness follows by contradiction taking into account that the limit in (2.6) exists and $H(\beta)$ is differentiable.

3. Wave Front Propagation

In this section we assume that $(Y_t; \bar{P}_y)$ satisfies conditions **(L.1)-(L.5)** introduced in Theorem 2.2 and $\widetilde{X}_t^\varepsilon$ satisfies the stochastic differencial equation

$$d\widetilde{X}_t^\varepsilon = \sqrt{\varepsilon a(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon)}\, dW_t.$$

The action functional on the space $(C_{[0,T]}(\mathrm{R}^2), \rho_{0T})$ for the two-dimensional family of processes $\left(\widetilde{X}_t^\varepsilon, \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds\right)$ is obtained similarly to section 2 in [2]. We will not go into details in this matter but just point out the main steps.

Define for each $\beta = (\beta_1, \beta_2) \in \mathbf{R}^2$ and $x \in \mathbf{R}$ the semigroup of operators $\{T_t^\beta\}_{t \geq 0}$ by

$$T_t^\beta f(y) = \bar{E}_y f(Y_t) \exp\left\{\int_0^t [\beta_1 a(x, Y_s) + \beta_2 c(x, Y_s)]\, ds\right\},$$

for $f \in C_D$ and $a(x,y)$, $c(x,y)$ the functions introduced in section 1. By Theorem 2.2,

$$\lim_{T\to\infty} \frac{1}{T} \log \bar{E}_y \exp\left\{\int_0^T [\beta_1 a(x, Y_s) + \beta_2 c(x, Y_s)]\, ds\right\} = H(x, \beta_1, \beta_2)$$

exists uniformly in y, $H(x, \beta_1, \beta_2)$ being the first eigenvalue of the operator $\mathcal{A}^\beta$ given by

$$\mathcal{A}^\beta f(y) = \mathcal{A}^1 f(y) + [\beta_1 a(x,y) + \beta_2 c(x,y)]\, f(y), \quad y \in (D).$$

Moreover, $H(x, \beta_1, \beta_2)$ is differentiable in $\beta = (\beta_1, \beta_2)$. Using the same proof of Theorem 7.4.1 in [6] one can show that, for each $\varphi \in C_{[0,T]}(\mathbf{R})$, the action functional for the family of processes $\left(\int_0^t a(\varphi_s, Y_s^\varepsilon)\, ds, \int_0^t c(\varphi_s, Y_s^\varepsilon)\, ds\right)$ is $\frac{1}{\varepsilon} S_{0T}^\varphi(\psi, \eta)$ with

$$S_{0T}^\varphi(\psi, \eta) = \begin{cases} \int_0^T L(\varphi_s; \dot{\psi}_s, \dot{\eta}_s)\, ds, & \text{if } \psi, \eta \text{ are a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}^2) \end{cases}$$

where $L(x, \alpha^1, \alpha^2)$ is the Legendre transform of $H(x, \beta_1, \beta_2)$ with respect to β.

Notice that the trajectories of the processes $\int_0^t a(\varphi_s, Y_s^\varepsilon)\, ds$ and $\int_0^t c(\varphi_s, Y_s^\varepsilon)\, ds$ belong, with probability one, respectively to $F_{\bar{a}}$ and $F_{\bar{c}}$ where

$$F_{\bar{k}} = \{\psi \in C_{[0,T]}(\mathbf{R}) : \psi_0 = 0, \exists \dot{\psi}_t, \underline{k} \leq \dot{\psi}_t \leq \bar{k}, t \in [0,T]\}.$$

Now, exactly as in [2], one can prove that $\left(\widetilde{X}_t^\varepsilon, \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds\right)$ has action functional $\frac{1}{\varepsilon} S_{0T}(\varphi, \eta)$ with

$$(3.1) \qquad S_{0T}(\varphi, \eta) = \begin{cases} \inf_{\psi \in F_a} \left\{\frac{1}{2}\int_0^T \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\, ds + \int_0^T L(\varphi_s; \dot{\psi}_s, \dot{\eta}_s)\, ds\right\}, & \text{if } \varphi \text{ is a.c., } \eta \in F_{\bar{c}} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}^2). \end{cases}$$

In particular, using Theorem 3.3.1 in [6], we conclude that the action functional for $\widetilde{X}_t^\varepsilon$ is $\frac{1}{\varepsilon} S_{0T}(\varphi)$ with

$$(3.2) \qquad S_{0T}(\varphi) = \begin{cases} \inf\limits_{\psi \in F_a} \left\{ \frac{1}{2} \int_0^T \frac{|\dot{\varphi}_s|^2}{\psi_s} ds + \int_0^T L(\varphi_s; \dot{\psi}_s)\, ds \right\}, & \text{if } \varphi \text{ is a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}). \end{cases}$$

In (3.2) the function $L(x, \alpha)$ is the Legendre transform of the first eigenvalue $\lambda(x, \beta)$ of the operator

$$\mathcal{A}^\beta = \mathcal{A}^1 + \beta a(x, y), \quad \beta \in \mathbf{R}$$

As in [2], the wave front propagation for the solution $u^\epsilon(t, x, y)$ of (1.2) is analyzed by means of the action functional (3.1). Define, for each $x \in \mathbf{R}$, $t > 0$, a function $V(t, x)$ by

$$(3.3) \qquad V(t, x) = \sup\{\eta_t - S_{0t}(\varphi, \eta) : \varphi \in C_{[0,t]}(\mathbf{R}),\ \varphi_0 = x,\ \varphi_t \in [G_0],\ \eta \in F_{\bar{c}}\}.$$

This function is analogous to a function $V(t, x)$ introduced in [2], section 3.

We say that Condition (N) (see Freidlin [4]) is fulfilled if for all (t, x) such that $V(t, x) = 0$,

$$V(t, x) = \sup\{\eta_t - S_{0t}(\varphi, \eta) : \varphi \in C_{[0,t]}(\mathbf{R}),\ \varphi_0 = x,\ \varphi_t \in G_0,\ V(t - s, \varphi_s) < 0 \text{ for } s \in (0, t),\ \eta \in F_{\bar{c}}\}$$

As in section 3 of [2] one can prove that, under Condition (N),

$$\lim_{\epsilon \downarrow 0} u^\epsilon(t, x, y) = \begin{cases} 0, & \text{if } V(t, x) < 0,\ y \in D \\ 1, & \text{if } V(t, x) > 0,\ y \in D. \end{cases}$$

