Universidade Federal do Rio Grande do Sul
Instituto de Matemática
Cadernos de Matemática e Estatística
Série A: Trabalho de Pesquisa

ORE EXTENSIONS AND JACOBSON RINGS

Miguel Ferrero
Eduardo Cisneros
María Inés González

Série A, nọ 27,
Porto Alegre, maio de 1992

ORE EXTENSIONS AND JACOBSON RINGS*

Miguel Ferrero
Instituto de Matemática
Universidade Federal do Rio Grande do Sul
90049 - Porto Alegre, RS, Brazil

Eduardo Cisneros and María Inés González
Departamento de Matemática
Universidad Nacional de Rosario
Avda. Pellegrini 250
2000 - Rosario, Argentina

Introduction

Throughout this paper R is a ring with an identity element and ρ is an automorphism of R. The skew Laurent polynomial ring $R\langle X;\rho\rangle$ is the ring whose elements are of the form $\sum_{i=m}^{n} X^i a_i$, $a_i \in R$, where the addition is defined as usually and the multiplication by $aX = X\rho(a)$, for every $a \in R$[5]. The skew polynomial ring $R[X;\rho]$ is the subring of $R\langle X;\rho\rangle$ whose elements are the polynomials $\sum_{i=0}^{n} X^i a_i$, $a_i \in R$.

A ring R is said to be a Jacobson ring if every prime ideal of R is an intersection of primitive (either left or right) ideals. It is well known that R is a Jacobson ring if and only if the polynomial ring $R[X]$ is a Jacobson ring [16]. This result has been extended to other classes of prime ideals in [10] and related questions for skew polynomial rings $R[X;\rho]$ (resp. $R[X;D]$, D a derivation of R) have been considered in [11], [13], [14] and [15] (resp. [7], [8]).

*This paper was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, and Programa de Matemática de Rosario (Conicet-UNR), Argentina.

In particular, in [10] the authors gave the notion of an $\mathcal{A}$-Jacobson ring, where $\mathcal{A}$ is a class of prime rings. An $\mathcal{A}$-Jacobson ring is a ring R such that every prime ideal is an intersection of $\mathcal{A}$-ideals (i.e. ideals P with $R/P \in \mathcal{A}$). It is proved that if R is an $\mathcal{A}$-Jacobson ring, then $R[X]$ is an $\mathcal{A}$-Jacobson ring, for many classes of prime rings $\mathcal{A}$.

The purpose of this paper is to study when every prime ideal of $R\langle X;\rho\rangle$ (resp. $R[X;\rho]$) is an intersection of prime ideals of some particular type following a similar treatment to that in [10]. First we study s-Jacobson rings, i.e., $\mathcal{A}$-Jacobson rings when $\mathcal{A}$ is the class of (right) strongly prime rings. Then we show that the results can be easily extended to other classes of prime rings. In particular, we recover well known results on classical Jacobson rings.

Section 1 is an introductory section. In section 2 we study when $R\langle X;\rho\rangle$ is an s-Jacobson ring. The main result of this section shows that this is the case if and only if R is an s_ρ-Jacobson ring, where an s_ρ- Jacobson ring is defined in a similarly way as an s-Jacobson ring.

In section 3 we consider the same question for $R[X;\rho]$. We prove that $R[X;\rho]$ is an s-Jacobson ring if and only if R is s-Jacobson and (R,ρ) is s'_ρ-Jacobson (where s'_ρ-Jacobson will be defined latter and is slightly weaker than s_ρ-Jacobson).

In section 4 we note that the results of sections 2 and 3 can be easily extended to other classes of prime rings. We obtain general theorems which include the above results. We apply these results to other classes of prime ideals in section 5, where we recover the results on classical Jacobson rings.

1 Prerequisites

Prime ideals of $R\langle X;\rho\rangle$ and $R[X;\rho]$ have been studied in several papers ([1], [3], [14]). In particular, a complete description of R-disjoint prime ideals has been given in [3]. We will use frequently these results. Now we recall some definitions and basic facts.

An ideal I of R is said to be a ρ-ideal (ρ- invariant ideal) if $\rho(I) \subseteq I$ $(\rho(I) = I)$. A ρ-invariant ideal P of R is said to be ρ-prime (resp. strongly ρ-prime) if $IJ \subseteq P$ for any ρ-invariant ideals I and J (ρ-ideal I and ideal J) of R implies either $I \subseteq P$ or $J \subseteq P$. A ρ-prime (strongly ρ-prime) ring R is defined obviously. The terminology is taken from [1] and [3] and does not agree with that of references [13] and [14]. It is convenient to remark that strongly ρ-prime is not the same as ρ-strongly prime (see § 2).

The automorphism ρ of R can be extended to $R\langle X;\rho\rangle$ (and $R[X;\rho]$) by the natural way. We denote the extension by ρ again. Every ideal of $R\langle X;\rho\rangle$ is ρ-invariant.

Let I be a non-zero R-disjoint ideal of $R[X;\rho]$. We denote by $\tau(I)$ (resp.

$\mu(I)$) the ideal of R consisting of 0 and all the leading coefficients (resp. constant terms) of all the polynomials of minimal degree in I, and we put $\gamma(I) = \tau(I) \cap \mu(I)$. For $f \in R[X;\rho]$ we denote by δf the degree of f and we define the minimality of I by $Min(I) = Min\{\delta f : 0 \neq f \in I\}$.

An element $f \in R\langle X;\rho\rangle$ is said to be a proper polynomial if $f = \sum_{i=0}^{n} X^i a_i$, where $a_0 \neq 0$. If I is an R-disjoint ideal of $R\langle X;\rho\rangle$ the ideals $\tau(I)$, $\mu(I)$ and $\gamma(I)$ are defined as above considering elements of minimal length and they are ρ-invariant ideals.

