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Vinicius Medeiros Gomes da Silveira

On CP violation and relativistic non-inertial effects

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Vinicius Medeiros Gomes da Silveira

On CP violation and relativistic non-inertial effects*

Sobre violação CP e efeitos não-inerciais relativísticos

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Supervisor: Dimiter Hadjimichef
Co-supervisor: César Augusto Zen Vasconcellos

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. . . just fields on fields.
—Carlo Rovelli, *Quantum Gravity*

Abstract

This dissertation investigates the impact that non-inertial effects predicted by quantum field theory in curved spacetimes have on the phenomenon of CP violation by analyzing the relative rate of CP-violating meson decays when the particles involved are accelerated. The results obtained agree with previous analyses, in that an increase in decay rates with increasing acceleration is predicted, and lead to the conclusion that the amplitude of CP violation in meson systems decreases very slightly with increasing acceleration. The problem of matter–antimatter asymmetry and its relationship with non-inertial effects and CP violation are briefly discussed.

Keywords: CP violation. Unruh effect. Matter–antimatter asymmetry. Quantum field theory in curved spacetimes.

Resumo

Esta dissertação investiga o impacto que efeitos não-inerciais previstos pela teoria quântica de campos em espaços-tempos curvos têm sobre o fenômeno da violação CP ao analisar a taxa relativa de decaimentos de mésons que violam a simetria CP quando as partículas envolvidas são aceleradas. Os resultados obtidos estão em acordo com análises anteriores, no sentido de que é previsto um aumento na taxa de decaimento com um aumento na aceleração, e levam à conclusão de que a amplitude de violação CP em sistemas de mésons diminui ligeiramente com um aumento na aceleração. O problema da assimetria matéria–antimatéria e a sua relação com efeitos não-inerciais e violação CP são brevemente discutidos.

Palavras-chave: Violação CP. Efeito Unruh. Assimetria matéria–antimatéria. Teoria quântica de campos em espaços-tempos curvos.

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List of Abbreviations and Initialisms

QFT	Quantum field theory
SM	Standard Model
QCD	Quantum chromodynamics
EW	Electroweak
GR	General relativity
KG	Klein–Gordon
CCR	Canonical commutation relations
CKM	Cabibbo–Kobayashi–Maskawa
FLRW	Friedmann–Lemaître–Robertson–Walker

Contents

Introduction

Quantum field theory (QFT) is the physical–mathematical formalism that grounds the Standard Model of particle physics (SM), composed of the Glashow–Salam–Weinberg model with the Higgs mechanism and quantum chromodynamics (QCD), which are, respectively, the best descriptions of the electroweak (EW) and strong interactions and of the kinds of matter that interact via these forces. QFT unifies the ideas at the core of classical field theory with quantum mechanics and special relativity, and its predictions have been tested with incredible accuracy (see the various reviews in [1]).

General relativity (GR) is the theory that embodies our best understanding of the gravitational interaction up to date. It subverts the very fundamental concepts of space and time to describe gravity as a field theory and, consequently, calls for profound changes in the ontologies espoused by most physical models. Its fantastic predictions have also been tested extensively, particularly in the last few years (see [2–4]).

The void created by these two frameworks lies at their interface. It seems reasonable to infer from QFT and GR that the fundamental entities of our Universe are fields, yet the gravitational field (as described by GR) is not a *quantum* field, nor are the electroweak and strong fields (as described by QFT) *generally covariant*[1] fields. While a complete solution to this incompatibility may be expected to arise from a theory of quantum gravity, it is possible to tackle important aspects of this problem with less radical techniques, given by the framework of *quantum field theory in curved spacetimes*. A semiclassical approach where the gravitational field remains classical and spacetime remains a background, QFT in curved spacetimes brings to light the effects of general covariance in QFTs, challenging the conventional concepts of particle and vacuum. Some of the most celebrated predictions of QFT in curved spacetimes are the *Hawking effect*—which predicts that black holes emit thermal radiation when they interact with the (inertial) vacuum of a QFT (see [5])—the phenomenon of particle creation from a vacuum state due to the expansion of the Universe (see [6]) and the *Unruh effect* (see [7–9]). The latter is one of the focal points of this text. Although a great deal of experimental evidence in favour of both QFT in flat spacetimes and GR is available, there is, essentially, no direct evidence for the existence of any of the effects predicted by QFT in curved spacetimes. If one is to trust these predictions based solely on the level of verification achieved by GR and QFT in flat spacetimes, a strong connection must be draw between these two frameworks and QFT in curved spacetimes, especially at the conceptual level. It is for this reason that great emphasis is put on the development of the basic tools of the formalism in what follows.

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[1] The meaning of general covariance is clarified in section 2.1.2.

The existence of non-inertial effects in QFT in curved spacetime becomes apparent with the Unruh effect. It predicts that accelerated observers do not agree with inertial observers when it comes to the particle content of spacetime[2]: if an inertial observer describes a quantum field as being in its vacuum state, an accelerated observer describes it as being in a thermal state. Besides undermining the, often thought as fundamental, notions of particle and vacuum, the Unruh effect leads to some very interesting conclusions concerning particle decay. Investigations concerning the decay rate of accelerating particles presented in [10, 11] conclude the that decay rate increases with increasing acceleration, even allowing for inertially forbidden transitions like protons decaying into neutrons.

An effect arising in a QFT in Minkowski spacetime is the violation of the CP symmetry by the weak interaction. It was first observed in the *Fitch–Cronin* experiment (see [12]) on kaon decays and one of its most important consequences was pointed out in a theoretical analysis presented in [13]: it is a necessary condition for a universe with differing amounts of matter and antimatter. Theoretical models that offer explanations for the quantum mechanism responsible for the existence of CP violation and for the cosmological processes leading to the observed *matter–antimatter asymmetry* are well established, but, when combined, lead to the conclusion that the amount of CP violation that is observed cannot account for the current asymmetry. It is for this reason that the impact of non-inertial effects on CP violation are investigated in this dissertation.

The system of neutral kaons is one of the better understood sources of CP violation and is the focus of the studies presented here. Two quantities are crucial in the analysis of CP violation in this system: the decay rate Γ for the process $K_\mathrm{L}^0 \to \pi\pi$, where a species of neutral kaon decays into two pions[3], and the ratio between this rate and the decay rate for the process $K_\mathrm{S}^0 \to \pi\pi$, called η. Employing an adaptation of the model for the decay of accelerating particles mentioned above, the oscillations of these two quantities with respect to the value of the acceleration is predicted. Furthermore, the possible impact of a change in the rate of CP violation on the matter–antimatter asymmetry is analyzed with the use of an analogy between the Unruh effect and the thermal state of the early Universe.

Each chapter of this dissertation covers a facet of the problem of describing the influence of non-inertial effects on CP violation. Chapter 1 covers matters of notation and convention in section 1.1 and expounds upon some of the mathematical ideas at the core of GR and QFT in section 1.2. The basics of these two theories are introduced in chapter 2, with GR being discussed in section 2.1 and QFT in section 2.2. The formalism of QFT in curved spacetimes is presented in subsection 2.2.3 and the Unruh effect is described

[2] The spacetime in question need not be curved. In fact, the Unruh effect was originally derived for QFTs in Minkowski spacetime (see subsection 2.2.4).

[3] Although there are differences in the processes whether the final state is composed of charged or neutral pions, they are essentially irrelevant for the purposes of the investigation conducted below (see subsection 3.2.2)

and derived in subsection 2.2.4. Chapter 3 tackles CP violation and its relation to the predictions of QFT in curved spacetimes. Section 3.1 contains a basic discussion on the fundamentals of CP violation and section 3.2 connects this phenomenon with the Unruh effect. The model for the decay of accelerated particles is presented in subsection 3.2.1 and its predictions are analyzed in subsection 3.2.2. Subsection 3.2.3 is composed of an investigation on the relationship between the Unruh effect, CP violation and baryon production as a model for explaining the matter–antimatter asymmetry.

1 Mathematical Preliminaries

Both GR and QFT have their conventions and notational quirks, along with complex underlying mathematical formalisms. QFT in curved spacetimes inherits these characteristics. The purpose of this chapter is to establish the adopted conventions, clarify notation and introduce the essential mathematical ideas that appear in this text.

1.1 Conventions and Notation

The conventions and notation used mostly follow those used in treatments of GR. The convention concerning the sign of the metric of spacetime in particular leads to a series of differences in definitions according to whether the “relativists’ convention” or the “particle physicists’ convention” is adopted.

1.1.1 Conventions

In what follows, natural units, $c = \hbar = 8\pi G = k_\mathrm{B} = 1$, are used (except where explicitly stated otherwise). Here c is the speed of light[4], $\hbar$ is the (reduced) Planck constant, G is the gravitational constant and k_B is the Boltzmann constant.

One of the fundamental mathematical structures of GR is a Lorentzian metric over a manifold. Since this metric is indefinite, there is no preferred choice for its sign. Here the spacelike convention, $(-, +, +, +)$, is adopted.