Further, the convergence is uniform in compact sets. The above result tells us that the wave front at time t is determined by the sets $G_t = \{(x, y) : V(t, x) = 0,\ y \in D\}$. In a more general situation, without Condition (N), the wave front is described in terms of a different function. As in [2], section 3, define a functional $\tau = \tau_F(t, \varphi^1, \varphi^2)$ on $(-\infty; +\infty) \times C_{[0,+\infty)}(\mathbf{R}) \times C_{[0,+\infty)}(D)$ with values in $[0; +\infty]$ by

$$\tau_F(t, \varphi^1, \varphi^2) = \inf\{s : (t - s, \varphi_s^1, \varphi_s^2) \in F \times D\}$$

where F is any closed subset of $(-\infty;+\infty)\times\mathbf{R}$. We denote by Θ the set of all such functionals. Let us define for each $x\in\mathbf{R}$ and $t>0$ a function $V^*(t,x)$

$$V^*(t,x)=\inf_{\tau\in\Theta}\sup_{\varphi,\eta}\{\eta_{t\wedge\tau}-S_{0,t\wedge\tau}(\varphi,\eta):\varphi\in C_{[0,t]}(\mathbf{R}),\ \varphi_0=x,\ \varphi_t\in G_0,\ \eta\in F_{\bar{c}}\}\tag{3.4}$$

where S_{0t} is the action functional in (3.1). Clearly $V^*(t,x)\le(0\wedge V(t,x))\le 0$ where $V(t,x)$ is the function defined in (2.1). Assuming that $a(x,y)\equiv a(y)$ and that the nonlinear term in (1.2) depends on x and y, one can prove, as in [2], that the wave front is described by the set $\partial M\times D$ where $M=\{(x,t):V^*(t,x)=0\}$ and ∂M is the frontier of M.

Using results from [5] and [7] one can show that $\bar{V}(t,x)=V^*(t,x)$ for $t>0,\ x\in\mathbf{R}$ with

$$\bar{V}(t,x)=\sup_{\varphi,\eta}\left\{\min_{0\le a\le t}\left[\eta_a-S_{0a}(\varphi,\eta):\varphi\in C_{[0,t]}(\mathbf{R}),\ \varphi_0=x,\ \varphi_t\in G_0,\ \eta\in F_{\bar{c}}\right]\right\}.$$

4. Examples

Example 4.1. Let us consider $(Y_t;\bar{P}_y)$ as a diffusion non-degenerated process in a bounded domain $D\subset\mathbf{R}^r$ with smooth boundary ∂D and normal reflection on the boundary.

It is known (see Freidlin [4]) that the infinitesimal generator $\mathcal{A}^1$ of this process is defined at least on the functions $f(y)$ having continuous first- and second-order derivatives up to the boundary ∂D for which $\frac{\partial f(y)}{\partial n(y)}|_{y\in\partial D}=0$, where $n(y)=(n_1(y),\cdots,n_r(y))$ is the inward normal to the boundary ∂D. For these functions,

$$\mathcal{A}^1 f(y)=\sum_{i=1}^r c^i(y)\frac{\partial f(y)}{\partial y^i}+\frac{1}{2}\sum_{i,j=1}^r d^{ij}(y)\frac{\partial^2 f(y)}{\partial y^i\partial y^j},\quad y\in(D)$$

where $d^{ij}(y)$ are assumed to be twice continuously differentiable up to the boundary and $\sum_{i,j=1}^r d^{ij}(y)\lambda_i\lambda_j>0$; the functions $c^i(y)$ are assumed to be Lipschitz continuous. A construction of such process is available, for example, in Freidlin [4] or, with more details, in Anderson and Orey [1]. In that construction the process $(Y_t;\bar{P}_y)$ is obtained as the solution of the stochastic differential equation

(4.1) $$dY_t^y = c(Y_t^y)\,dt + \sigma(Y_t^y)\,dW_t + \mathcal{X}_{\partial D}(Y_t^y) n(Y_t^y)\,d\xi_t^y,\ \ Y_0^y = y,\, \xi_0^y = 0$$

where $\mathcal{X}_{\partial D}(y)$ is the indicator of the set ∂D, W_t is a Wiener process in $\mathbf{R}^r$ adapted to an increasing family of σ-fields $\mathcal{N}_t$, $(d^{ij}(y))_{i,j=1,\cdots,r} = \sigma(y)\sigma^*(y)$. The process ξ_t^y is a non-decreasing process which increases only for $t \in \Gamma = \{t : Y_t^y \in \partial D\}$, Γ having Lebesgue measure zero a.s. The random function ξ_t^y is referred as the local time on the boundary.

From the construction of $(Y_t; \bar{P}_y)$ it is derived that Y_t is a strong Feller-Markov process, it is uniformly stochastically continuous, and its transition function has density $p(t, y, z)$ with $p(t, y, z) > 0$ for $t > 0$. Therefore, conditions **(L.1)-(L.3)** are satisfied. Conditions **(L.4)** and **(L.5)** are easily verified if we take into account that $(Y_t; \bar{P}_y)$ is a Feller-Markov family and satisfies the stochastic differential equation (4.1). ∎

Relying on Theorem 2.2, conditions **(L.1)-(L.5)** allow us to apply Theorem 7.4.1 in [6] to conclude that the action functional for the family of processes $\left(\widetilde{X}_t^\varepsilon, \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\,ds\right)$ is $\frac{1}{\varepsilon} S_{0T}(\varphi, \eta)$ with $S_{0T}(\varphi, \eta)$ given in (3.1). But now the function $L(x; \alpha^1, \alpha^2)$ in (3.1) is the Legendre transform of the first eigenvalue $\lambda(x; \beta_1, \beta_2)$ of the operator $\mathcal{A}^\beta$ defined by

$$\mathcal{A}^\beta f(y) = \sum_{i=1}^r c^i(y) \frac{\partial f(y)}{\partial y^i} + \frac{1}{2} \sum_{i,j=1}^r d^{ij}(y) \frac{\partial^2 f(y)}{\partial y^i \partial y^j} + [\beta_1 a(x,y) + \beta_2 c(x,y)]\, f(y), \quad y \in (D)$$

with $\frac{\partial f(y)}{\partial n(y)}|_{y \in \partial D} = 0$.

Problem (1.2) reduces to

(4.2) $$\begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \dfrac{1}{\varepsilon} \displaystyle\sum_{i=1}^r c^i(y) \frac{\partial u^\varepsilon(t,x,y)}{\partial y^i} + \frac{1}{2\varepsilon} \sum_{i,j=1}^r d^{ij}(y) \frac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^i \partial y^j} + \\ \qquad + \dfrac{\varepsilon}{2} a(x,y) \dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon} f(x,y,u^\varepsilon), \quad x \in \mathbf{R},\ y \in (D),\ t > 0 \\ u^\varepsilon(0,x,y) = g(x) \\ \dfrac{\partial u^\varepsilon(t,x,y)}{\partial n(y)}|_{y \in \partial D} = 0. \end{cases}$$