The following results on prime ideals will be very useful (c.f. [1], section 1 and [3], sections 2, 3).

LEMMA 1.1 Let P be an R-disjoint ideal of $R\langle X;\rho\rangle$. Then P is prime if and only if R is ρ- prime and one of the following conditions is fulfilled

i) $P = 0$.

ii) P is maximal with respect to $P \cap R = 0$.

LEMMA 1.2 Let P be an R-disjoint ideal of $R[X;\rho]$. Then P is prime if and only if one of the following conditions is fulfilled

i) R is prime and $X \in P$.

ii) R is strongly ρ-prime, $X \notin P$ and either $P = 0$ or P is maximal with respect to $P \cap R = 0$.

In the first case $P = XR[X;\rho]$ and in the second case X is regular modulo P and P is ρ-invariant.

2 Skew Laurent polynomial rings and s-Jacobson rings.

For the notions of (right) strongly prime (s.prime, for shortness) and ρ-s.prime rings and ideals and the s.prime (ρ-s.prime) radical $s(R)$ $(s_\rho(R))$ of R we refer to [6].

The ring R is said to be s-Jacobson if every prime ideal of R is an intersection of s.prime ideals. Similarly, the pair (R,ρ) (a ring R with an automorphism ρ) is said to be s_ρ-Jacobson if every ρ-prime ideal of R is an intersection of ρ-s.prime ideals.

The main result of this section is the following

THEOREM 2.1 Let R be a ring and ρ an automorphism of R. Then $R\langle X;\rho\rangle$ is s-Jacobson if and only if R is s_ρ-Jacobson.

There are two essentially different parts in the proof of Theorem 2.1. The "only if" part is straightforward and specific for this case. We need the following.

LEMMA 2.2 If P is an s.prime ideal of $R\langle X;\rho\rangle$, then $P\cap R$ is a ρ-s.prime ideal of R.

PROOF: Since $P\cap R$ is a ρ-invariant ideal of R we may factor out this ideal and assume that $P\cap R=0$. If I is a non-zero ρ-invariant ideal of R, then $I\langle X;\rho\rangle$ is an ideal of $R\langle X;\rho\rangle$ which is not contained in P. Hence there exists a finite set $F\subseteq I\langle X;\rho\rangle$ which is an insulator modulo P. Thus the set of all the coefficients of all the polynomials in F is an insulator in R which is contained in I.

The "if" part of Theorem 2.1 is more interesting. In fact, this part is non-trivial and, furthermore, it follows the lines of ([10], Theorem 5). In this way we give a proof which can be repeated for several other classes of Jacobson rings.

In [10], there are two main points: the key Lemma 3 and the condition (A). Lemma 3 is general and can be applied to use the going up argument: given a non-zero prime ideal Q of R and an R-disjoint ideal P of $R[X]$ with $\tau(P)\not\subseteq Q$, to find a prime ideal Q^* of $R[X]$ such that $Q^*\cap R=Q$ and $Q^*\supseteq P$. The corresponding lemma in our case is the following.

LEMMA 2.3 Let P be an R-disjoint ideal of $R\langle X;\rho\rangle$ and let Q be a non-zero ρ-prime ideal of R. If $\gamma(P)\not\subseteq Q$, then $(P+Q\langle X;\rho\rangle)\cap R=Q$.

PROOF: Assume to the contrary that there exists $r\in R\setminus Q$ such that $r=h_1+h_2$, for some $h_1\in P$ and $h_2\in Q\langle X;\rho\rangle$. It follows that there exists $g=\sum_{i=t}^{m}X^ia_i\in P$ such that $a_i\in Q$ for $i\neq 0$ and $a_0\notin Q$. Take such a g of minimal length $l(g)=m-t$.

Assume $m>0$. Since $\gamma(P)\not\subseteq Q$ we have $\tau(P)\not\subseteq Q$ and there exists $f=\sum_{j=u}^{n}X^jb_j\in P$ of minimal length in P with $b_n\notin Q$. We may also assume $n=m$ because X is invertible. Further, since Q is ρ-prime there exists an integer number v and $r\in R$ such that $\rho^v(b_n)ra_0\notin Q$. Then $h=\rho^v(b_n)rg-\rho^{n+v}(f)\rho^n(r)a_n\in P$ has length $l(h)<n-t=l(g)$, which contradicts the minimality of $l(g)$.

The case $m=0$ can be handled similarly using $\mu(P)\not\subseteq Q$.

REMARK 2.4 A similar result holds if Q is a maximal ρ-invariant left ideal of R instead of a ρ-prime ideal. The proof is the same as in Lemma 2.3. This fact will be used in section 5.

The class of strongly prime rings satisfy condition (A) of [10]. In our case this condition takes the following form:

LEMMA 2.5 If R is a ρ-s.prime ring and P is an R-disjoint ideal of $R\langle X;\rho\rangle$ which is maximal with respect to $P \cap R = 0$, then P is an s.prime ideal.

PROOF: Let I be an ideal of $R\langle X;\rho\rangle$ with $I \supsetneq P$. By the maximality of P we have $I \cap R \neq 0$ and so there exists a finite set $F \subseteq I \cap R$ such that $Fa = 0,\quad a \in R$, implies $a = 0$. We show that F is an insulator modulo P in $R\langle X;\rho\rangle$. This is clear if $P = 0$. So we may assume $P \neq 0$.