Whenever an inner product $\langle \cdot \,|\, \cdot \rangle$ on a complex vector space $\mathcal{H}$ is present, it is linear in the second variable in order to make it compatible with bra-ket notation,

$$\langle \psi' | \psi \rangle = \langle \psi' | \left(| \psi \rangle \right), \tag{1.1}$$

with $|\psi\rangle \in \mathcal{H}$ and $\langle \psi'| \in \mathcal{H}^*$, where $\mathcal{H}^*$ is the dual space to $\mathcal{H}$. The Riesz–Fréchet lemma is the key result surrounding the bra-ket notation and this convention, since it gives an isomorphism between the dual space $\mathcal{H}^*$ and the complex conjugate space $\overline{\mathcal{H}}$ (see, e.g., [14]).

The Fourier transform $\hat{f}$ of a function $f\colon \mathbb{R} \to \mathbb{C}$ is defined in terms of the angular frequency and normalized so as to be a unitary operator:

$$\hat{f}(\omega) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t\, f(t) e^{i\omega t}. \tag{1.2}$$

[4] Or the Lorentz group parameter.

1.1.2 Notation

Tensors are extensively used in this work, so a brief exposition with respect to the notation used in dealing with them is in order. A tensor is an element of the tensor products of vector spaces or, as implied by the fundamental property of tensor product spaces, a multilinear map between vector spaces. Abstract index notation is used in this text and a detailed treatment of it can be found in [15]. It differs from standard index notation in that the symbol $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell}$ does not denote the components of a tensor of type (k, ℓ), but the tensor itself. Therefore, the expression

$$T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} \in \bigotimes_{i=1}^{k} V \bigotimes_{j=1}^{\ell} V^*, \tag{1.3}$$

where V is a vector space, is well defined. Whenever tensor components must be addressed they are denoted by underlined indices, so that $v^{\underline{b}}$ is the b-th component of v^a.

Index contraction, also called the *interior product*, is given by the evaluation map on the space corresponding to the pair of indices to be contracted, and the summation convention of component notation is mirrored in the abstract notation:

$$T^{ab}{}_b \coloneqq \sum_{\underline{b}} T^{a\underline{b}}{}_{\underline{b}}. \tag{1.4}$$

Tensor products are denoted by concatenation, so that

$$T^{ab} S^c{}_d \coloneqq T^{ab} \otimes S^c{}_d. \tag{1.5}$$

Raising and lowering of indices is done using the musical isomorphisms induced by a nondegenerate bilinear form $G\colon V \times V \to \mathbb{F}$ and its inverse. Let g_{ab} be the tensor associated to the bilinear form, i.e.[5], $G(v, w) = g_{ab} v^a w^b$. Then the musical isomorphisms are given by

$$\flat\colon V \to V^* \quad , \quad v^a \mapsto v_a = g_{ab} v^b; \tag{1.6a}$$

$$\sharp\colon V^* \to V \quad , \quad v_a \mapsto v^a = g^{ab} v_b, \tag{1.6b}$$

where g^{ab} is the inverse of g_{ab}, i.e., $g^{ac} g_{cb} = \delta^a{}_b$.

The symmetrization and antisymmetrization in the indices $\{a_1, \ldots, a_k\}$ of a tensor are denoted, respectively, by

$$T_{(a_1 \cdots a_k)} \coloneqq \frac{1}{k!} \sum_{\pi \in S_k} T_{\pi(a_1) \cdots \pi(a_k)}\,, \tag{1.7a}$$

$$T_{[a_1 \cdots a_k]} \coloneqq \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sgn}(\pi) T_{\pi(a_1) \cdots \pi(a_k)}\,, \tag{1.7b}$$

[5] When a tensor is a variable of a function, it is conventional to leave out the indices to avoid confusion.

where S_k is the symmetric group of order k. A tensor is said to be symmetric in the indices $\{a_1, \ldots, a_k\}$ if $T_{(a_1 \cdots a_k)} = T_{a_1 \cdots a_k}$ and antisymmetric in the same indices if $T_{[a_1 \cdots a_k]} = T_{a_1 \cdots a_k}$. The symmetrization and antisymmetrization of the tensor product (note the lack of normalization factors) are called, respectively, the *symmetric tensor product* and the *exterior product* and denoted as

$$v_a \otimes_{\mathrm{s}} w_b \coloneqq v_a \otimes w_b + w_b \otimes v_a, \tag{1.8a}$$

$$v_a \wedge w_b \coloneqq v_a \otimes w_b - w_b \otimes v_a. \tag{1.8b}$$

A totally antisymmetric (antisymmetric in all indices) type $(0, k)$ tensor is called a *k-form*, while a totally antisymmetric type $(k, 0)$ tensor is called a *k-vector*. The antisymmetrization of the derivative of a k-form field $\omega_{a_1 \cdots a_k}$, called a *differential k-form*, is called the *exterior derivative* and denoted, in the notation of exterior calculus (which suppresses indices), by

$$\mathrm{d}\omega = (k+1)\, \partial_{[a_0} \omega_{a_1 \cdots a_k]}. \tag{1.9}$$

It satisfies the graded Leibniz identity with respect to the exterior product,

$$\mathrm{d}(\eta \wedge \omega) = \mathrm{d}\eta \wedge \omega + (-1)^k \eta \wedge \mathrm{d}\omega, \tag{1.10}$$

where η is a 1-form.

When dealing with tensor fields on a manifold, lowercase Greek letters, e.g., μ, ν, ρ, are used for the indices. In the special case where the manifold is Minkowski spacetime the indices are denoted by uppercase Latin letters, e.g., I, J, K. Submanifold indices are denoted by lowercase Latin letters from the middle of the alphabet, e.g., i, j, k, while indices indicating internal spaces, e.g., Fock Space, are denoted by lowercase Latin letters from the beginning of the alphabet, e.g., a, b, c.

1.2 Geometrical Tools

Geometry has played a crucial role in GR, from its inception to modern developments, but it also turns out to be a very useful tool in the development of certain QFT techniques. This section aims to introduce the ideas used in this development.

1.2.1 Fiber Bundles

A key concept in modern differential geometry is that of a fiber bundle. It generalizes the idea of a product space and clarifies the definitions of various common geometric objects. A series of definitions are given in what follows, but more comprehensive and detailed introductions to differential geometry can be found in [16, 17]. A more in-depth treatment of the theory of fiber bundles is presented in [18].

Definition 1.1 (Differentiable manifold)**.** An n-dimensional *differentiable manifold* M is a Hausdorff topological space[6] equipped with a *differential structure*, i.e., an equivalence relation of *atlases*, which are, in turn, families of compatible differentiable *charts*—maps to[7] $\mathbb{R}^n$—which cover M.

Definition 1.2 (Fiber bundle)**.** A *fiber bundle* is a short exact sequence $F \hookrightarrow E \xrightarrow{\pi} M$ of manifolds where E is called the *total space*, π is called the *projection*, M is called the *base space* and F is called the *standard fiber*, subject to a local triviality condition, i.e., having *bundle charts*—maps to $M \times F$—that cover E and commute with the natural projection of $M \times F$.

Definition 1.3 (Principal fiber bundle)**.** A *principal fiber bundle* is a pair (P, G) consisting of a manifold P, called the *total space*, a Lie group G, called the *structure group*, and a right action of G on P such that:

1. G acts freely on P, i.e., if $g \in G$ and there is $q \in P$ such that $qg = q$ then $g = e$.
2. The canonical projection $\pi \colon P \to M$ to the quotient space of P by G, $M = P/G$, is differentiable.
3. E is locally trivial, i.e., for every $p \in M$ there is a pair (U, ψ) such that U is a neighborhood of p, $\psi \colon \pi^{-1}(U) \to M \times G$ and $\psi(q) = (\pi(q), \varphi(q))$, with $\varphi \colon \pi^{-1}(U) \to G$ satisfying the condition that, for every $q \in \pi^{-1}(U)$ and $g \in G$, $\varphi(qg) = \varphi(q)g$.

It can be shown (under the assumption that the action is properly discontinuous) that the quotient space M, called the *base space*, is a manifold (see [18]) and that $G \hookrightarrow P \xrightarrow{\pi} M$ is a fiber bundle. The definition of principal bundles given above does not, *a priori*, depend on a choice of base space. This allows for the interpretation that the total space is a fundamental geometric entity while the base space is merely a consequence of the action of G on P. The physical implications of this statement are elaborated in section 2.1.3 and in [19].

The preimage $\pi^{-1}(p) = \{pg : g \in G\}$ of a point $p \in M$ via the projection π is called the *fiber at* p. Every fiber of a principal bundle is diffeomorphic to G. Given a group G, a manifold for the base space and a set of bundle charts, it is possible to construct a principal bundle (see [16, 18]).

Example 1.1 (Frame bundle)**.** Let M be manifold and FM the disjoint union of all frames (ordered bases of the tangent space at a point) at all points of M. The general linear group $\mathrm{GL}(n, \mathbb{R})$ acts on FM freely via the change of basis. In fact, $\mathrm{GL}(n, \mathbb{R})$ is

6 A topological space consists of a set M with a topology τ. It is customary to call M the manifold rather than the triple (M, τ, D), where D is a differential structure.

7 All definitions given in the chapter are for structures over the field of real numbers. Suitable modifications can be made to contemplate other fields.

isomorphic to the set of all frames at a point—the fiber—and FM can be made into a principal bundle[8], called the *frame bundle*.

Another important kind of fiber bundle is the bundle, with a certain standard fiber, associated to a principal bundle.