The wave front propagation for the solution $u^\varepsilon(t,x,y)$ of (4.2) is analyzed as in [2], but using the action functional in (3.1). The functions in (3.3) and (3.4) become respectively

$$V(t,x) = \sup\{\eta_t - \frac{1}{2}\int_0^t \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\,ds - \int_0^t L(\varphi_s;\dot{\psi}_s,\dot{\eta}_s)\,ds :$$
$$\varphi_0 = x,\ \varphi_t \in G_0,\ \eta \in F_{\bar{c}},\ \psi \in F_{\bar{a}}\}.$$

and

$$V^*(t,x) = \inf_{\tau\in\Theta}\sup_{\varphi,\eta}\{\eta_{t\wedge\tau} - \frac{1}{2}\int_0^{t\wedge\tau} \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\,ds - \int_0^{t\wedge\tau} L(\varphi_s;\dot{\psi}_s,\dot{\eta}_s)\,ds :$$
$$\varphi \in C_{[0,t]}(\mathbf{R}),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \eta \in F_{\bar{c}}\}.$$

Example 4.2. Let $(Y_t;\bar{P}_y)$ be a homogeneous Markov chain with continuous time and states $\{1,2,\cdots,n\}$ for which

$$\bar{P}\{Y_{t+\Delta} = j/Y_t = i\} = q_{ij}\,\Delta + O(\Delta),\quad i\neq j,\quad \Delta\downarrow 0$$

with $q_{ij} > 0$ for $i \neq j$. The phase space is $(\{1,2,\cdots,n\},\mathcal{B}(\{1,2,\cdots,n\}))$, $\mathcal{B}(\{1,2,\cdots,n\})$ being the class of all subsets of $\{1,2,\cdots,n\}$. The semigroup $\{T_t\}_{t\geq 0}$ is written as

$$T_t f(i) = \bar{E}_i f(Y_t) = \sum_{j=1}^{n} f(j)p_{ij}(t)$$

with $p_{ij}(t) = \bar{P}(t,i,j)$ being the transition function of the process and $f:\{1,2,\cdots,n\}\to\mathbf{R}$. Hence, we can identify the domain of T_t with $\mathbf{R}^n$. It is easily seen that the infinitesimal generator of $\{T_t\}_{t\geq 0}$ is

$$\mathcal{A}^1 f(i) = \sum_{j=1,j\neq i}^{n} [f(j)-f(i)]\,q_{ij},\qquad i = 1,\cdots,n.$$

Let $Q = (q_{ij})_{i,j=1,\cdots,n}$ with $q_{ii} = -\sum_{j\neq i} q_{ij}$. Then

$$\mathcal{A}^1[f(1),\cdots,f(n)]^T = Q[f(1),\cdots,f(n)]^T$$

and the infinitesimal generator is identified with the matrix Q.

Condition **(L.1)** is obvious. Condition **(L.2)** follows from

$$
\lim_{t \to 0} p_{ij}(t) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{4.3}
$$

It is easily verified that (4.3) implies that $(Y_t; \bar{P}_y)$ is uniformly stochastically continuous. The assumption that $q_{ij} > 0$ for $i \neq j$ ensures that $T_t f > 0$ for every $f \geq 0$, $f \neq 0$, i.e., condition **(L.3)** is fulfilled.

For each $h : \{1, \cdots, n\} \to \mathbf{R}$, the semigroup $\{T_t^{(h)}\}_{t \geq 0}$ in (2.5) is a semigroup of matrices acting in the n-dimensional space of vectors $f = (f(1), \cdots, f(n))$. The infinitesimal generator is

$$
\mathcal{A}^h = (q_{ij} + \delta_{ij} h(i)) \equiv Q^{(h)}.
$$

The semigroup $\{T_t^{(h)}\}_{t \geq 0}$ can be represented in the form $T_t^{(h)} = \exp\{t\, Q^{(h)}\}$. Clearly condition **(L.4)** is fulfilled. The operator $T_t^{(h)}$ is compact because the space is finite-dimensional and then condition **(L.5)** is also verified.

Problem (1.2) reduces to the system

$$
\begin{cases} \dfrac{\partial u_k^\varepsilon(t,x)}{\partial t} = \dfrac{\varepsilon a_k(x)}{2} \dfrac{\partial^2 u_k^\varepsilon(t,x)}{\partial x^2} + \dfrac{1}{\varepsilon} \left[f_k(x, u_k^\varepsilon) + \displaystyle\sum_{j=1}^{n} q_{kj}(u_k^\varepsilon - u_j^\varepsilon) \right], \quad x \in \mathbf{R},\, t > 0 \\ u_k^\varepsilon(0,x) = g_k(x), \quad k = 1, \cdots, n. \end{cases}
$$

This problem is a particular case of the reaction-diffusion system studied by Freidlin in [5]. We included this example here just to make more natural the construction of the next example.

Example 4.3. Let $(Y_t; \bar{P}_y)$ be a Wiener process in $[-b; b]$ with instantaneous reflection at the end-points (particular case of Example 4.1). Let ν_t be a step random process with states $\{1, \cdots, n\}$ and $P_{ij}(\Delta) = q_{ij}\, \Delta + O(\Delta)$ as $\Delta \downarrow 0$, $i \neq j$, $q_{ij} \geq 0$ (as in Example 4.2). We consider the homogeneous right-continuous Markov process (Y_t, ν_t) in the phase-space $[-b; b] \times \{1, \cdots, n\}$.

The semigroup on the space B of bounded, measurable functions on $[-b; b] \times \{1, \cdots, n\}$ into $\mathbf{R}$ associated with the process (Y_t, ν_t) is

$$T_t f(y,i) = E_{y,i} f(Y_t, \nu_t) = \sum_{j=1}^{n} \bar{E}_y f(Y_t, j) P_i(\nu_t = j).$$

The above semigroup can be regarded as acting on the space of bounded, measurable functions on $[-b;b]$ into $\mathbf{R}^n$, i.e., $f(y) = (f_1(y), \cdots, f_n(y))$, $y \in [-b;b]$. In this case, $T_t[f^T(y)] = [E_{y,1} f(Y_t, \nu_t), \cdots, E_{y,n} f(Y_t, \nu_t)]^T$. The infinitesimal generator of $\{T_t\}_{t\geq 0}$ is

$$\mathcal{A}^1 f(y) = \frac{1}{2} \frac{\partial^2 f(y)}{\partial y^2} + Q f^T(y)$$

where $Q = (q_{ij})_{i,j=1,\cdots,n}$. Conditions **(L.1)**-**(L.5)** can be easily verified similarly to examples 4.1 and 4.2.