By ([3], Theorem 1.8), there exists a monic irreducible proper polynomial $f_0 \in Q\langle X;\rho\rangle$ such that $P = f_0Q\langle X;\rho\rangle \cap R\langle X;\rho\rangle$, where Q is here the right (Martindale) ρ-quotient ring of R. Let h be an element of $R\langle X;\rho\rangle$ with $Fh \subseteq P$. Take an integer number t such that $X^t h \in R[X;\rho]$. Since f_0 is monic there exist $g, r \in Q[X;\rho]$ with $X^t h = f_0 g + r$, where $\delta r < \delta f_0 = Min(P)$. Hence $h = X^{-t} f_0 g + X^{-t} r$ and so $FX^{-t} r \subseteq f_0 Q\langle X;\rho\rangle$. Take a ρ-invariant ideal H of R such that $rH \subseteq R[X;\rho]$. It follows that $FX^{-t}rH \subseteq P$, hence $FX^{-t}rH = 0$ and we easily obtain $r = 0$. Consequently $h = X^{-t} f_0 g \in P$. The proof is complete.

REMARK 2.6 It is not difficult to show that if R is a ρ-s.prime ring, then every prime ideal P of $R\langle X;\rho\rangle$ with $P \cap R = 0$ is s.prime . But for our purposes all what we need is Lemma 2.5 because of the following.

LEMMA 2.7 Let R be a ρ-prime ring. Then the intersection of all the R-disjoint prime ideals P of $R\langle X;\rho\rangle$ which are maximal with respect to $P \cap R = 0$ is zero.

PROOF: If there is no non-zero R-disjoint ideal of $R\langle X;\rho\rangle$ the result is clear . The other case follows from ([3], Corollary 2.11).

As an immediate consequence of Lemmas 2.5 and 2.7 we have.

COROLLARY 2.8 If R is a ρ-s.prime ring, then $s(R\langle X;\rho\rangle) = 0$.

Now we are in position to prove the theorem.

PROOF OF THEOREM 2.1 Assume $R\langle X;\rho\rangle$ is an s- Jacobson ring and P is a ρ-prime ideal of R. Then $P\langle X;\rho\rangle$ is a prime ideal of $R\langle X;\rho\rangle$ and we have $P\langle X;\rho\rangle = \cap_{i\in\Omega} Q_i$, where $(Q_i)_{i\in\Omega}$ is a family of s.prime ideals of $R\langle X;\rho\rangle$. Thus, $P = \cap_{i\in\Omega}(Q_i \cap R)$ and $(Q_i \cap R)_{i\in\Omega}$ is a family of ρ-s.prime ideals of R

by Lemma 2.2. Therefore R is an s_ρ-Jacobson ring.

Conversely, assume that R is an s_ρ-Jacobson ring and let P be a prime ideal of $R\langle X;\rho\rangle$. Then $P \cap R$ is a ρ-prime ideal of R and by factoring out this ideal we may assume $P \cap R = 0$ and R is ρ-prime. By the assumption we have $0 = \cap_{i\in\Omega} Q_i$, where $(Q_i)_{i\in\Omega}$ is a family of ρ-s.prime ideals of R. Hence $\cap_{i\in\Omega} Q_i\langle X;\rho\rangle = 0$, where $Q_i\langle X;\rho\rangle$ is an intersection of s.prime ideals of $R\langle X;\rho\rangle$ by Corollary 2.8. This takes care of the case $P = 0$. Thus we may assume $P \neq 0$ and so P is maximal with respect to $P \cap R = 0$.

Since R is ρ-prime we have $\gamma(P) \neq 0$. Thus there exists a subfamily $(Q_j)_{j\in\theta}$ of $(Q_i)_{i\in\Omega}$ such that $\cap_{j\in\theta} Q_j = 0$ and $\gamma(P) \not\subseteq Q_j$ for every $j \in \theta$. By Lemma 2.3, $(P + Q_j\langle X;\rho\rangle) \cap R = Q_j$ and so there exists an ideal Q_j^* of $R\langle X;\rho\rangle$ which is maximal with respect to $Q_j^* \supseteq P$ and $Q_j^* \cap R = Q_j$, $j \in \theta$. Lemma 2.5 tells us that Q_j^* is an s.prime ideal of $R\langle X;\rho\rangle$ and since $(\cap_{j\in\theta} Q_j^*) \cap R = 0$ and $\cap_{j\in\theta} Q_j^* \supseteq P$ we obtain $\cap_{j\in\theta} Q_j^* = P$. The proof is complete.

Now we give some additional remarks. First we show that $R\langle X;\rho\rangle$ need not be an s-Jacobson ring when R is an s-Jacobson ring.

EXAMPLE 2.9 (c.f. [2], Example 2.4). Let K be a field, $X = (X_i)_{i\in\mathbf{Z}}$ a set of indeterminates and $A = K[X]$ the polynomial ring over K. Let ρ be the K- automorphism of A defined by $\rho(X_i) = X_{i+1}$ for all $i \in \mathbf{Z}$. For an integer $n \geq 2$ we put $R = A/P$, where P is the ideal of A generated by $\{X_i^n : i \in \mathbf{Z}\}$ and we denote by ρ again the automorphism induced by ρ. Then R is a local ring with the maximal ideal M generated by $\{X_i + P : i \in \mathbf{Z}\}$. Further, the ring R is ρ-prime ([2], Lemma 2.5). Thus, R is s-Jacobson and $R\langle X;\rho\rangle$ is not an s-Jacobson ring.

In fact, clearly R is an s-Jacobson ring. If there exists a ρ-invariant ideal Q of R such that R/Q is ρ-s.prime we easily obtain $Q = M$, because M/Q does not contain an insulator ([12], Corollary 2.2). It follows that R is not an s_ρ-Jacobson ring and so $R\langle X;\rho\rangle$ is not s-Jacobson, by Theorem 2.1.

The former example shows that it is natural to ask for additional conditions under which $R\langle X;\rho\rangle$ is an s-Jacobson ring when R is an s-Jacobson ring. To give these conditions we introduce a notation. Let I be an ideal (left ideal) of R. We denote by $\Gamma(I)$ the largest subideal of I which is ρ-invariant. We easily see $\Gamma(I) = \cap_{j\in\mathbf{Z}} \rho^j(I) = \{a \in R : \rho^j(a) \in I, \forall j \in \mathbf{Z}\}$.