Definition 1.4 (Associated bundle)**.** Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal bundle with structure group G and F a manifold on which G acts on the left. Denote by $E = P \times_G F$ the G-product of P and F, i.e., $P \times_G F = (P \times F)/\sim$, where $(q, f)g \coloneqq (qg, g^{-1}f)$ and $(q, f)g \sim (q, f)$. Then the fiber bundle $F \hookrightarrow E \xrightarrow{\pi_E} M$, where $\pi_E([q, f]) = \pi(q)$ and $[q, f]$ is the equivalence class of (q, f), is called the *fiber bundle associated* to P with standard fiber F.

Associated bundles are a way of obtaining a fiber bundle E with a certain standard fiber F on which the action of the principal bundle P is well defined. They play a crucial role in gauge theories (see, e.g., [16, 20]) and can be used to construct very natural geometric objects. A very special case is when F is a vector space.

Definition 1.5 (Vector bundle)**.** Let ρ be a representation of a group G into $\mathrm{GL}(n, \mathbb{R})$ and $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi_E} M$ be the bundle associated to $G \hookrightarrow P \xrightarrow{\pi} M$ with fiber $\mathbb{R}^n$ on which G acts via ρ. The bundle $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi_E} M$ is called a *vector bundle*.

Example 1.2 (Tangent and cotangent bundles)**.** The vector bundle associated to FM with standard fiber $\mathbb{R}^n$ is called the *tangent bundle* TM, and the fiber at $p \in M$ of TM is the tangent space T_pM at p. The *dual bundle*[9] to TM is the cotangent bundle T^*M whose fibers are the cotangent spaces T_p^*M at points p.

Example 1.3 (Tensor product bundle)**.** The group $\mathrm{GL}(n, \mathbb{R})$ has a natural action on the tensor product space of tensors of type (k, ℓ) acting on $\mathbb{R}^n$. The tensor bundle of type (k, ℓ) of TM and T^*M is the bundle associated to FM whose fiber at a point p is the space of tensor products of tangent and cotangent spaces T_pM and T_p^*M.

The constructions above are particularly useful in the study of tensor fields. It is of interest to see these objects as sections of bundles, a concept introduced below.

Definition 1.6 (Cross section)**.** A *cross section* of a fiber bundle E is a continuous map $s\colon U \to E$, where $U \subset M$ is open, such that $\pi \circ s = \iota\colon U \hookrightarrow M$, the natural embedding of U in M. If $U \neq M$, the section is said to be *local*, while if $U = M$ the section is said to be *global*[10]. The space of sections of a fiber bundle is denoted $\Gamma(E)$.

8 Similarly to the case of manifolds (see footnote 6), one may refer to the total space of a principal bundle as the bundle.

9 Vector bundles allow for constructions such as dual bundles and tensor bundles, obtained by taking dual and tensor spaces as fibers.

10 If a principal bundle E has a global section, it is trivial, i.e., $E = M \times F$. If $E = TM$ is trivial, M is said to be *parallelizable*.

Example 1.4 (Tensor fields)**.** A section of the tensor tensor bundle of type (k, ℓ) over M is a tensor field on M. Of particular interest are vector fields, i.e., sections of the tangent bundle TM, and differential forms, i.e., sections of the cotangent bundle T^*M.

Example 1.5 (Gauge transformations)**.** Let P be a principal bundle with structure group G. Sections of P may be called *gauge transformations*, since they may be seen as local transformations of some associated bundle.

When a fiber bundle has some kind of extra structure it might be possible to reduce its structure group to elements that preserve this structure.

Definition 1.7 (Reduction of the structure group)**.** If the principal bundle $G' \hookrightarrow P' \xrightarrow{\pi'} M'$ is a subbundle of the principal bundle $G \hookrightarrow P \xrightarrow{\pi} M$, i.e., there is a principal bundle morphism (ψ, ρ, φ) satisfying the following commutative diagram,

$$\begin{tikzcd} G' \arrow[r, hook, "\rho"] \arrow[d, hook] & G \arrow[d, hook] \\ P' \arrow[r, hook, "\psi"] \arrow[d, two heads, "\pi'"] & P \arrow[d, two heads, "\pi"'] \\ M' \arrow[r, hook, "\varphi"] & M \end{tikzcd}, \tag{1.11}$$

$M' = M$ and $\varphi = \mathrm{id}$, it is said that the structure group G can be *reduced* to the subgroup G' and that $G' \hookrightarrow P' \xrightarrow{\pi'} M$ is the *reduced bundle.*

Bilinear forms are common structures in geometry. Whenever one is present, the study of transformations that preserve this structure is in order.

Example 1.6 (Reduction to the orthogonal group)**.** Let $\mathrm{GL}(n, \mathbb{R}) \hookrightarrow FM \xrightarrow{\pi} M$ be the frame bundle of M and let M be equipped with a nondegenerate symmetric bilinear form—a section of the bundle of symmetric tensors of type $(0, 2)$—of signature (p, q). The group of transformations that preserve this bilinear form (in every tangent space) is $\mathrm{O}(p, q)$, the orthogonal group of signature (p, q). The reduction of $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(p, q)$ results in the *bundle of orthogonal frames* or simply the *orthogonal bundle* $\mathrm{O}(p, q) \hookrightarrow OM \xrightarrow{\pi} M$. A further reduction from $\mathrm{O}(p, q)$ to the special orthogonal group, $\mathrm{SO}(p, q)$, allows for the preservation of an orientation.

Example 1.7 (Triads and tetrads)**.** A common tool in GR are *triads* and *tetrads*, i.e., orthonormal frame fields in 3 and 4-dimensional manifolds, respectively. These fields map the metric on a manifold M to the Euclidean and Minkowski metrics and are the mathematical counterparts to the physical statements that space is locally Euclidean and spacetime is locally Minkowskian. As orthonormal frame fields, they are local sections of the orthogonal bundle OM.

1.2.2 Connections

When dealing with sections of a fiber bundle, e.g., vector fields, it is desirable to be able to compare their values at different points of the base manifold. No natural definition of a derivative operator on a fiber bundle is suitable for all purposes, so the whole class of objects that can play this role, called connections, needs to be studied.

Definition 1.8 (Koszul connection)**.** Let $V \hookrightarrow E \overset{\pi}{\twoheadrightarrow} M$ be a vector bundle. A *Koszul connection* or *linear connection* on E is a linear map $\mathrm{D}\colon \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M)$ satisfying the Leibniz identity for all $s^a \in \Gamma(E)$ and[11] $f \in C^\infty(M)$:

$$\mathrm{D}_\mu(f s^a) = \big(\partial_\mu f\big) s^a + f\,\mathrm{D}_\mu s^a. \tag{1.12}$$

The object $\mathrm{D}_\mu s^a$ is called the *covariant derivative* of s^a.

Other kinds of connections exist, such as the Ehresmann connection on a principal bundle (see [18]), which induces a connection on associated bundles, but the Koszul connection suffices for the purposes of this text. A very useful definition, especially for computations, is that of the connection form.

Definition 1.9 (Connection form)**.** Let D be a connection on the bundle E over M. Its *connection form* $A_\mu{}^a{}_b$ is given by the difference between the covariant derivative and a preferential connection, the *flat connection*, given by the derivative operator on M:

$$\big(\mathrm{D}_\mu - \partial_\mu\big) s^a = A_\mu{}^a{}_b s^b. \tag{1.13}$$

The definition above rests on the fact that the difference of two connections $\mathrm{D}' - \mathrm{D}$ is C^∞-linear (see [15]), i.e.,

$$\big(\mathrm{D}'_\mu - \mathrm{D}_\mu\big)(f s^a) = f\big(\mathrm{D}'_\mu - \mathrm{D}_\mu\big) s^a. \tag{1.14}$$

The connection form allows for the computation of the curvature F by the usual expression:

$$F^a{}_{b\mu\nu} = 2\,\partial_{[\mu} A_{\nu]}{}^a{}_b + 2 A_{[\mu}{}^a{}_{|c|} A_{\nu]}{}^c{}_b, \tag{1.15}$$

where the vertical bars indicate that the antisymmetrization does not affect the index c (which should be clear). In terms of the exterior derivative, one has

$$F^a{}_b = \mathrm{d}A^a{}_b + A^a{}_c \wedge A^c{}_b. \tag{1.16}$$

Connections and curvatures play very special roles in gauge theory and GR, as illustrated by the following examples.

[11] The definition can be easily adapted for objects in other differentiability classes.

Example 1.8 (Yang–Mills theory)**.** One the most successful theories of interactions in physics is the *Yang–Mills theory*, describing all the interactions that are part of the SM. Let $V \hookrightarrow E \xrightarrow{\pi}\!\!\!\!\rightarrow M$ be the vector bundle associated to the principal bundle $G \hookrightarrow P \xrightarrow{\pi}\!\!\!\!\rightarrow M$. Then the connection form $igA_\mu{}^a{}_b$ of a certain connection on E can be written in terms of the *gauge potential* $A_\mu{}^a{}_b$, which takes values in the representation of the lie algebra $\mathfrak{g}$ of G associated to the representation with which G acts on V. The curvature is called the *field strength*. The (free) theory has its dynamics given by the action

$$S_{\mathrm{YM}} = \frac{1}{2}\int_M F^a{}_b \wedge \star F^b{}_a\,, \tag{1.17}$$

where $\star$ denotes the Hodge dual, and its dynamic equation is

$$\mathrm{D}\star F = 0, \tag{1.18}$$

where D denotes the exterior covariant derivative.