Problem (1.2) reduces to a weakly coupled R-D equation:

$$(4.4) \qquad \begin{cases} \dfrac{\partial u_k^\varepsilon(t,x,y)}{\partial t} = \dfrac{1}{2\varepsilon} \dfrac{\partial^2 u_k^\varepsilon(t,x,y)}{\partial y^2} + \dfrac{\varepsilon a_k(x,y)}{2} \dfrac{\partial^2 u_k^\varepsilon(t,x,y)}{\partial x^2} + \\ \qquad + \dfrac{1}{\varepsilon} \left[f_k(x,y,u_k^\varepsilon) + \displaystyle\sum_{j=1}^{n} q_{kj}(u_k^\varepsilon - u_j^\varepsilon) \right], \quad x \in \mathbf{R}, |y| < b, t > 0 \\ u_k^\varepsilon(0,x,y) = g_k(x) \\ \dfrac{\partial u_k^\varepsilon(t,x,y)}{\partial y} |_{y=\pm b} = 0 \quad \text{for } k = 1, \cdots, n \end{cases}$$

where $q_{ij} \geq 0$ for $i,j \in \{1, \cdots, n\}$. For each k, the functions $a_k(x,y)$, $f_k(x,y,u)$, $g_k(x)$ satisfy the same conditions given in the introduction of this paper. Here, $G_0 = \operatorname{supp}(\sum_{k=1}^{n} g_k)$. We assume that G_0 is contained in the closure of the set (G_0) of its interior points.

System (4.4) is associated with a right-continuous strong Markov process

$$\left(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon, \nu_t^\varepsilon; \widetilde{P}_{xyk}^\varepsilon \right)$$

in the phase-space $\mathbf{R} \times [-b;b] \times \{1, \cdots, n\}$. The process ν_t^ε is obtained from ν_t by taking $\nu_t^\varepsilon \equiv \nu_{\frac{t}{\varepsilon}}$, the process Y_t^ε is defined by $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$, and the first component $\widetilde{X}_t^\varepsilon$ satisfies the stochastic differential equation

$$d\widetilde{X}_t^\varepsilon = \sqrt{\varepsilon\, a_{\nu_t^\varepsilon}(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon)}\, dW_t, \qquad \widetilde{X}_0^\varepsilon = x.$$

The probabilistic representation of the solution of (4.4) is obtained from the Feynman-Kac formula: One can prove that

$$u_k^\varepsilon(t,x,y) = \widetilde{E}_{xyk} g_{\nu_t^\varepsilon}(\widetilde{X}_t^\varepsilon) \exp\left\{ \frac{1}{\varepsilon} \int_0^t c_{\nu_t^\varepsilon}\left(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon, u_{\nu_s^\varepsilon}^\varepsilon(t-s, \widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\right) ds \right\}. \tag{4.5}$$

The asymptotic behavior, as $\varepsilon \downarrow 0$, of the solution of (4.4) is analyzed by means of the action functional for the family of random processes $\left(\widetilde{X}_t^\varepsilon, \int_0^t c_{\nu_s^\varepsilon}(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds\right)$. Since (Y_t, ν_t) satisfies conditions **(L.1)-(L.5)** we can apply the results obtained in section 3.

Notice that using Example 4.1, problem (4.4) can be generalized to

$$\begin{cases} \dfrac{\partial u_k^\varepsilon(t,x,y)}{\partial t} = \dfrac{1}{\varepsilon} \displaystyle\sum_{i=1}^{r} c_k^i(y) \dfrac{\partial u_k^\varepsilon(t,x,y)}{\partial y^i} + \dfrac{1}{2\varepsilon} \displaystyle\sum_{i,j=1}^{r} d_k^{ij}(y) \dfrac{\partial^2 u_k^\varepsilon(t,x,y)}{\partial y^i \partial y^j} + \\ \qquad + \dfrac{\varepsilon}{2} a_k(x,y) \dfrac{\partial^2 u_k^\varepsilon(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon} \left[f_k(x,y,u_k^\varepsilon) + \displaystyle\sum_{j=1}^{n} q_{kj}(u_k^\varepsilon - u_j^\varepsilon) \right], \\ \qquad x \in \mathbf{R},\ y \in (D),\ t > 0 \\ u_k^\varepsilon(0,x,y) = g_k(x) \\ \dfrac{\partial u^\varepsilon(t,x,y)}{\partial n(y)}\Big|_{y \in \partial D} = 0 \end{cases}$$

for $k = 1, \cdots, n$ and $q_{ij} \geq 0$ for $i,j \in \{1, \cdots, n\}$.

Now, the action functional for $\left(\widetilde{X}_t^\varepsilon, \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds\right)$ is given in (3.1) but the function $L(x; \alpha^1, \alpha^2)$ is the Legendre transform of the first eigenvalue $\lambda(x; \beta_1, \beta_2)$ of the operator

$$\frac{1}{2} \frac{\partial^2}{\partial y^2} + Q + [\beta_1 a(x,y) + \beta_2 c(x,y)], \quad y \in (D),\ x \in \mathbf{R}.$$

5. Slow motion dependent of the fast variable

Let us consider the following mixed problem:

$$
(5.1) \qquad \begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \mathcal{A}^{1,\varepsilon} u^\varepsilon(t,x,y) + \mathcal{A}^{2,\varepsilon} u^\varepsilon(t,x,y) + \dfrac{1}{\varepsilon} f(x,y,u^\varepsilon), \\ \text{for } t>0,\ x \in \mathbf{R},\ y \in (D),\ D \subset \mathbf{R}^r \\ u^\varepsilon(0,x,y) = g(x,y). \end{cases}
$$

The nonlinear term $f(x,y,u)$ and the initial function $g(x,y)$ satisfy the conditions specified in the introduction of this paper. The operator $\mathcal{A}^{1,\varepsilon}$ is the infinitesimal generator of the fast process $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$ where $(Y_t; \bar{P}_y)$ is a homogeneous Markov process in the phase space $(D; \mathcal{B}(D))$, $D \subset \mathbf{R}^r$ being a compact set and $\mathcal{B}(D)$ the σ-field of the Borel subsets of D in the topology inherited from the Euclidean norm in $\mathbf{R}^r$. We assume that $(Y_t; \bar{P}_y)$ satisfies conditions **(L.1)-(L.5)** formulated in Theorem 2.2.

The operator $\mathcal{A}^{2,\varepsilon}$ is the infinitesimal generator of the slow process $\widetilde{X}_t^\varepsilon$ and is defined in (1.6). We assume that $a(y)$ and $b(y)$ in (1.6) are real-valued continuous functions and $0 < \underline{a} \le a(y) \le \bar{a}$, $\underline{b} \le b(y) \le \bar{b}$. It is important to observe and keep in mind that the infinitesimal characteristics of $\widetilde{X}_t^\varepsilon$ depend only on the fast variable y and the measure $\Pi(\cdot)$ does not depend neither of x or y.

The strong Markov process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$ is associated with the operator $\mathcal{A}^{1,\varepsilon} + \mathcal{A}^{2,\varepsilon}$. Moreover, one can prove (see for example Freidlin [4]) that there exists a unique generalized solution of problem (5.1) in the sense that it satisfies the unique solution of the generalized Feynman-Kac formula (1.3).