Assume that $R\langle X;\rho\rangle$ is an s-Jacobson ring. Then R is an s_ρ-Jacobson ring and so if Q is a ρ-prime ideal of R we have $Q = \cap_{i\in\Omega} P_i$, where $(P_i)_{i\in\Omega}$ is a family of ρ-s.prime ideals. Further, for every $i \in \Omega$ there exists an s.prime ideal L_i of R such that $\Gamma(L_i) = P_i$ ([6], Lemma 1.2(ii)). Thus $Q = \cap_{i,j} \rho^j(L_i)$ is semiprime. Also, if P is an s-prime ideal of R it is clear that $s_\rho(R/\Gamma(P)) = 0$ since $\Gamma(P)$ is ρ-prime. Consequently the following conditions are necessary for

$R\langle X;\rho\rangle$ to be an s-Jacobson ring.

(C_1) For every ρ-prime ideal Q of $R, R/Q$ is semiprime.

(C_2) For every s.prime ideal P of R, $s_\rho(R/\Gamma(P)) = 0$.

COROLLARY 2.10 Let R be an s-Jacobson ring. Then $R\langle X;\rho\rangle$ is s-Jacobson if and only if (C_1) and (C_2) hold.

PROOF: Assume that (C_1) and (C_2) hold. If Q is a ρ-prime ideal of R, then Q is an intersection of prime ideals and so an intersection of s.prime ideals. Then $Q = \cap_{i\in\Omega}\Gamma(P_i)$, where $(P_i)_{i\in\Omega}$ is a family of s.prime ideals. Thus, condition (C_2) tells us that $s_\rho(R/Q) = 0$ and it follows that R is s_ρ-Jacobson. Theorem 2.1 completes the proof.

REMARK 2.11 Note that condition (C_1) holds if the ρ-prime radical of R/Q equals the prime radical, for every ρ-prime ideal Q of R. Similarly, condition (C_2) holds if the s_ρ-prime radical of $R/\Gamma(P)$ equals the s.prime radical, for every s.prime ideal P of R. Assumptions on R and ρ can be given in order to have the validity of these facts. Some results in this direction have been obtained in several papers ([2], [4], [6] and [14]). We mention here just one more example.

The automorphism ρ is said to be left locally integral on R if for every $a \in R$ the ρ-invariant left ideal $[a]$ of R generated by a is finitely generated [9]. In this case there exists an integer number $n > 0$ such that $[a]$ is generated by $\{\rho^{-n}(a), ..., a, ..., \rho^n(a)\}$. It is not difficult to show that if ρ is left locally integral, then for every ρ-prime ideal Q of R there exists a prime ideal L with $\Gamma(L) = Q$. Also, if P is an s.prime ideal of R, then $\Gamma(P)$ is ρ-s.prime. Therefore the conditions (C_1) and (C_2) hold when the automorphism ρ is left locally integral on R.

3 Skew polynomial rings.

The purpose of this section is to show that the method of the former section can be adapted to study the skew polynomial ring $R[X;\rho]$.

The definition of an s_ρ-Jacobson ring has to be slightly weaker than the one used in section 2. We say that (R,ρ) is an s'_ρ-Jacobson ring if every strongly ρ-prime ideal of R is an intersection of ρ-s.prime ideals. Clearly, if R is an s_ρ-Jacobson ring, then R is an s'_ρ-Jacobson ring.

We will prove the following.

THEOREM 3.1 Let R be a ring and ρ an automorphism of R. Then $R[X;\rho]$ is an s-Jacobson ring if and only if R is an s-Jacobson ring and an

s'_ρ-Jacobson ring.

As in the former section first we prove some lemmas.

LEMMA 3.2 If P is an s.prime ideal of $R[X;\rho]$ with $X \notin P$, then $P \cap R$ is a ρ-s.prime ideal of R.

PROOF: It is similar to the proof of Lemma 2.2.

The next lemma is the corresponding to Lemma 2.3.

LEMMA 3.3 Let P be an R-disjoint ρ-invariant ideal of $R[X;\rho]$ and let Q be a non-zero strongly ρ-prime ideal of R. If $\tau(P) \not\subseteq Q$, then $(P+Q[X;\rho]) \cap R = Q$.

PROOF: By the assumption there exists a polinomial $g = a_0 + Xa_1 + ... + X^m a_m \in P$ which is of minimal degree with respect to $a_0 \notin Q, a_1, ..., a_n \in Q$. Since $\tau(P) \not\subseteq Q$ there exists $f = b_0 + Xb_1 + ... + X^n b_n \in P$ of minimal degree in P with $b_n \notin Q$. Also, since Q is strongly ρ-prime there exists $j \geq 0$ and $r \in R$ such that $\rho^j(b_n)ra_0 \notin Q$. Therefore $h = \rho^j(b_n)rg - X^{m-n}\rho^{j+m}(f)\rho^m(r)a_m \in P$ which is a contradiction to the minimality of δg.

REMARK 3.4 The same result holds if Q is a maximal left ideal of R instead of a strongly ρ-prime ideal.

The corresponding to Condition (A) of [10] is not as simple as Lemma 2.5, in this case. The reason is that the analogous of Lemma 2.7 is no more true, in general. We state this as follows.

LEMMA 3.5 Let R be a ρ-s.prime ring and let P be an R-disjoint prime ideal of $R[X;\rho]$. Then $s(R[X;\rho]/P) = 0$.

PROOF: Assume that $X \notin P$ and $P \neq 0$. Then $L = \sum_{i \geq 0} X^{-i}P$ is an R-disjoint ideal of $R\langle X;\rho\rangle$ which is maximal with respect to $L \cap R = 0$ ([3], Corollary 3.3). Hence L is s.prime by Lemma 2.5 and we easily show that P is s.prime.