Example 1.9 (Levi-Civita connection)**.** Let $\mathbb{R}^n \hookrightarrow TM \xrightarrow{\pi}\!\!\!\!\rightarrow M$ be the tangent bundle of M and $g_{\mu\nu}$ a metric on M. There is a unique connection ∇, called the *Levi-Civita connection*, that preservers this metric, i.e.,

$$\nabla_\rho g_{\mu\nu} = 0, \tag{1.19}$$

and is torsion-free, i.e., for every $f \in C^\infty(M)$,

$$\nabla_{[\mu}\nabla_{\nu]} f = 0. \tag{1.20}$$

The Levi-Civita connection is commonly taken as the connection on the tangent bundle of spacetime in GR.

1.2.3 Symplectic Geometry

Classical mechanics, notwithstanding the developments of physics in the 20th century, remains a very important formalism, both as a description of the physical world in everyday scales and as the limit of theoretical models under some regime. One of its many reformulations, based on symplectic geometry, has applications to QFT methods. A study of the geometric structure underlying this formalism makes itself necessary. A great introduction to symplectic geometry and its role in physics is [21]. Another reference that puts more emphasis on geometrical aspects is [22].

Definition 1.10 (Symplectic manifold)**.** A $2n$-dimensional manifold is called a *symplectic manifold* if it is equipped with a *symplectic form* $\omega_{\mu\nu}$, i.e., a nondegenerate closed 2-form. A 2-form $\omega_{\mu\nu}$ is said to be *closed* if

$$\mathrm{d}\omega = 3\,\partial_{[\rho}\omega_{\mu\nu]} = 0. \tag{1.21}$$

The requirement that the manifold be $2n$-dimensional does not imply a loss of generality, since every 2-form in an odd dimensional manifold is degenerate. A symplectic manifold that arises quite naturally is the cotangent bundle of a manifold.

Example 1.10 (Cotangent bundle as a symplectic manifold)**.** Let $\mathcal{C}$ be a manifold and $\Omega = T^*\mathcal{C}$ its cotangent bundle. There is a special 1-form $\theta \in \Gamma(T^*\Omega)$, called the *canonical 1-form*, defined pointwise in terms of $\theta_s := \pi_* s = s \circ \mathrm{d}\pi$, where $\mathrm{d}\pi \colon T(T^*\mathcal{C}) = T\Omega \to T\mathcal{C}$ is the derivative of the projection $\pi \colon T^*\mathcal{C} = \Omega \to \mathcal{C}$ and $s \in \Omega$ is a 1-form acting in $T_{\pi(s)}\mathcal{C}$. The canonical 1-form is given by $\theta(s) = \theta_s$. To get a symplectic structure in Ω, define the symplectic form ω as

$$\omega = \mathrm{d}\theta. \tag{1.22}$$

It is clear that this 2-form is closed, since

$$\mathrm{d}\omega = \mathrm{d}^2\theta = 6\,\partial_{[\rho}\,\partial_{\mu}\theta_{\nu]} = 0, \tag{1.23}$$

a general property of the exterior derivative stemming from the commutativity of second partial derivatives.

In any $2n$-dimensional symplectic manifold there is a special set of coordinates $\left(q^\mu, p_\mu\right)$, called *Darboux coordinates* or *canonical coordinates*, that diagonalizes the blocks of ω. In this set of coordinates,

$$\theta = -\sum_{\underline{\mu}=1}^{n} p_{\underline{\mu}}\,\mathrm{d}q^{\underline{\mu}}, \tag{1.24}$$

$$\omega = \sum_{\underline{\mu}=1}^{n} \mathrm{d}q^{\underline{\mu}} \wedge \mathrm{d}p_{\underline{\mu}}. \tag{1.25}$$

Classical physical systems are often dealt with using coordinates of this kind, with q^μ coordinatizing $\mathcal{C}$, the configuration space of a system, and $\left(q^\mu, p_\mu\right)$ coordinatizing $\Omega = T^*\mathcal{C}$, the phase space of the system.

The symplectic form, being nondegenerate, induces a set of musical isomorphisms as in equations (1.6). A special notation is introduced for the following mapping:

$$\left(\partial_\mu f\right)^\sharp = \omega^{-1}(\mathrm{d}f, \cdot) = \omega^{\mu\nu}\partial_\nu f =: X_f^\mu. \tag{1.26}$$

The vector field X_f^μ is called the *Hamiltonian vector field* of f. The *Lie derivative* in the direction of X_f^μ of the symplectic form, $\mathcal{L}_{X_f}\omega_{\mu\nu}$, is zero:

$$\begin{aligned}
\mathcal{L}_{X_f}\omega_{\mu\nu} &= X_f^\rho\,\partial_\rho\omega_{\mu\nu} + \omega_{\rho\nu}\,\partial_\mu X_f^\rho + \omega_{\mu\rho}\,\partial_\nu X_f^\rho \\
&= \omega^{\rho\sigma}\partial_\sigma f\,\partial_\rho\omega_{\mu\nu} + \partial_\mu\left(\omega_{\rho\nu}\omega^{\rho\sigma}\right)\partial_\sigma f - \omega^{\rho\sigma}\,\partial_\mu\omega_{\rho\nu}\partial_\sigma f - \partial_\mu\partial_\nu f \\
&\qquad + \partial_\nu\left(\omega_{\mu\rho}\omega^{\rho\sigma}\right)\partial_\sigma f - \omega^{\rho\sigma}\,\partial_\nu\omega_{\mu\rho}\partial_\sigma f + \partial_\nu\partial_\mu f \\
&= \omega^{\rho\sigma}\left(\partial_\rho\omega_{\mu\nu} - \partial_\mu\omega_{\rho\nu} - \partial_\nu\omega_{\mu\rho}\right)\partial_\sigma f = 3\omega^{\rho\sigma}\,\partial_{[\rho}\omega_{\mu\nu]}\partial_\sigma f = 0.
\end{aligned} \tag{1.27}$$

This means that the symplectic form is invariant with respect to the flows generated by Hamiltonian vector fields. Transformations that preserve the symplectic form are called *symplectomorphisms*. That the flow generated by the Hamiltonian vector field of a function H leaves H invariant (see [21]) is a consequence of this fact and corresponds to the physical laws of conservation of energy and momentum.

A very important object in symplectic geometry is the *Poisson bracket*, which establishes a homomorphism of the Lie algebra of functions on Ω onto the Lie algebra of Hamiltonian vector fields on Ω.

Definition 1.11 (Poisson bracket)**.** Let Ω be a symplectic manifold and $f, g \in C^\infty(\Omega)$. The *Poisson bracket* $\{f\,, g\}$ of f and g is defined by

$$\{f\,,g\} \coloneqq \omega(X_f, X_g) = \omega_{\mu\nu} X_g^\mu X_f^\nu. \tag{1.28}$$

The following relations are readily verified:

$$\begin{aligned}\{f\,,g\} &= \omega_{\mu\nu} X_g^\mu X_f^\nu = \omega_{\mu\nu}\omega^{\mu\rho}\partial_\rho g\,\omega^{\nu\sigma}\partial_\sigma f = \delta_\mu{}^\sigma \omega^{\mu\rho}\partial_\rho g\partial_\sigma f = \omega^{\sigma\rho}\partial_\rho g\partial_\sigma f \\ &= X_g^\sigma\partial_\sigma f = -X_f^\rho\partial_\rho g = \mathcal{L}_{X_g} f = -\mathcal{L}_{X_f} g,\end{aligned} \tag{1.29}$$

$$\begin{aligned}X^\mu_{\{f,\,g\}} &= \omega^{\mu\nu}\partial_\nu\big(X_g^\rho\,\partial_\rho f\big) = \omega^{\mu\nu}\big(\partial_\nu X_g^\rho\,\partial_\rho f + X_g^\rho\,\partial_\rho\partial_\nu f\big) \\ &= \omega^{\mu\nu}\big[\partial_\nu X_g^\rho\omega_{\rho\sigma}X_f^\sigma + X_g^\rho\partial_\rho\omega_{\nu\sigma}X_f^\sigma + X_g^\rho\omega_{\nu\sigma}\partial_\rho X_f^\sigma\big] \\ &= \omega^{\mu\nu}\big[\omega_{\nu\sigma}\big(X_f^\rho\,\partial_\rho X_g^\sigma - X_g^\rho\,\partial_\rho X_f^\sigma\big) \\ &\qquad + X_g^\sigma\big(X_f^\rho\,\partial_\rho\omega_{\nu\sigma} + \omega_{\nu\rho}\,\partial_\sigma X_f^\rho + \omega_{\rho\sigma}\,\partial_\nu X_f^\rho\big)\big] \\ &= \omega^{\mu\nu}\big(\omega_{\nu\sigma}[X_f\,,X_g]^\sigma + X_g^\sigma\mathcal{L}_{X_f}\omega_{\nu\sigma}\big) = [X_f\,,X_g]^\mu,\end{aligned} \tag{1.30}$$