As in problem (1.1) which was studied in [2], the asymptotic behavior of the solution $u^\varepsilon(t,x,y)$ of (5.1) as $\varepsilon \downarrow 0$ is related with probabilities of large deviations for the two-dimensional family of processes $(\widetilde{X}_t^\varepsilon, \widetilde{Z}_t)$ where $\widetilde{Z}_t^\varepsilon = \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds$. To determine the action functional for $(\widetilde{X}_t^\varepsilon, \widetilde{Z}_t)$ we shall express $\widetilde{X}_t^\varepsilon$ as the unique solution of a stochastic differential equation.

Let us consider the stochastic differential equation

$$
(5.2) \qquad d\widetilde{X}_t^\varepsilon = b(Y_t^\varepsilon)\, dt + \sqrt{\varepsilon}\sigma(Y_t^\varepsilon)\, dW_t + \varepsilon \int_{\mathbf{R}} u\, \mu_q^\varepsilon(dt, du), \quad \widetilde{X}_0^\varepsilon = x
$$

where $\sigma^2(y) = a(y)$ and q is a $(\mathcal{F}_t)$-stationary Poisson point process in $\mathbf{R}$ (see definition in Ikeda & Watanabe [9]) with characteristic measure $\frac{1}{\varepsilon}\Pi(\cdot)$. Let $\nu_q^\varepsilon(t,A)$ be the integervalued random measure associated with q and $\mu_q^\varepsilon(t,A)$ the corresponding orthogonal local martingale measure. It is known (see [8], Vol 3) that $\nu_q^\varepsilon(t,A) = \mu_q^\varepsilon(t,A) + \frac{1}{\varepsilon}\Pi(A)t$ for all $A \in \mathcal{B}(\mathbf{R})$ with $\Pi(A) < \infty$ and $E\nu_q^\varepsilon(t,A) = \frac{1}{\varepsilon}\Pi(A)t$. The process W_t is a $\mathbf{R}$-Wiener process starting at zero, $(\mathcal{F}_t)$-adapted, and independent of q.

One can prove similarly to the proof of Theorem VI.9.1 in [9] that, for a fixed trajectory of Y_t^ε, there exists a unique solution $\widetilde{X}_t^\varepsilon$ of (5.2) which can be written as

$$\widetilde{X}_t^\varepsilon = x + \int_0^t b(Y_s^\varepsilon)\,ds + \sqrt{\varepsilon}\int_0^t \sqrt{a(Y_s^\varepsilon)}\,dW_s + \varepsilon\int_{\mathbf{R}} u\,\mu_q^\varepsilon(t,du). \tag{5.3}$$

Using the generalized Itô's formula (see [8], Vol 3) one can verify that the infinitesimal generator of $\widetilde{X}_t^\varepsilon$ is given by (1.6).

We shall now derive the action functional for $\left(\widetilde{X}_t^\varepsilon, \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\,ds\right)$ on $(\mathcal{D}_{[0,T]}(\mathbf{R}^2), \rho_{0T})$. Here $\mathcal{D}_{[0,T]}(\mathbf{R}^m)$ is the space of right-continuous functions with limit on the left and ρ_{0T} is defined by $\rho_{0T}((\varphi^1,\cdots,\varphi^m),(\psi^1,\cdots,\psi^m)) = \sum_{i=1}^m \|\varphi^i - \psi^i\|$ where $\|\cdot\|$ is the supremum norm in $\mathcal{D}_{[0,T]}(\mathbf{R})$.

Let us introduce the processes

$$\Upsilon_t^\varepsilon = \int_0^t a(Y_s^\varepsilon)\,ds, \quad \xi_t^\varepsilon = \int_0^t b(Y_s^\varepsilon)\,ds. \tag{5.4}$$

Taking into account the proprieties of the functions $a(y)$ and $b(y)$ we can see that the trajectories of the processes in (5.4) belong respectively to the sets $F_{\bar{a}}$ and $F_{\bar{b}}$ a.s. where

$$F_{\bar{k}} = \left\{\psi \in C_{[0,T]}(\mathbf{R}) : \psi_0 = 0,\ \exists\dot{\psi}_t \text{ a.e. },\underline{k} \le \dot{\psi}_t \le \bar{k},\ t\in[0,T]\right\}.$$

It is known (see McKean [11]) that there exists a Wiener process $\widetilde{W}_t$ in $\mathbf{R}$, starting at zero, and independent of Y_t^ε satisfying the relation

$$\sqrt{\varepsilon}\int_0^t \sqrt{a(Y_s^\varepsilon)}\,dW_s = \sqrt{\varepsilon}\,\widetilde{W}_{\int_0^t a(Y_s^\varepsilon)\,ds}.$$

Then $x + \sqrt{\varepsilon}\int_0^t \sqrt{a(Y_s^\varepsilon)}\,dW_s = X_{\Upsilon_t^\varepsilon}^\varepsilon$ where $X_t^\varepsilon = x + \sqrt{\varepsilon}\,\widetilde{W}_t$. Therefore, the process $\widetilde{X}_t^\varepsilon$ satisfying (5.2) can be written as

(5.5) $$\widetilde{X}_t^\varepsilon = \xi_t^\varepsilon + X_{\Upsilon_t^\varepsilon}^\varepsilon + \zeta_t^\varepsilon$$

where

(5.6) $$\zeta_t^\varepsilon = \varepsilon \int_{\mathbf{R}} u\, \mu_q^\varepsilon(t, du).$$

The process ζ_t^ε is a process in $\mathbf{R}$ with frequent small jumps and trajectories belonging to $\mathcal{D}_{[0,T]}(\mathbf{R})$ with probability one. Moreover, it is independent of Y_t^ε and X_t^ε and has infinitesimal generator given by

$$\Pi f(x) = \frac{1}{\varepsilon} \int_{\mathbf{R}} \left[f(x+\varepsilon\beta) - f(x) - \varepsilon\beta \frac{df(x)}{dx} \right] \Pi(d\beta)$$

where $\Pi(\cdot)$ is a σ- finite measure with $\Pi(\{0\}) = 0$ and $\int_{-\infty}^{+\infty} \beta^2\, \Pi(d\beta) < \infty$. The cumulant of the process ζ_t^ε (see [15]) is

$$G^\varepsilon(z) = \frac{1}{\varepsilon} \int_{\mathbf{R}} [e^{\varepsilon z\beta} - 1 - \varepsilon z\beta]\, \Pi(d\beta).$$

Then

$$G^0(z) = \lim_{\varepsilon \downarrow 0} \varepsilon\, G^\varepsilon(\frac{z}{\varepsilon}) = \int_{\mathbf{R}} [e^{z\beta} - 1 - z\beta]\, \Pi(d\beta).$$

The function $G^0(z)$ is measurable with respect to z and $G^0(0) \equiv 0$. It is downward convex and lower semicontinuous ; in the interior of its domain of finiteness it is analytic and the second-order derivative is strictly positive (see [15]). Let $H_0(u)$ be the Legendre transform of $G^0(z)$. This function is also lower semicontinuous and downward convex.