Since $s_\rho(R) = 0$ we have $s(R) = 0$ ([6], Lemma 1.2(iii)). Then the case $P = 0$ is covered by ([6], Theorem 3.8). In the remaining case $P = XR[X;\rho]$ and the result follows from $R[X;\rho]/P \simeq R$.

Now we prove the Theorem

PROOF OF THEOREM 3.1 Assume that $R[X;\rho]$ is an s-Jacobson ring. Then $R \simeq R[X;\rho]/XR[X;\rho]$ is also an s- Jacobson ring. Let P be an strongly ρ-prime ideal of R. By factoring out the ideal P we may assume $P = 0$ and R is a strongly ρ-prime ring. Hence $R[X;\rho]$ is prime and so $0 = \cap_{i\in\Omega} L_i$, where $(L_i)_{i\in\Omega}$ is a family of s.prime ideals of $R[X;\rho]$. Let $(L_j)_{j\in\theta}$ be the subfamily consisting of all the members of $(L_i)_{i\in\Omega}$ with $X \notin L_j, j \in \theta$. We easily obtain $\cap_{j\in\theta} L_j = 0$ and hence $\cap_{j\in\theta}(L_j \cap R) = 0$, where $(L_j \cap R)_{j\in\theta}$ is a family of ρ-s.prime ideals of R, by Lemma 3.2. Consequently, R is an s'_ρ-Jacobson ring.

Conversely, assume that R is an s-Jacobson and an s'_ρ-Jacobson ring and let P be a prime ideal of $R[X;\rho]$.

If $X \in P$, then $P\cap R$ is a prime ideal of R and we obtain $s(R[X;\rho]/P) = 0$, since $R[X;\rho]/P \simeq R/(P \cap R)$ and R is s-Jacobson. So we may assume that $X \notin P$ and, by factoring out $P\cap R$, R is strongly ρ-prime and $P\cap R = 0$. Then there exists a family $(Q_i)_{i\in\Omega}$ of ρ-s.prime ideals of R such that $\cap_{i\in\Omega} Q_i = 0$. It is easy to see that every ρ- s.prime ideal of R is strongly ρ-prime. This implies that $Q_i[X;\rho]$ is prime. By Lemma 3.5 we have $s(R[X;\rho]) \subseteq \cap_{i\in\Omega} Q_i[X;\rho] = 0$, which covers the case $P = 0$.

It remains to consider the case $P \neq 0$. Then P is maximal with respect to $P\cap R = 0$. Since $\gamma(P) \neq 0$ there exists a subfamily $(Q_j)_{j\in\theta}$ of $(Q_i)_{i\in\Omega}$ such that $\gamma(P) \nsubseteq Q_j$ (for $j \in \theta$) and $\cap_{j\in\theta} Q_j = 0$. Lemma 3.3 can be applied because P is ρ-invariant and Q_j is strongly ρ-prime. Thus, there exists an ideal Q_j^* of $R[X;\rho]$ such that $Q_j^* \supseteq P$ and it is maximal with respect to $Q_j^* \cap R = Q_j$. If $X \in Q_j^*$ for some $j \in \theta$, we easily obtain $\mu(P) \subseteq Q_j$, a contradiction. Therefore $X \notin Q_j^*$ for all $j \in \theta$ and it follows that Q_j^* is prime. By Lemma 3.5, $s(R[X;\rho]/Q_j^*) = 0$ and by the maximality of P we have $\cap_{j\in\theta} Q_j^* = P$. Thus $s(R[X;\rho]/P) = 0$ and the proof is complete.

The following corollary is an easy consequence of Corollary 2.10.

COROLLARY 3.6 Assume that R is an s-Jacobson ring and (C_1) and (C_2) hold. Then $R[X;\rho]$ is an s- Jacobson ring.

4 Generalization

.As we have pointed out in section 2 we can repeat the proof of Theorem 2.1 for other classes of prime ideals. In this way, every time we choose convenient classes of prime and ρ-prime rings such that some condition like condition (A) in [10] is satisfied, we obtain a theorem giving sufficient conditions for every prime ideal of $R\langle X;\rho\rangle$ to be an intersection of prime ideals of some special type. The treatment is similar to the one given in [8] for differential rings and can be summarized as follows.

Let $\mathcal{A}$ be a class of prime rings and let $\mathcal{B}$ be a class of ρ-prime rings (with automorphism). We say that an ideal (resp. ρ-invariant ideal) P of R is an $\mathcal{A}$-ideal (resp. $\mathcal{B}$-ideal) if $R/P \in \mathcal{A}$ (resp. $R/P \in \mathcal{B}$). We denote by $\mathcal{A}(R)$ (resp. $\mathcal{B}(R)$) the intersection of all the $\mathcal{A}$-ideals (resp. $\mathcal{B}$-ideals) of R. The ring R is said to be an $\mathcal{A}$-Jacobson ring if every prime ideal of R is an intersection of $\mathcal{A}$-ideals. A $\mathcal{B}$-Jacobson ring is defined similarly ([10], [8]). Examples of this classes of rings are Jacobson, s-Jacobson, ρ-Jacobson and s_ρ- Jacobson rings.

Now suppose that a class of prime rings $\mathcal{A}$ and a class of ρ-prime rings $\mathcal{B}$ are given. We say that $(\mathcal{A}, \mathcal{B})$ satisfies the condition (A) if the following holds

(A) If $(R, \rho) \in \mathcal{B}$, then $R\langle X; \rho\rangle/P \in \mathcal{A}$ for every ideal P of $R\langle X; \rho\rangle$ which is maximal with respect to $P \cap R = 0$.

By Lemma 2.5 the condition (A) is satisfied if $\mathcal{A}$ is the class of s.prime rings and $\mathcal{B}$ is the class of ρ- s.prime rings.