$$\begin{aligned}\mathrm{Jac}(f,g,h) &= \{f\,,\{g\,,h\}\} + \{h\,,\{f\,,g\}\} + \{g\,,\{h\,,f\}\} \\ &= \frac{1}{2}\big[\omega_{\mu\nu}X^\mu_{\{g\,,h\}}X_f^\nu + \omega_{\rho\mu}X^\rho_{\{f\,,g\}}X_h^\mu + \omega_{\nu\rho}X^\nu_{\{h\,,f\}}X_g^\rho \\ &\qquad - X_f^\mu\,\partial_\mu\big(\omega_{\nu\rho}X_g^\nu X_h^\rho\big) - X_h^\rho\,\partial_\rho\big(\omega_{\mu\nu}X_f^\mu X_g^\nu\big) - X_g^\nu\,\partial_\nu\big(\omega_{\rho\nu}X_h^\rho X_f^\mu\big)\big] \\ &= \frac{1}{2}\big[\omega_{\mu\nu}X_f^\nu[X_g\,,X_h]^\mu + \omega_{\rho\mu}X_h^\mu[X_f\,,X_g]^\rho + \omega_{\nu\rho}X_g^\rho[X_h\,,X_f]^\nu \\ &\qquad - X_f^\mu\,\partial_\mu\big(\omega_{\nu\rho}X_g^\nu X_h^\rho\big) - X_h^\rho\,\partial_\rho\big(\omega_{\mu\nu}X_f^\mu X_g^\nu\big) - X_g^\nu\,\partial_\nu\big(\omega_{\rho\nu}X_h^\rho X_f^\mu\big)\big] \\ &= \frac{3}{2}\,\partial_{[\mu}\big(X_f^\mu X_g^\nu X_h^\rho\big)\omega_{\nu\rho]} - \frac{3}{2}\,\partial_{[\mu}\big(\omega_{\nu\rho]}X_f^\mu X_g^\nu X_h^\rho\big) = -\frac{3}{2}X_f^\mu X_g^\nu X_h^\rho\,\partial_{[\mu}\omega_{\nu\rho]} \\ &= -\frac{1}{2}\,\mathrm{d}\omega(X_f, X_g, X_h) = 0,\end{aligned} \tag{1.31}$$

$$\{f\,,g\} = \omega_{\mu\nu}X_g^\mu X_f^\nu = -\omega_{\nu\mu}X_f^\nu X_g^\mu = -\{g\,,f\}. \tag{1.32}$$

Here $[\cdot\,,\cdot]^\mu$ is the commutator (or Lie bracket) of two vector fields and Jac is the Jacobiator of three functions. Equation (1.31) is called the *Jacobi identity* and, together with equation (1.32), establishes that $C^\infty(\Omega)$ equipped with the Poisson bracket forms a Lie algebra. A homomorphism of this Lie algebra onto the Lie algebra of Hamiltonian vector fields is given by equation (1.30).

In Darboux coordinates a Hamiltonian vector field is given by $X_f = \big(\partial_{p_\mu} f, -\partial_{q^\mu} f\big)$ and the Poisson bracket takes the familiar form

$$\{f, g\} = \omega(X_f, X_g) = \sum_{\underline{\mu}=1}^{n} \left(\frac{\partial f}{\partial q^{\underline{\mu}}} \frac{\partial g}{\partial p_{\underline{\mu}}} - \frac{\partial g}{\partial q^{\underline{\mu}}} \frac{\partial f}{\partial p_{\underline{\mu}}} \right). \tag{1.33}$$

The Poisson bracket between the Darboux coordinates is then given by

$$\big\{q^{\underline{\mu}}, q^{\underline{\nu}}\big\} = 0, \tag{1.34a}$$

$$\big\{p_{\underline{\mu}}, p_{\underline{\nu}}\big\} = 0, \tag{1.34b}$$

$$\big\{q^{\underline{\mu}}, p_{\underline{\nu}}\big\} = \delta^{\underline{\mu}}{}_{\underline{\nu}}. \tag{1.34c}$$

These relations are sometimes called *fundamental*.

The simplest concrete example of a symplectic manifold is an even dimensional vector space, where some useful properties arise.

Example 1.11 ($\mathbb{R}^{2n}$ as a symplectic manifold)**.** The vector space $\Omega = \mathbb{R}^{2n}$ with a symplectic structure can naturally be seen as a symplectic manifold. It is also isomorphic to the cotangent bundle $T^*\mathbb{R}^n$ of the vector space $\mathcal{C} = \mathbb{R}^n$. Since $T\Omega = \Omega \times \Omega$, the symplectic form is now a map $\omega\colon \Omega \times \Omega \to \mathbb{R}$ and takes the form

$$\omega\big((q^\mu, p_\mu), (q'^\mu, p'_\mu)\big) = \sum_{\underline{\mu}=1}^{n} \big(q^{\underline{\mu}} p'_{\underline{\mu}} - q'^{\underline{\mu}} p_{\underline{\mu}}\big) \tag{1.35}$$

in Darboux coordinates. Since a bilinear form maps a vector space into its dual, the action of the symplectic form on a vector $y \in \Omega$, $\omega(y, \cdot)$, is a function on Ω and it is meaningful to take the Poisson bracket of entities of this kind. It is, then, easy to check that equations (1.34) can be written as

$$\big\{\omega\big((q^\mu, p_\mu), \cdot\big), \omega\big((q'^\mu, p'_\mu), \cdot\big)\big\} = \omega\big((q^\mu, p_\mu), (q'^\mu, p'_\mu)\big). \tag{1.36}$$

In fact, defining $y^\alpha = (q^\mu, p_\mu)$ and $y'^\alpha = (q'^\mu, p'_\mu)$ and taking care not to confuse indices tangent to $\mathcal{C}$ with those tangent to Ω,

$$\{\omega(y, \cdot), \omega(y', \cdot)\} = \omega_{\zeta\varepsilon}\omega^{\zeta\delta}\omega_{\delta\beta}y'^\beta\omega^{\varepsilon\gamma}\omega_{\gamma\alpha}y^\alpha = \omega_{\zeta\varepsilon}y'^\zeta y^\varepsilon = \omega(y, y'). \tag{1.37}$$

2 Quantum Field Theory in Curved Spacetimes

The theoretical framework underpinning the results of the present dissertation is QFT in curved spacetimes, which calls for an account of its fundamentals and main results. Short introductions to GR and QFT in flat spacetimes are presented with a focus on the concepts that ground the basic ideas of QFT in curved spacetimes. A formulation of QFT in curved spacetimes is expounded here and a derivation of the Unruh effect, one of the centerpieces of this work, is given.

2.1 General Relativity

Gravitation is best described by GR, a theory renowned for its insights into the nature of space and time. Despite being a centennial theory, much of its conceptual foundation has only been built in the last few decades—in particular, the *principle of general covariance* has been a point of contention [23]. This foundation is the source of a wealth of phenomena in QFT in curved spacetimes, and, so, a brief exposition is in order. Thorough references on technical aspects of GR are [15, 24]. Conceptual aspects are discussed, from philosophical and physical standpoints, in [19].

2.1.1 The Structure of Spacetime

Spacetime is the main object of the traditional formulation of GR. It is the combination of the concepts of space and time and possesses dynamical properties and a geometric structure reflecting causality. In most treatments, spacetime is considered to be a four-dimensional manifold M equipped with a Lorentzian metric $g_{\mu\nu}$, representing the set of physical events subject to causality conditions. This is the point of view introduced in this section, for convenience. A slightly different approach revolving around the frame bundle of spacetime, the one presented in [19], is introduced in subsection 2.1.3.

All gravitational phenomena may be seen as stemming from the metric tensor. Objects that describe the geometry of spacetime are obtained from the metric and its derivatives, and are introduced in what follows, along with brief descriptions of how to interpret their role in the theory. This serves to make explicit the link between gravity and the geometry of spacetime.

Causality is locally encoded in the metric tensor as in the context of special relativity. A vector $v^\mu \in T_pM$ is called *timelike* if $g_{\mu\nu}v^\mu v^\nu < 0$, *null* if $g_{\mu\nu}v^\mu v^\nu = 0$ and *spacelike* if $g_{\mu\nu}v^\mu v^\nu > 0$. The set of null vectors at a point p forms the *light cone* at p. The light cone is oriented such that timelike vectors reside at its inside and spacelike vectors at its outside. One of the halves of the region inside of the light cone may be chosen to consist of *future directed* vectors, with the other consisting of *past directed* vectors, and if this choice can be made continuously throughout M the spacetime is said to be *time orientable*. Every spacetime to be considered in this work is time orientable.

Global causal considerations follow by extending the definitions given above to curves. A curve in M has any of the properties defined above for vectors if its tangent vector field has them. It is called *causal* if its tangent vector field is not spacelike at any point. The *chronological future (past)* $I^\pm(p)$ of a point p is defined as the set of points that can be connected to p with a future (past) directed timelike curve. The *causal future (past)* $J^\pm(p)$ of p is defined analogously but with causal instead of timelike curves. These definitions are extended to arbitrary sets via

$$I^\pm\left(\bigcup_\alpha O_\alpha\right) = \bigcup_\alpha I^\pm(O_\alpha), \tag{2.1a}$$

$$J^\pm\left(\bigcup_\alpha O_\alpha\right) = \bigcup_\alpha J^\pm(O_\alpha), \tag{2.1b}$$

where O_α is an arbitrary family of sets. A set Σ is said to be *achronal* if $I^+(\Sigma) \cup \Sigma = \emptyset$.