Let us assume that G^0 and H_0 satisfy the following conditions:

(S.1) $G^0(z) \le \bar{G}^0(z)$ for all z where $\bar{G}^0$ is a downward convex nonnegative function, finite for all z, and $G^0(0) \equiv \bar{G}^0(0) = 0$. This condition means that $\bar{H}_0(u) \le H_0(u)$ for all u where $\bar{H}_0(u)$ is the Legendre transform of $\bar{G}^0(z)$; the condition of finiteness of $\bar{G}^0$ becomes $\lim_{|u| \to \infty} \frac{\bar{H}_0(u)}{|u|} = \infty$.

(S.2) $H_0(u) < \infty$ for the same u for which $\bar{H}_0(u)$ is finite.

(S.3) The set $\{u : \bar{H}_0(u) < \infty\}$ is open.

(S.4) For any compactum $K \subset \{u : \bar{H}_0(u) < \infty\}$ the derivative $\frac{\partial H_0}{\partial u}$ is bounded and continuous in $u \in K$.

Then, relying on Theorem 4.3.1 in Wentzell **[15]**, we conclude that the action functional for the family of processes $(\zeta_t^\varepsilon; P_x^\varepsilon)$ is $\frac{1}{\varepsilon}S_{0T}(\nu)$ where

$$S_{0T}(\nu) = \begin{cases} \int_0^T H_0(\dot{\nu}_s)\,ds, & \text{if } \nu \text{ is a.c.} \\ +\infty, & \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbf{R}). \end{cases} \tag{5.7}$$

Now we shall determine the action functional for the two-dimensional family of processes $(X^\varepsilon_{\Upsilon^\varepsilon_t}, \xi^\varepsilon_t)$ in the space $(\mathcal{D}_{[0,T]}(\mathbf{R}^2), \rho_{0T})$. Let G_1 be the operator from $(\mathcal{D}_{[0,T]}(\mathbf{R}^3), \rho_{0T})$ into $(\mathcal{D}_{[0,T]}(\mathbf{R}^2), \rho_{0T})$ defined by $G_1(\varphi, \psi, \eta) = (\varphi, \eta)$; it is easily seen that G_1 is a continuous operator. Using Theorem 3.3.1 in **[6]** and taking into account that $(X^\varepsilon_{\Upsilon^\varepsilon_t}, \xi^\varepsilon_t) = G_1(X^\varepsilon_{\Upsilon^\varepsilon_t}, \Upsilon^\varepsilon_t, \xi^\varepsilon_t)$, we can see that it suffices to obtain the action functional for $(X^\varepsilon_{\Upsilon^\varepsilon_t}, \Upsilon^\varepsilon_t, \xi^\varepsilon_t)$.

Using the same proof of Propositions 2.1, 2.2, and 2.3 in **[2]**, one can show that the action functional for $(X^\varepsilon_{\Upsilon^\varepsilon_t}, \Upsilon^\varepsilon_t, \xi^\varepsilon_t)$ is $\frac{1}{\varepsilon}\widetilde{S}_{0T}(\varphi, \psi, \eta)$ with

$$\widetilde{S}_{0T}(\varphi, \psi, \eta) = \begin{cases} \frac{1}{2}\int_0^T \frac{|\dot{\varphi}_s|^2}{\psi_s}\,ds + \int_0^T L(\dot{\psi}_s, \dot{\eta}_s)\,ds, & \text{if } \varphi \text{ is a.c. }, \psi \in F_{\bar{a}},\ \eta \in F_{\bar{b}} \\ +\infty, & \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbf{R}^3). \end{cases}$$

The function $L(\alpha^1, \alpha^2)$ is the Legendre transform of the first eigenvalue $\lambda(\beta_1, \beta_2)$ of the operator

$$\mathcal{A}^1 + [\beta_1 a(y) + \beta_2 b(y)]$$

where $\mathcal{A}^1$ is the infinitesimal generator of the process $(Y_t; \bar{P}_y)$.

Now, using Theorem 3.3.1 in **[6]**, we conclude that the action functional for $(X^\varepsilon_{\Upsilon^\varepsilon_t}, \xi^\varepsilon_t)$ is $\frac{1}{\varepsilon}S_{0T}(\varphi, \eta)$ with

$$(5.8) \qquad S_{0T}(\varphi,\eta)=\begin{cases}\inf\limits_{\psi\in F_a}\left\{\frac{1}{2}\int_0^T\frac{|\dot\varphi_s|^2}{\dot\psi_s}\,ds+\int_0^T L(\dot\psi_s,\dot\eta_s)\,ds\right\}, & \text{if } \varphi \text{ is a.c. },\\ & \eta\in F_{\bar b}\\ +\infty, & \text{in the rest of}\\ & \mathcal{D}_{[0,T]}(\mathbb{R}^2).\end{cases}$$

The action functional for $X^{\varepsilon}_{\Upsilon^\varepsilon_t}+\xi^\varepsilon_t$ is easily obtained taking into account that $X^{\varepsilon}_{\Upsilon^\varepsilon_t}+\xi^\varepsilon_t = G_2(X^{\varepsilon}_{\Upsilon^\varepsilon_t},\xi^\varepsilon_t)$ where $G_2(\varphi,\eta)=\varphi+\eta$. Clearly G_2 is a continuous operator from $(\mathcal{D}_{[0,T]}(\mathbb{R}^2),\rho_{0T})$ into $(\mathcal{D}_{[0,T]}(\mathbb{R}),\rho_{0T})$. Using Theorem 3.3.1 in [5] once again, we obtain the action functional for $X^{\varepsilon}_{\Upsilon^\varepsilon_t}+\xi^\varepsilon_t$ which is given by $\frac{1}{\varepsilon}S_{0T}(\varphi)$ with

$$(5.9) \qquad S_{0T}(\varphi)=\begin{cases}\inf\limits_{\eta\in F_{\bar b},\psi\in F_a}\left\{\int_0^T\frac{|\dot\varphi_s-\dot\eta_s|^2}{\dot\psi_s}\,ds+\int_0^T L(\dot\psi_s,\dot\eta_s)\,ds\right\}, & \text{if } \varphi \text{ is a.c}\\ +\infty, & \text{in the rest of}\\ & \mathcal{D}_{[0,T]}(\mathbb{R}).\end{cases}$$

The process ζ^ε_t in (5.6) is independent of $X^{\varepsilon}_{\Upsilon^\varepsilon_t}+\xi^\varepsilon_t$. Then the normalized action functional for $(X^{\varepsilon}_{\Upsilon^\varepsilon_t}+\xi^\varepsilon_t,\zeta^\varepsilon_t)$ is the sum of the functionals in (5.9) and (5.7), i.e, it is given by

$$S_{0T}(\varphi,\nu)=\begin{cases}\inf\limits_{\eta\in F_{\bar b},\psi\in F_a}\left\{\frac{1}{2}\int_0^T\frac{|\dot\varphi_s-\dot\eta_s|^2}{\dot\psi_s}\,ds+\int_0^T L(\dot\psi_s,\dot\eta_s)+\int_0^T H_0(\dot\nu_s)\,ds\right\},\\ \text{if } \varphi,\ \nu \text{ are a.c}\\ +\infty, \quad \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbb{R}^2).\end{cases}$$