As an easy consequence of Lemma 2.7 we have

LEMMA 4.1 Assume that $\mathcal{A}$ and $\mathcal{B}$ are given and the condition (A) is satisfied. If $(R, \rho) \in \mathcal{B}$, then $\mathcal{A}(R\langle X; \rho\rangle) = 0$.

Thus, using the same arguments as in Theorem 2.1 we easily obtain

THEOREM 4.2 Suppose that classes of prime and ρ-prime rings $\mathcal{A}$ and $\mathcal{B}$ are given and the condition (A) is satisfied. If (R, ρ) is a $\mathcal{B}$-Jacobson ring, then $R\langle X; \rho\rangle$ is an $\mathcal{A}$-Jacobson ring.

The results of section 3 can also be generalized. Assume that $\mathcal{B}$ is a class of strongly ρ-prime rings. We say that a ring R (with the automorphism ρ) is a weakly $\mathcal{B}$- Jacobson (w.$\mathcal{B}$-Jacobson for shortness) ring if every strongly ρ-prime ideal of R is an intersection of $\mathcal{B}$-ideals.

We say that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the condition (B) if the following holds

(B) If $(R, \rho) \in \mathcal{B}$, then $\mathcal{A}(R[X; \rho]/P) = 0$ for every R-disjoint prime ideal P of $R[X; \rho]$.

By Lemma 3.5 the condition (B) is satisfied in the s.prime case.

Repeating the proof of Theorem 3.1 we have

THEOREM 4.3 Assume that classes of prime and strongly ρ-prime rings $\mathcal{A}$ and $\mathcal{B}$ are given and the condition (B) is satisfied. If (R, ρ) is an $\mathcal{A}$- Jacobson and a w.$\mathcal{B}$-Jacobson ring, then $R[X; \rho]$ is an $\mathcal{A}$-Jacobson ring.

It is also possible to give a generalization of Corollaries 2.10 and 3.6. For example, suppose that the pair $(\mathcal{A}, \mathcal{B})$ satisfies (A). We say that (R, ρ) satisfies (C) if the following conditions hold.

(C_1) For every ρ-prime ideal Q of R, R/Q is semiprime.

(C_2) For every $\mathcal{A}$-ideal P of R, $\mathcal{B}(R/\Gamma(P)) = 0$.

As in Corollary 2.10 we easily obtain

COROLLARY 4.5 Assume that classes of prime and ρ-prime rings $\mathcal{A}$ and $\mathcal{B}$ are given and the condition (A) is satisfied. If R is an $\mathcal{A}$-Jacobson ring and ρ is an automorphism of R such that the condition (C) holds, then $R\langle X;\rho\rangle$ is an $\mathcal{A}$-Jacobson ring.

5 Applications.

The results of the former section can be applied to several particular cases. First we recover the classical and well known case of Jacobson rings.

Recall that an ideal P of R is said to be (left) ρ- primitive if there exists a maximal ρ-invariant left ideal L of R such that $(L : R) = \{a \in R : aR \subseteq L\} = P$ [14]. The ideal $(L : R)$ is the largest ideal of R contained in L. Further, $(L : R)$ is ρ-invariant. The ring R is said to be a ρ-Jacobson ring if every ρ-prime ideal of R is an intersection of ρ-primitive ideals and is said to be a w.ρ-Jacobson ring if every strongly ρ- prime ideal of R is an intersection of ρ-primitive ideals.

Note that the definition of a ρ-Jacobson ring given in [14] coincides with the above definition of a w.ρ-Jacobson ring (a strongly ρ-prime ideal is called simple a ρ-prime ideal in [14]).

First we prove condition (A).

LEMMA 5.1 Assume that R is a ρ-primitive ring and P is an ideal of $R\langle X;\rho\rangle$ which is maximal with respect to $P \cap R = 0$. Then P is a primitive ideal.

PROOF: Let L be a maximal ρ-invariant left ideal of R with $(L : R) = 0$.

If $R\langle X;\rho\rangle$ has no non-zero R-disjoint ideal, then $P = 0$ and $L\langle X;\rho\rangle$ is a left ideal of $R\langle X;\rho\rangle$ with $L\langle X;\rho\rangle \cap R = L$. In the other case $P \neq 0$ and so $\gamma(P) \neq 0$. It follows that $\gamma(P) \not\subseteq L$ and so $(L\langle X;\rho\rangle + P) \cap R = L$, by Remark 2.4.

Thus, in both cases there exists a left ideal L^* of $R\langle X;\rho\rangle$ which is maximal with respect to $L^* \supseteq P$ and $L^* \cap R = L$. Hence L^* is a maximal left ideal of $R\langle X;\rho\rangle$ and we easily see that $(L^* : R\langle X;\rho\rangle) = P$. The proof is complete.

As a consequence of Lemma 5.1 and Theorem 4.2 we have

COROLLARY 5.2 If R is a ρ-Jacobson ring, then $R\langle X;\rho\rangle$ is a Jacobson ring.

REMARK 5.3 The above corollary can also be obtained using the arguments in [14]. Because of this it is well known.

Now we consider the skew polynomial ring case.

LEMMA 5.4 Let R be a ρ-primitive ring. Then the Jacobson radical $J(R[X;\rho]/P) = 0$ for every R-disjoint prime ideal P of $R[X;\rho]$.

PROOF: Let L be a maximal ρ-invariant left ideal of R with $(L:R) = 0$. First note that R is semiprimitive. In fact, let H be a maximal left ideal of R such that $\Gamma(H) = L$. Then $P = (H:R)$ is primitive and so $0 = \Gamma(P) = \cap_{i\in\mathbb{Z}}\rho^i(P)$ is semiprimitive. Thus, if $X \in P$ we have $J(R[X;\rho]/P) = J(R) = 0$.