The Levi-Civita connection (q.v. subsection 1.2.2) associated to $g_{\mu\nu}$ is denoted by ∇_μ and its connection form, called the *Christoffel symbol*, is denoted[12] by $\Gamma^\lambda{}_{\mu\nu}$. From the torsion-free condition of the Levi-Civita connection, the symmetry on the lower indices, $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{(\mu\nu)}$, may be deduced and an expression for the Christoffel symbol in terms of the derivatives of the metric,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}\big(\partial_\nu g_{\alpha\mu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\big), \tag{2.2}$$

can be derived from the metric preserving property.

Gravitational effects are associated to the Christoffel symbol with the notion of *parallel transport*. A tensor field $V^{\alpha\cdots}_{\beta\cdots}$ on M is said to be parallel transported along a curve γ with tangent vector field γ^μ if

$$\gamma^\mu \nabla_\mu V^{\alpha\cdots}_{\beta\cdots} = 0. \tag{2.3}$$

A curve is said to be a *geodesic* if its tangent vector field is parallel transported along itself, i.e.,

$$\gamma^\mu \nabla_\mu \gamma^\lambda = \frac{\mathrm{d}\gamma^\lambda}{\mathrm{d}s} + \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}s} = 0, \tag{2.4}$$

[12] The indices for the Christoffel symbol differ from the one introduced for connection forms in section 1.2.2. Besides the internal indices now being tangent indices, the first lower index has its position exchanged with the upper index.

where s is an *affine parameter* for the curve, i.e., $\gamma^\mu \nabla_\mu s = 1$. It is clear, then, that the Christoffel symbol corresponds to, in the classical limit, the gravitational force field (see [24]). Likewise, inertial movement corresponds to geodesic trajectories in spacetime.

To describe the dynamics of the gravitational field, one needs the notion of the curvature of spacetime. The *Riemann curvature tensor* plays this role and is defined as the curvature (q.v. subsection 1.2.2) associated to the Levi-Civita connection and is denoted by $R^\mu{}_{\nu\rho\sigma}$. In terms of the Christoffel symbol,

$$R^\rho{}_{\sigma\mu\nu} = 2\,\partial_{[\mu}\Gamma^\rho{}_{\nu]\sigma} + 2\Gamma^\rho{}_{[\mu|\alpha|}\Gamma^\alpha{}_{\nu]\sigma}. \tag{2.5}$$

Since the Riemann curvature tensor is the linear approximation of the *holonomy*, the operator mapping a vector field to its parallel transported vector field along a closed loop, it can be associated to tidal effects, as evidenced by the *geodesic deviation* equation:

$$\gamma^\mu \nabla_\mu \big(\gamma^\sigma \nabla_\sigma J^\lambda\big) = R^\lambda{}_{\sigma\mu\nu}\gamma^\sigma\gamma^\mu J^\nu, \tag{2.6}$$

where J^λ is the *Jacobi field*

$$J^\lambda := \left.\frac{\mathrm{d}\gamma_u}{\mathrm{d}u}\right|_{u=0}, \tag{2.7}$$

i.e., the field giving the "infinitesimal" difference between a geodesic γ and family of geodesics γ_u with $\gamma_0 = \gamma$. The left side of equations (2.6) may be seen as the relative acceleration between γ and some close geodesic (see [15]).

The dynamics of the gravitational field is given by the *Einstein equation*,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}\,, \tag{2.8}$$

where

$$R_{\mu\nu} := R^\rho{}_{\mu\rho\nu}, \tag{2.9}$$

$$R := R^\mu{}_\mu \tag{2.10}$$

are, respectively, the *Ricci curvature tensor* and the *Ricci curvature scalar*, Λ is the cosmological constant and $T_{\mu\nu}$ is the energy-momentum tensor of matter. The Einstein equation is a constraint on the geometry of spacetime, or gravity, related to its matter content. The interpretation that the equation describes the dynamics of the gravitational field is clear if a foliation of M into spacelike slices is taken (see [25]).

2.1.2 General Covariance

It is clear from section 2.1.1 that GR is very dissimilar to most other physical theories, e.g., Yang–Mills theories, since it does not admit an *a priori* defined geometry for spacetime—it is said to be *background independent*—by the nature of the Einstein equation. This raises a series of questions regarding the nature of spacetime and what kinds of quantities can be measured, which may be partially answered in light of the discussion presented in what follows.

The requirement of no prior geometry stems from the principle of general covariance, which, roughly, states that the laws of physics are indifferent to a choice of frame of reference. A more precise statement can be obtained using a more adequate language, provided by modern differential geometry. A frame of reference is frequently identified with a choice of coordinates on the spacetime manifold M, i.e., with an atlas of charts φ on M, and the requirement of indifference then translates to invariance with respect to changes of coordinates, i.e., diffeomorphisms $h\colon \mathbb{R}^4 \to \mathbb{R}^4$ between the images (subsets of $\mathbb{R}^4$) of the charts, as illustrated in the following commutative diagram:

$$\begin{tikzcd} & M \arrow[dl, "\varphi_k"'] \arrow[dr, "\varphi_\ell"] & \\ \mathbb{R}^4 \arrow[rr, "h"] & & \mathbb{R}^4 \end{tikzcd} . \tag{2.11}$$

The meaning of invariance remains obscure but is clarified below.

Further refinements of the statement of general covariance can be obtained in the spirit of the coordinate-free approach to geometry, replacing transformations between coordinate charts with diffeomorphisms between manifolds. A diffeomorphism between two manifolds M and N is an invertible map $f\colon M \to N$ such that the compositions $\varphi' \circ f \circ \varphi^{-1}\colon \mathbb{R}^4 \to \mathbb{R}^4$, where φ and φ' are charts on M and N respectively, are diffeomorphisms, and is illustrated in the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"] & N \arrow[d, "\varphi'"'] \\ \mathbb{R}^4 \arrow[r, "\varphi'\circ f\circ\varphi^{-1}"] & \mathbb{R}^4 \end{tikzcd} . \tag{2.12}$$

The notion of change of coordinates can, then, be substituted by the notion of a diffeomorphism from M to itself in the statement of the principle.

A diffeomorphism f of M into N induces a transformation between the tangent bundles TM and TN,

$$\begin{tikzcd} TM \arrow[r, "\mathrm{d}f"] \arrow[d, "\pi"] & TN \arrow[d, "\pi"'] \\ M \arrow[r, "f"] & N \end{tikzcd} , \tag{2.13}$$

called the *differential* $\mathrm{d}f$ of f. It maps the vector fields on M, sections of TM, into vector fields on N. Intuitively, this map drags a vector over a point x of M along f to its image $f(x)$ on N. A precise definition of the action of the differential on a vector field $v \in \Gamma(TM)$ is given by

$$\mathrm{d}f(v)s \coloneqq v(s \circ f) = \partial_\nu f^\mu v^\nu \, \partial_\mu s, \tag{2.14}$$

where the vector field $\mathrm{d}f(v)$ acts on $s\colon N \to \mathbb{R}$ by the directional derivative. The action of the differential on vector fields is called the *pushforward* of v by f and denoted $f_* v \coloneqq \mathrm{d}f(v)$. The composition $s \circ f$, appearing in equation (2.14), is called the *pullback* of s by f and denoted $f^* s$. The pullback can be extended to a 1-form $\omega \in \Gamma(T^*N)$ via

$$f^*\omega(v) \coloneqq \omega(f_* v) = \omega_\mu \, \partial_\nu f^\mu v^\nu, \tag{2.15}$$

and both maps can be extended to general tensor fields by demanding the compatibility between the tensor product and the pushforward or pullback, e.g., $f_*(v \otimes w) = f_* v \otimes f_* w$. Care must be taken when dealing with pushforwards and pullbacks, since the transformations induced by them differ considerably:

$$\begin{tikzcd}
TM \arrow[rr, "f_*"] \arrow[d, "\pi"] & & TN \arrow[d, "\pi"'] \\
M \arrow[rr, bend left, "f"] \arrow[dr, "f^* s"] & & N \arrow[dl, "s"'] \\
& \mathbb{R} & \\
T^*M \arrow[uu, "\pi"'] & & T^*N \arrow[ll, "f^*"] \arrow[uu, "\pi"]
\end{tikzcd} . \tag{2.16}$$

These tools allow for the precise formulation of the principle of general covariance.

Principle of General Covariance. Let $g_{\mu\nu}(x)$ be a solution of the Einstein equation and f a diffeomorphism of M into itself. Then[13] $\tilde{g}_{\mu\nu}(x) \coloneqq f^{-1*} g_{\mu\nu}(x)$ is a solution of the Einstein equation corresponding to *the same* physics, i.e., $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ are part of an equivalence class of solutions corresponding to a single physical model.