The process $\widetilde{X}^\varepsilon_t$ in (5.5) satisfies the relation $\widetilde{X}^\varepsilon_t = G_3(X^{\varepsilon}_{\Upsilon^\varepsilon_t}+\xi^\varepsilon_t,\zeta^\varepsilon_t)$ where $G_3(\varphi,\nu)=\varphi+\nu$. It is easily seen that the operator G_3 from $(\mathcal{D}_{[0,T]}(\mathbb{R}^2),\rho_{0T})$ into $(\mathcal{D}_{[0,T]}(\mathbb{R}),\rho_{0T})$ is a continuous operator. Relying on Theorem 3.3.1 in [6] we obtain the action functional for $\widetilde{X}^\varepsilon_t$ which is $\frac{1}{\varepsilon}S_{0T}(\varphi)$ with

$$
(5.10) \qquad S_{0T}(\varphi) = \begin{cases} \inf_{\psi\in F_{\bar a},\eta\in F_{\bar b},\nu \text{ a.c}} \left\{ \frac{1}{2}\int_0^T \frac{|\dot\varphi_s - \dot\eta_s - \dot\nu_s|^2}{\dot\psi_s}\, ds + \int_0^T L(\dot\psi_s,\dot\eta_s)\, ds + \right. \\ \qquad\qquad \left. + \int_0^T H_0(\dot\nu_s)\, ds \right\}, \text{ if } \varphi \text{ is a.c.} \\ +\infty, \quad \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbf{R}). \end{cases}
$$

The action functional for the two-dimensional family of processes $\left(\widetilde{X}_t^\epsilon; \int_0^t c(\widetilde{X}_s^\epsilon, Y_s^\epsilon)\, ds\right)$ is obtained in a similar way. Here we shall just point out the differences.

For each $\varphi \in C_{[0,T]}(\mathbf{R})$, define

$$
Z_t^{\epsilon,\varphi} = \int_0^t c(\varphi_s, Y_s^\epsilon)\, ds.
$$

The trajectories of $Z_t^{\epsilon,\varphi}$ belong to $F_{\bar c}$ with probability one. Moreover, $(Y_t; \bar P_y)$ satisfies conditions **(L.1)-(L.5)** introduced in Theorem 2.2. Then we can apply Theorem 7.4.1 in **[6]** to conclude that the action functional for the three-dimensional family of random processes $(\Upsilon_t^\epsilon, \xi_t^\epsilon, Z_t^{\epsilon,\varphi})$ is given by $\frac{1}{\epsilon} S_{0T}^\varphi(\psi,\eta,\phi)$ with

$$
S_{0T}^\varphi(\psi,\eta,\phi) = \begin{cases} \int_0^T L(\varphi_t, \dot\psi_t, \dot\eta_t, \dot\phi_t)\, dt, & \text{if } \psi\in F_{\bar a},\ \eta \in F_{\bar b},\ \phi\in F_{\bar c} \\ +\infty, & \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbf{R}^3) \end{cases}
$$

where $L(x,\alpha^1,\alpha^2,\alpha^3)$ is the Legendre transform of the first eigenvalue $\lambda(x,\beta_1,\beta_2,\beta_3)$ with respect to β_1, β_2, β_3 of the operator

$$
\mathcal{A}^1 + [\beta_1 a(y) + \beta_2 b(y) + \beta_3 c(x,y)].
$$

Using the same arguments as in [2] one can obtain the action functional for $(X_{\Upsilon_t^\epsilon}^\epsilon, \Upsilon_t^\epsilon, \xi_t^\epsilon, Z_t^\epsilon)$ and then, as before, to prove that the normalized action functional for $\left(\widetilde{X}_t^\epsilon, \int_0^t c(\widetilde{X}_s^\epsilon, Y_s^\epsilon)\, ds\right)$ is

$$
(5.11) \qquad S_{0T}(\varphi,\phi)=\begin{cases} \inf_{\psi,\eta,\nu}\left\{\frac{1}{2}\int_0^T \frac{|\dot\varphi_s-\dot\eta_s-\dot\nu_s|^2}{\dot\psi_s}\,ds+\int_0^T L(\varphi_s,\dot\psi_s,\dot\eta_s,\dot\phi_s)\,ds \right. \\ \qquad \left. +\int_0^T H_0(\dot\nu_s)\,ds : \psi\in F_{\bar a},\ \eta\in F_{\bar b},\ \nu \text{ a.c.}\right\}, \\ \text{if } \varphi \text{ is a.c }, \ \phi\in F_{\bar c} \\ +\infty, \quad \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbf{R}^2) \end{cases}
$$

with normalizing coefficient $\frac{1}{\varepsilon}$.

To analyze the asymptotic behavior of the solution $u^\varepsilon(t,x,y)$ of problem (5.1) we shall follow the same approach used in [2]. Let us define for each $t>0$ and $x\in\mathbf{R}$ a function $V(t,x)$,

$$
(5.12) \qquad V(t,x)=\sup\{\phi_t-S_{0t}(\varphi,\phi):\varphi\in C_{[0,t]}(\mathbf{R}),\ \varphi_0=x,\ \varphi_t\in G_0,\ \phi\in F_{\bar c}\}.
$$

where S_{0t} is defined in (5.11).

Exactly as in [2] one can prove that $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t,x,y)=0$ if $(t,x,y)\in Q_-\times D$ where $Q_-=\{(t,x):V(t,x)<0\}$; further, the convergence is uniform in any compact subset of $Q_-\times D$. Also, if Condition (N) (introduced in section 3) is fulfilled then $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t,x,y)=1$ for $(t,x,y)\in Q_+\times D$ where $Q_+=\{(t,x):V(t,x)>0\}$; this convergence is uniform in compact subsets of $Q_+\times D$.

Condition (N) is a restriction. One can construct an example similar to Example 3.1 in [2] showing that Condition (N) is not fulfilled necessarily. Using the same approach as in [2], one can analyze the wave front propagation of $u^\varepsilon(t,x,y)$ as $\varepsilon\downarrow 0$ without Condition (N).

All the results in section 3 of [2] can be proved in a similar way in this new context. The wave front is described by means of the function $V^*(t,x)$ in (3.4) but using the functional S_{0T} in (5.11).