So we may assume $X \notin P$. We consider two cases.

CASE I: There exists a non-zero R-disjoint ideal which does not contain X.

Suppose that $P \neq 0$. Then P is ρ-invariant and maximal with respect to $P \cap R = 0$. Also $\gamma(P) \not\subseteq H$ and by Remark 3.4 there exists a left ideal H^* of $R[X;\rho]$ containing P which is maximal with respect to $H^* \cap R = H$. So H^* is a maximal left ideal and the ideal $Q = (H^* : R[X;\rho])$ is primitive and contains P. If $X \in Q$ we easily obtain that $\mu(P) \subseteq Q \cap R \subseteq H$. Therefore $X \notin Q$ and it follows that Q is ρ-invariant. Then $Q \cap R = 0$ and we obtain $Q = P$. So P is primitive.

Since the intersection of all the non-zero prime ideals P with $X \notin P$ is zero in this case ([3], Corollary 3.4(ii)), we also have $J(R[X;\rho]) = 0$.

CASE II: There is no a non-zero R-disjoint ideal which does not contain X.

Since $X \notin P$ we have $P = 0$. Also, $J(R[X;\rho]) \cap R = 0$ since R is semiprimitive. So we have either $J(R[X;\rho]) = 0$ or $X \in J(R[X;\rho])$. The last possibility implies that there exists $f \in R[X;\rho]$ with $X + f + Xf = 0$, which is clearly a contradiction. The proof is complete.

It is easy to see that every ρ-primitive ideal is strongly ρ-prime. Since condition (B) holds we have the following corollary which is already contained in ([14], Proposition 4.15).

COROLLARY 5.5 If R is a Jacobson ring and a w.ρ- Jacobson ring, then $R[X;\rho]$ is a Jacobson ring.

REMARK 5.6 Corollary 5.5 can also be obtained using ([14], Theorem

4.4). In fact, if R is a Jacobson and a w.ρ- Jacobson ring, then it is not difficult to show that conditions [4.1] and [4.3] of [14] hold.

Finally, we discuss some other examples.

EXAMPLE 5.7 If R is a ρ-simple ring and P is an ideal of $R\langle X;\rho\rangle$ which is maximal with respect to $P \cap R = 0$, then P is a maximal ideal of $R\langle X;\rho\rangle$. So applying the results of section 4 to the classes of simple and ρ-simple rings we obtain that if R is a ρ-Brown-McCoy ring, then $R\langle X;\rho\rangle$ is a Brown-McCoy ring.

Now, the Brown-McCoy radical $G(R[X;\rho])$ of $R[X;\rho]$ is not necessarily zero when R is ρ-simple. For example, if ρ^k is not an inner automorphism of the right ρ- quotient ring Q of R for all $k \geq 1$, then the unique non- zero R-disjoint ideal of $R[X;\rho]$ is $XR[X;\rho] = G(R[X;\rho])$ ([3], Lemma 1.3 and Corollary 3.4). Therefore $R[X;\rho]$ is not necessarily a Brown-McCoy ring when R is a Brown-McCoy ring and a ρ-Brown-McCoy ring. This is the case, for example, if R is simple and no power of ρ is inner on R. Perhaps these remarks add some new light on why is so important the condition on ρ being a "power inner" automorphism of R in [15].

EXAMPLE 5.8 We say that R is Z-Jacobson (Z_ρ-Jacobson) if every prime (ρ-prime) ideal of R is an intersection of non-singular prime (ρ-prime) ideals. If R is a non-singular ρ-prime ring and P is an ideal of $R\langle X;\rho\rangle$ which is maximal with respect to $P \cap R = 0$, then P is a prime non-singular ideal ([3], Lemma 4.4). Consequently, if R is a Z_ρ-Jacobson ring, then $R\langle X;\rho\rangle$ is a Z-Jacobson ring. The converse is also true in this case.

Similarly, R is a Z-Jacobson ring and a w.Z_ρ-Jacobson ring if and only if $R[X;\rho]$ is a Z-Jacobson ring, where a w.Z_ρ-Jacobson ring is defined by the obvious way.

REMARK 5.9 The results of section 4 can also be applied to other classes of prime rings as prime nil (locally nilpotent) semisimple rings, prime G-rings and prime Noetherian rings. We leave the statements of these results to the reader.

References

[1]. A. D. Bell, "When are all prime ideals in an Ore extension Goldie?", Comm. Algebra 13 (8) (1985), 1743-1762.

[2]. R. Carbajo, E. Cisneros, M. I. González, "The prime radical of a skew group ring", Rev. U. Mat. Argent 32 (1985), 87-92.

[3]. E. Cisneros, M. Ferrero, M. I. González, "Prime ideals of skew polynomial rings and skew Laurent polynomial rings", Math. J. Okayama Univ. 32 (1990), 61-72.

[4]. M. Ferrero, "A note on liberal extensions with automorphisms", Rev. U. Mat. Argent. 32 (1986), 196-205.

[5]. M. Ferrero, "Radicals of skew polynomial rings and skew Laurent polynomial rings", Math. J. Okayama Univ. 29 (1987), 119-126.

[6]. M. Ferrero, "The strongly prime radical of an Ore extension", Comm. Algebra 17 (2) (1989), 351-376.

[7]. M. Ferrero, K. Kishimoto, "On differential rings and skew polynomials", Comm. Algebra 13 (2) (1985), 285-304.

[8]. M. Ferrero, J. Matczuk, "Prime ideals in skew polynomial rings of derivation type", Comm. Algebra 18 (3) (1990), 689-710.

[9]. M. Ferrero, A. Nowicki, "Locally integral derivations and automorphisms", Math. J. Okayama Univ, to appear.