An important remark must be made. The principle of general covariance should not be confused with the statement of coordinate independence in the definitions of the various geometric objects in GR, called the *trivial identity* in [19]. The trivial identity states that the value of a field at a point is invariant if both are dragged by a diffeomorphism, i.e., $f_* v(f(x)) = v(x)$. General covariance involves the dragging of fields and not points (or vice versa). It also makes use of the dynamics of the field, in its evocation of the Einstein equation, which elucidates it as a relation between solutions of the field equation, not between generic geometric objects. It is in this sense that general covariance imposes the requirement of background independence.

Historically, general covariance has been tied to the *hole argument* and to the *point-coincidence argument* (see [26] for a review). The former was formulated as an argument against general covariance, while the latter signifies a return to it, which culminated with the proposal that the field equation of GR be (2.8) (see [27]). An outline of these arguments (cf. [28]) may shine a light on the technical aspects of the principle of general covariance, and is presented bellow.

[13] See diagram (2.16) as to why the pullback by the inverse mapping must be used.

Consider a solution $g_{\mu\nu}(x)$ of the Einstein equation on a manifold M. Suppose also that there is a region $H \subset M$, called the *hole*, devoid of matter. There is some smooth diffeomorphism f of M into itself such that f is the identity outside the hole (and on its boundary) but differs from it on H, and the pullback $\tilde{g}_{\mu\nu}(x) = f^{-1*}g_{\mu\nu}(x)$ of the solution via f must also be a solution, according to the principle of general covariance. The hole argument consist on the observation that $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x)$ outside H but $\tilde{g}_{\mu\nu}(x) \neq g_{\mu\nu}(x)$ inside H, which implies a loss of determinism for the theory (the two solutions agree on the causal past of H but disagree on H itself) if the requirement that these two solutions belong to an equivalence class is dropped. The point-coincidence argument asserts that physical observations are not affected by a choice between the solutions, i.e., GR has a sort of gauge freedom (see [28]), since all physical fields must be dragged along with the gravitational field. The situation is illustrated in figure 1.

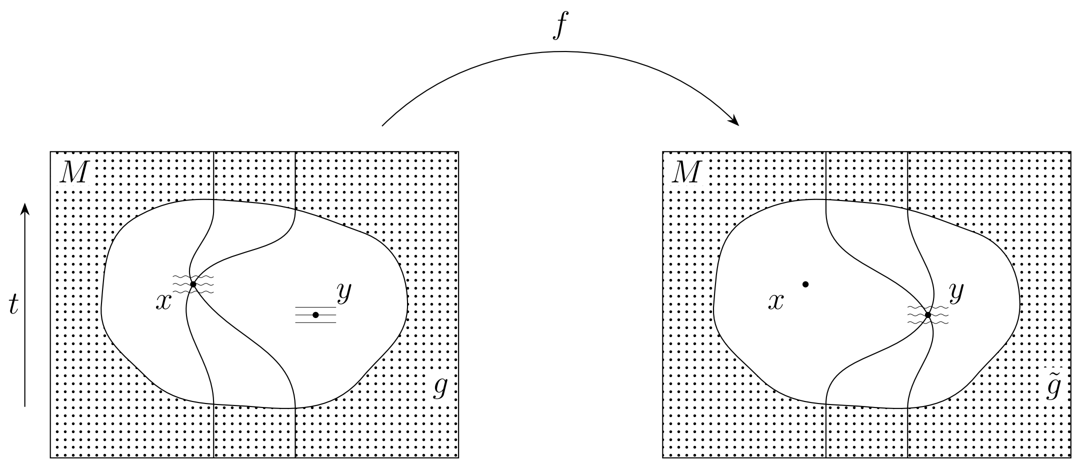

Figure 1 – The diffeomorphism f is such that the hole argument applies and $f(x) = y$. The field originally at x is dragged to y, but so are any other physical objects, e.g., the event corresponding to a crossing of worldlines. Adapted from [28].

Accepting the conclusion of the point-coincidence argument, that GR is a generally covariant theory, puts into question the reality of points of spacetime and, therefore, of spacetime itself. Statements referencing points such as x and y in figure 1, e.g., referring to the value $g_{\mu\nu}(x)$ of the metric at a point, must be devoid of physical meaning if the condition of diffeomorphism invariance (as stated in the principle) is respected, since the quantities referenced will not be gauge invariant, e.g., different values of the metric at the same point may correspond to the same solution of the Einstein equation. If points of the manifold M can be individuated, on the other hand, general covariance must be replaced by the trivial identity, which places no constraints on any physical theory with a geometrical interpretation.

If spacetime is not fundamental, the elaboration of a formulation of GR that makes no direct reference to the manifold M is desirable in light of the principle of ontological parsimony. This can be achieved using the tools presented in section 1.2.

2.1.3 Spacetime from Gravity and Observables

Spacetime may not be a fundamental entity of GR, but gravity is. The task tackled in this subsection is, thus, finding the geometric structure that properly encodes the information of the gravitational field and allows for the interpretation of spacetime as an emergent entity. The structure in question is the bundle of oriented orthonormal frames, introduced in subsection 1.2.1.

A tetrad ${e^\mu}_I$ (cf. examples 1.1 and 1.7) corresponds to a set of 4 orthonormal vector fields on M, i.e.,

$$g_{\mu\nu}{e^\mu}_I{e^\nu}_J = \eta_{IJ}, \tag{2.17}$$

where η_{IJ} is the Minkowski metric. Along with its inverse ${e_\mu}^I$, which satisfies

$$\eta_{IJ}{e_\mu}^I{e_\nu}^J = g_{\mu\nu}, \tag{2.18}$$

the tetrad can be made into an isomorphism from the space of (local) vector fields on M to the space of (local) vector fields on $\mathbb{R}^4$ and, therefore, can be used to exchange tangent and Minkowski indices via

$$v^I = {e_\mu}^I v^\mu, \tag{2.19a}$$
$$v^\mu = {e^\mu}_I v^I. \tag{2.19b}$$

General relativity can be formulated with the tetrad taking the place of the metric, so long as a spin connection[14] D_μ with the torsion-free property with respect to ${e_\mu}^I$,

$$\mathrm{D}_{[\mu}{e_{\nu]}}^I = 0, \tag{2.20}$$

is introduced, along with the curvature tensors that derive from it (see [28]). Tetrads can naturally be associated with frames of reference and introduce an extra kind of gauge freedom, since the metric is invariant with respect to the action of Lorentz transformations ${\Lambda^I}_J \in \mathrm{SO}(3,1)$ on the tetrad:

$$g'_{\mu\nu} = {e'_\mu}^I{e'_\nu}^J\eta_{IJ} = {\Lambda^I}_K{e_\mu}^K{\Lambda^J}_L{e_\nu}^L\eta_{IJ} = {e_\mu}^K{e_\nu}^L\eta_{KL} = g_{\mu\nu}. \tag{2.21}$$

Since the tetrad can now be taken as a dynamical field, GR can be interpreted as theory of certain frame bundles rather than a theory of Lorentzian manifolds. Taking into account the observation presented following definition 1.3, the spacetime manifold M can be defined as the quotient of the bundle of oriented orthonormal frames by the Lorentz group $\mathrm{SO}(3,1)$. This allows for the interpretation that spacetime arises from the dynamics of the gravitational field rather than being a fundamental entity. A better developed version of this idea results in the philosophical position called *dynamic structural realism*, discussed in [19].

[14] A spin connection on TM is one that is associated to an Ehresmann connection on the spin bundle $\mathrm{Spin}(3,1) \hookrightarrow E \xrightarrow{\pi_E} M$. It takes values on $\mathfrak{spin}(3,1)$, the Lie algebra of the spin group.

The position introduced above forces one to reevaluate what is meant by an observable. These are entities that, in a given model, play the role of quantities that can be obtained as the result of a measurement. In most physical theories, observables are functions of elements of some set of background structures, usually taken to be spacetime, possibly endowed with some additional structure. In GR, as argued above, observables cannot depend exclusively on spacetime points, i.e., the value of some function at a point cannot be an observable in GR. To elaborate on this point, a precise notion of an observable must be introduced. The one to be considered is that of a *Dirac observable*, which requires the definition of a gauge transformation in the sense of *Dirac–Bergmann*. Consider two (mathematically) different solutions to a set of equations of motion evolving in some parameter t that are, nonetheless, equal for every $t < \hat{t}$. A mapping between these two solutions is called a *gauge transformation* and a quantity that is invariant with respect to gauge transformations is called a Dirac observable. This ensures that the system is deterministic and introduces a mutual dependence on the solutions to the equations of motion, the gauge transformations and the observables.

General covariance introduces the diffeomorphism group as a gauge transformation group for GR, implying that all Dirac observables must be diffeomorphism invariant quantities, which clarifies the statement that there are no local Dirac observables in GR, i.e., that Dirac observables cannot be functions of spacetime points alone. There is, however, another notion of locality that does apply to Dirac observables in GR. Describing it requires the introduction of physical coordinates, in the spirit of the point-coincidence argument. Consider four scalar matter fields (for more on field theory, see section 2.2) coupled to the gravitational field and such that their configurations do not present so many degeneracies as to make taking them as coordinates impossible. These fields can be used to individuate spacetime points. i.e., as physical coordinates, and the value of a scalar function of these fields is a Dirac observable, since diffeomorphisms drag the gravitational and matter fields alike. Locality follows if the value of a scalar function at a configuration s of the matter fields is determined (in the dynamical sense) entirely by the configurations in $J^{-}(s)$, the causal past of s. This is the case with the *GPS observables*, constructed in [29]. A more detailed discussion on observables and physical coordinates can be found in [28], including the description of *partial observables*, which clarify the relationship between observables and coordinates (see also [30]). For a more technical approach, in the context of the canonical formalism of GR, see [31].