6. Slow motion independent of the fast variable

In this section we study the wave front propagation as $\varepsilon\downarrow 0$ for the solution $u^\varepsilon(t,x,y)$ of the following Cauchy problem:

$$(6.1) \qquad \begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \mathcal{A}^{1,\varepsilon} u^\varepsilon(t,x,y) + \mathcal{A}^{2,\varepsilon} u^\varepsilon(t,x,y) + \dfrac{1}{\varepsilon} f(x,y,u^\varepsilon), \\ \text{for } t>0,\ x \in \mathbf{R},\ y \in (D),\ D \subset \mathbf{R}^r \\ u^\varepsilon(0,x,y) = g(x) \end{cases}$$

where $f(x,y,u)$ and $g(x)$ satisfy the conditions formulated in section 1. The operator $\mathcal{A}^{1,\varepsilon}$ is the same as in (5.1). We assume that conditions **(L.1)-(L.5)** (see Theorem 2.2) are fulfilled.

The operator $\mathcal{A}^{2,\varepsilon}$ describes the motion of the slow variable. In this section we assume that the slow motion is a time-homogeneous locally infinitely divisible process $(\widetilde{X}_t^\varepsilon; \widetilde{P}_x^\varepsilon)$ (see definition in section 1) with infinitesimal generator $\mathcal{A}^{2,\varepsilon}$ defined in (1.7). We assume that $b(x)$ and $a(x)$ in (1.7) are bounded, measurable, and Lipschitz continuous real-valued functions satisfying $0 < \underline{a} \le a(x) \le \bar{a}$ and $\underline{b} \le b(x) \le \bar{b}$. Notice that the infinitesimal characteristics of the slow motion are independent of the fast variable y.

As in section 5, the strong Markov process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$ is associated with the operator $L^\varepsilon = \mathcal{A}^{1,\varepsilon} + \mathcal{A}^{2,\varepsilon}$. Besides, there exists a unique generalized solution of (6.1) and it satisfies the generalized Feynman-Kac formula in (1.3). Again, the asymptotic behavior of $u^\varepsilon(t,x,y)$ as $\varepsilon \downarrow 0$ is analyzed by means of the action functional for $(\widetilde{X}_t^\varepsilon, \widetilde{Z}_t^\varepsilon)$ where $\widetilde{Z}_t^\varepsilon = \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds$.

The fact that $\widetilde{X}_t^\varepsilon$ and Y_t^ε are independent simplifies significantly the derivation of the action functional for $(\widetilde{X}_t^\varepsilon, \widetilde{Z}_t^\varepsilon)$. We will not go into details in this matter but just point out the main steps.

First, we use the result obtained by Wentzell [15] in Theorem 4.3.1. By assuming the hypothesis of that theorem, we can say that the action functional for $\widetilde{X}_t^\varepsilon$ is $\frac{1}{\varepsilon} S_{0T}(\varphi)$ with

$$(6.3) \qquad S_{0T}(\varphi) = \begin{cases} \int_0^T H_0(\varphi_t; \dot{\varphi}_t)\, dt, & \text{if } \varphi \text{ is a.c.} \\ +\infty, & \text{in the rest of } \mathcal{D}_{[0,T]}(\mathbf{R}). \end{cases}$$

The function $H_0(x;u)$ is the Legendre transform of $G^0(x;z)$ with respect to z and $G^0(x;z) = \lim_{\varepsilon \downarrow 0} \varepsilon\, G^\varepsilon(x; \frac{z}{\varepsilon})$ where

$$G^\varepsilon(x;z) = b(x)z + \frac{\varepsilon a(x)}{2} z^2 + \frac{1}{\varepsilon} \int_{\mathbf{R}} [e^{\varepsilon\beta z} - 1 - \varepsilon\beta z]\, \Pi_x(d\beta).$$

Notice that G^ε is the cumulant of the process $\widetilde{X}_t^\varepsilon$ (see **[15]**).

Secondly, for each $\varphi \in C_{[0,T]}(\mathrm{R})$, define $Z_t^{\varepsilon,\varphi} = \int_0^t c(\varphi_s, Y_s^\varepsilon)\,ds$. Recall that $(Y_t; \bar{P}_y)$ satisfies conditions **(L.1)**-**(L.5)**. Then, from Theorem 7.4.1 in **[6]** we obtain the action functional $\frac{1}{\varepsilon} S_{0T}^{\varphi}(\phi)$ for $Z_t^{\varepsilon,\varphi}$ with

$$S_{0T}^{\varphi}(\phi) = \begin{cases} \int_0^T L(\varphi_t; \dot{\phi}_t)\,dt, & \phi \in F_{\bar{\varepsilon}} \\ +\infty, & \text{in the rest of } \mathcal{D}_{[0,T]}(\mathrm{R}) \end{cases}$$

where $L(x;\alpha)$ is the Legendre transform of the first eigenvalue $\lambda(x;\beta)$ of the operator $\mathcal{A}^1 + \beta\, c(x,y)$.

It turns out that the action functional for $(\widetilde{X}_t^\varepsilon, \widetilde{Z}_t^\varepsilon)$ is $\frac{1}{\varepsilon}\widetilde{S}_{0T}(\varphi,\phi)$ where

$$(6.4) \qquad \widetilde{S}_{0T}(\varphi,\phi) = \begin{cases} \int_0^T H_0(\varphi_t; \dot{\varphi}_t)\,dt + \int_0^T L(\varphi_t; \dot{\phi}_t)\,dt, & \text{if } \varphi \text{ is a.c., } \phi \in F_{\bar{\varepsilon}} \\ +\infty, & \text{in the rest of } \mathcal{D}_{[0,T]}(\mathrm{R}^2). \end{cases}$$

To prove this fact we shall verify conditions **(A.0)**-**(A.2)** introduced in **[2]**.

The compactness of the level sets (condition **(A.0)**) can be proved similarly to Proposition 2.1 in [2] and Theorem 3.1.1 (b) in Wentzell **[15]**. The lower and upper bounds (conditions **(A.1)** and **(A.2)**) are easily obtained by taking into account that $\widetilde{X}_t^\varepsilon$ and Y_t^ε are independent and $c(x,y)$ is Lipschitz continuous in x.

The wave front propagation of $u^\varepsilon(t,x,y)$ as $\varepsilon \downarrow 0$ is described by means of the function

$$(6.5) \qquad \begin{aligned} V(t,x) = \sup\{\phi_t - \int_0^t H_0(\varphi_s; \dot{\varphi}_s)\,ds - \int_0^t L(\varphi_s; \dot{\phi}_s)\,ds : \\ \varphi_0 = x,\ \varphi \in C_{[0,T]}(\mathrm{R}),\ \varphi_t \in G_0,\ \phi \in F_{\bar{\varepsilon}}\}. \end{aligned}$$

Observe that the function in (6.5) is the same function $V(t,x)$ introduced in (3.3) but using the action functional in (6.4).

As in [2] one can prove that $u^\varepsilon(t,x,y)$ converges to zero as $\varepsilon \downarrow 0$ in the region $\{(t,x) : V(t,x) < 0\} \times D$. Moreover, if Condition (N) is fulfilled, then $u^\varepsilon(t,x,y)$ converges to one in $\{(t,x) : V(t,x) : V(t,x) > 0\} \times D$.

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