[10]. M. Ferrero, M. M. Parmenter, "A note on Jacobson rings and polynomial rings", Proc. Amer. Math. Soc. 104, $n^{\underline{o}}$ 4 (1988), 281-286.

[11]. A. Goldie, G. Michler, "Ore extensions and polycyclic group rings", J. London Math. Soc. (2) 9 (1974), 337-345.

[12]. M. M. Parmenter, D. S. Passman, P. N. Stewart, "The strongly prime radical of crossed products", Comm. Algebra 12 (9) (1984), 1099-1113.

[13]. K. R. Pearson, W. Stephenson, "A skew polynomial ring over a Jacobson ring need not be a Jacobson ring", Comm. Algebra 5 (1977), 783-794.

[14]. K. R. Pearson, W. Stephenson, J. F. Watters, "Skew polynomials and Jacobson rings", Proc. London Math. Soc. (3) 42 (1981), 559-576.

[15]. K. R. Pearson, W. Stephenson and J. F. Watters, "Skew polynomials and Brown-McCoy rings", Comm. Algebra 10 (15) (1982), 1669-1681.

[16]. J. F. Watters, "Polynomial extensions of Jacobson rings", J. Algebra 36 (1975), 302-308.

Publicações do Instituto de Matemática da UFRGS
Cadernos de Matemática e Estatística

Série A: Trabalho de Pesquisa

1. Marcos Sebastiani - Transformation des Singularités - MAR/89.
2. Jaime Bruck Ripoll - On a Theoremof R. Langevin About Curvature and Complex Singularities - MAR/89.
3. Eduardo Cisneros, Miguel Ferrero e Maria Inés Gonzales - Prime Ideals of Skew Polynomial Rings and Skew Laurent Polynomial Rings - ABR/89.
4. Oclide José Dotto - ϵ - Dilations - JUN/89.
5. Jaime Bruck Ripoll - A Characterization of Helicoids - JUN/89.
6. Mark Thompson, V. B. Moscatelli - Asymptotic Distribution of Liusternik-Schnirelman Eigenvalues for Elliptic Nonlinear Operators -JUL/89.
7. Mark Thompson - The Formula of Weyl for Regions with a Self- Similar Fractal Boundary - JUL/89.
8. Jaime Bruck Ripoll - A Note on Compact Surfaces with Non Zero Constant Mean Curvature - OUT/89.
9. Jaime Bruck Ripoll - Compact ϵ - Convex Hypersurfaces - NOV/89.
10. Jandyra Maria G. Fachel - Coeficientes de Correlação Tipo -Contingência - JAN/90.
11. Jandyra Maria G. Fachel - The Probability of Ocurrence of Heywood Cases - JAN/90.
12. Jandyra Maria G. Fachel - Heywood Cases in Unrestricted Factor Analysis - JAN/90.
13. Julio Cesar R. Claeyssen, Tereza Tsukazan - Dynamical Solutions of Linear Matrix Differential Equations - JAN/90.
14. Maria T. Albanese - Behaviour of de Likelihood in latent Analysis of Binary Data - ABR/91
15. Maria T. Albanese - Measuremente of de Latent Variable in Latent Trait Analysis of Binary Data - ABR/91
16. M. T. Albanese - Adequacy of the Asymptotic Variance-Covariance Matrix using Bootstrap Jackknife Techniques in Latent Trait Analysis of Binary Data - ABR/91

17. Maria T. Albanese - Latent Variable Models for Binary Response - ABR/91

18. Mark Thompson - Kinematic Dynamo in Random Flows - DEZ/90.

19. Jaime Bruck Ripoll, Marcos Sebastiani - The Generalized Map and Applications - AGO/91

20. Jaime Bruck Ripoll, Suzana Fornari, Katia Frensel - Hypersurfaces with Constant Mean Curvature in the Complex Hyperbolic Space - AGO/91

21. Suzana Fornari, Jaime Bruck Ripoll - Stability of Compact Hypersurfaces with Constant Mean Curvature - JAN/92

22. Marcos Sebastiani - Une Généralisation de L'Invariant de Malgrange - FEV/92

23. Cornelis Kraaikamp, Artur Lopes - The Theta Group and the Continued Fraction with even partial quotients - MAR/92

24. Silvia Lopes - Amplitude Estimation in Multiple Frequency Spectrum - MAR/92

25. Silvia Lopes, Benjamin Kedem - Sinusoidal Frequency Modulated Spectrum Analysis - MAR/92

26. Silvia Lopes, Benjamin Kedem - Iteration of Mappings and Fixed Points in Mixed Spectrum Analysis - MAR/92

27. Miguel Ferrero, Eduardo Cisneros, María Inés Gonzáles - Ore Extensions and Jacobson Rings - MAI/92

28. Sara C. Carmona - An Asymptotic Problem for a Reaction-Diffusion Equation with fast Diffusion Component - JUN/92

Universidade Federal do Rio Grande Sul
Reitor: Professor Tuiskon Dick

Instituto de Matemática
Diretor: Professor Aron Taitelbaum
Núcleo de Atividades Extra Curriculares
Coordenador: Professor Jandyra M. G. Fachel
Secretária: Faraildes Beatriz da Silva

Os Cadernos de Matemática e Estatística publicam as seguintes séries:

Série A: Trabalho de Pesquisa
Série B: Trabalho de Apoio Didático
Série C: Colóquio de Matemática SBM/UFRGS
Série D: Trabalho de Graduação
Série E: Dissertações de Mestrado
Série F: Trabalho de Divulgação
Série G: Textos para Discussão

Toda correspondência com solicitação de números publicados e demais informações deverá ser enviada para:

NAEC - Núcleo de Atividades Extra Curriculares
Instituto de Matemática - UFRGS
Av. Bento Gonçalves, 9500
91.500 - Agronomia - POA/RS
Telefone: 36.98.22 ou 39.13.55 Ramal: 6176