2.2 Quantum Field Theory

Every known physical interaction is well described by a field theory. Familiarity with its fundamental entities, fields, is crucial to understanding the phenomena addressed in this work. This section seeks to present the basics of QFT in flat and curved spacetimes.

Standard references for QFT in flat spacetimes are, e.g., [32–34]. For QFT in curved spacetimes, see [35–37] and especially [14], since the following sections are mostly based on it.

2.2.1 Free Fields in Flat Spacetime

A *field theory* assigns dynamics to sections of some fiber bundle, usually associated to a principal bundle. The ontology of most treatments of field theories includes a gravitational field with trivial dynamics, that is, the dynamics of the fields described by the theory take place on Minkowski or Newtonian spacetime, and are, hence, called field theories in flat spacetimes.

A special class of field theories will be the one presented here: that of *scalar fields*. A scalar field is a section of a *line bundle*, a vector bundle whose fiber is a one-dimensional vector space, typically taken to be $\mathbb{R}$ or $\mathbb{C}$. The field represents the spacial and temporal configurations of a system since the base manifold is taken to be a spacetime M.

The dynamics of a scalar field ϕ in Minkowski spacetime can be specified in a variety of manners. One is by specifying an action functional S, which can be obtained from a Lagrangian density $\mathscr{L}$ via

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \mathscr{L}, \tag{2.22}$$

whose stationary points correspond to solutions of the dynamics, i.e., such that

$$\delta S[\phi] = 0, \tag{2.23}$$

where δ is the functional differential, if ϕ solves the dynamic equations. Another consists in giving a Hamiltonian functional[15] H, written in terms of a Hamiltonian density $\mathscr{H}$ as

$$H = \int_{\mathbb{R}^3} \mathrm{d}^3x \, \mathscr{H}, \tag{2.24}$$

whose Poisson bracket generates the evolution of physical quantities. The latter is often called the *canonical formalism*, since symplectomorphisms may be called *canonical transformations* (cf. [21]). The two densities are related by the Legendre transform,

$$\mathscr{H} = \dot{\phi}\pi - \mathscr{L}, \tag{2.25}$$

where

$$\dot{\phi} := \partial_t \phi, \tag{2.26}$$

$$\pi := \partial_{\dot{\phi}} \mathscr{L}, \tag{2.27}$$

[15] Assumed here to be time-independent.

are, respectively, the velocity and conjugate momentum of the field.

The only Poincaré invariant Lagrangian density for a real scalar field on Minkowski spacetime yielding a second-order linear equation of motion is the Klein–Gordon (KG) Lagrangian density

$$\mathscr{L} = -\frac{1}{2}\big(\partial_K\phi\,\partial^K\phi + m^2\phi^2\big). \tag{2.28}$$

Its associated equation of motion is the KG equation

$$\big(\partial_K\partial^K - m^2\big)\phi = 0, \tag{2.29}$$

and its solutions are said to be *free* KG fields. Fields with nonlinear equations of motion are said to be *interacting*, and are treated at the end of this subsection. The conjugate momentum is $\pi = \dot\phi$ and the Hamiltonian density is

$$\mathscr{H} = \frac{1}{2}\big(\dot\phi^2 + \partial_k\phi\,\partial^k\phi + m^2\phi^2\big). \tag{2.30}$$

To obtain the time flow, the Poisson bracket must be defined. A canonical form θ on the phase space $\Omega = H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ (where $H^1(\mathbb{R}^3)$ is the Sobolev space $W^{1,2}(\mathbb{R}^3)$, see [38]) of initial data of finite energy at some time t, e.g., $t=0$, is given by

$$\theta(\phi_0,\pi_0)(\phi,\pi) = -\int_{\mathbb{R}^3}\mathrm{d}^3x\,\pi_0\phi, \tag{2.31}$$

from which arises a strong symplectic form ω given by

$$\omega\big((\phi,\pi),(\phi',\pi')\big) = \int_{\mathbb{R}^3}\mathrm{d}^3x\,(\phi\pi' - \phi'\pi). \tag{2.32}$$

To every functional F on Ω, since the symplectic form is strong, there is a corresponding Hamiltonian vector field X_F such that

$$\omega(X_F,\cdot) = \delta F. \tag{2.33}$$

The Poisson bracket of two functionals F and G is, thus, defined as

$$\{F\,,G\} \coloneqq \omega(X_F,X_G) = \int_{\mathbb{R}^3}\mathrm{d}^3x\left(\frac{\delta F}{\delta\phi}\frac{\delta G}{\delta\pi} - \frac{\delta G}{\delta\phi}\frac{\delta F}{\delta\pi}\right). \tag{2.34}$$

The time flow is the flow generated by the Hamiltonian functional and the time derivative of an observable[16] is the Lie derivative with respect to this flow, which can be written in terms of the Poisson bracket:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F\,,H\}. \tag{2.35}$$

[16] Taking observables to evolve in time is the classical equivalent of the Heisenberg picture, to be adopted in what follows.

The fundamental Poisson brackets given in equations (1.34) are then equivalent, in the distributional sense, to[17]

$$\{\phi(t,\mathbf{x})\,,\phi(t,\mathbf{x}')\} = 0, \tag{2.36a}$$

$$\{\pi(t,\mathbf{x})\,,\pi(t,\mathbf{x}')\} = 0, \tag{2.36b}$$

$$\{\phi(t,\mathbf{x})\,,\pi(t,\mathbf{x}')\} = \delta(\mathbf{x}-\mathbf{x}'). \tag{2.36c}$$

A concise introduction to symplectic geometry and Hamiltonian dynamical systems in infinite dimensional manifolds, including the definitions of strong and weak symplectic forms, can be found in [39]. A more detailed treatment with a wealth of results is [40].

The space of solutions of the KG equation can naturally be identified with the phase space of initial data, since every solution corresponds to a pair (ϕ, π) of initial data. The symplectic form defined on Ω induces a symplectic structure on $\mathcal{S}$, the space of solutions arising from initial data in Ω, given by

$$\omega(\phi,\phi') = \int_{\mathbb{R}^3} \mathrm{d}^3x\,(\phi\,\partial_t\phi' - \phi'\,\partial_t\phi). \tag{2.37}$$

It must be emphasized that the fields are now seen as functions on spacetime rather than on Ω. An inner product on the complexification $\mathcal{S}_{\mathbb{C}}$ of $\mathcal{S}$ can be obtained from the symplectic form using a standard procedure. Given a antisymmetric bilinear form ω and a complex structure J, a conjugate symmetric sesquilinear form m may defined as $m(\phi,\phi') \coloneqq \omega(\phi^*, J\phi')$. The inner product on $\mathcal{S}_{\mathbb{C}}$ is, thus, given by

$$\langle\phi|\phi'\rangle_{\mathrm{KG}} \coloneqq i\omega(\phi^*,\phi') = i\int_{\mathbb{R}^3} \mathrm{d}^3x\,(\phi^*\,\partial_t\phi' - \phi'\,\partial_t\phi^*). \tag{2.38}$$

The KG field may be treated as a family of harmonic oscillators. It can be written in terms of its spatial Fourier transform,

$$\phi(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{d}^3k\,\hat{\phi}(t,\mathbf{k})e^{ik_\ell x^\ell}, \tag{2.39}$$

and the reality of ϕ implies $\hat{\phi}^*(t,\mathbf{k}) = \hat{\phi}(t,-\mathbf{k})$. The Hamiltonian may, thus, be rewritten as

$$H = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^3k\,\left(|\dot{\hat{\phi}}|^2 + \omega^2|\hat{\phi}|^2\right), \tag{2.40}$$

where $\omega^2(\mathbf{k}) = k_\ell k^\ell + m^2$, which may be seen as the Hamiltonian of a family of decoupled harmonic oscillators. The fundamental Poisson brackets for the Fourier transforms can be obtained from equations (2.36):

$$\{\hat{\phi}(t,\mathbf{k})\,,\hat{\phi}(t,\mathbf{k}')\} = 0, \tag{2.41a}$$

$$\{\hat{\pi}(t,\mathbf{k})\,,\hat{\pi}(t,\mathbf{k}')\} = 0, \tag{2.41b}$$

$$\{\hat{\phi}(t,\mathbf{k})\,,\hat{\pi}(t,\mathbf{k}')\} = \delta(\mathbf{k}-\mathbf{k}'), \tag{2.41c}$$

[17] The boldface letter $\mathbf{x}$ denotes the three-dimensional position vector, a convention adopted for other three-dimensional vectors.

where[18] $\hat{\pi}(t,k) := \dot{\hat{\phi}}^*(t,k)$. It is useful to introduce the Fourier coefficients

$$a(t,\mathbf{k}) := \sqrt{\frac{\omega(\mathbf{k})}{2}}\hat{\phi}(t,\mathbf{k}) + i\sqrt{\frac{1}{2\omega(\mathbf{k})}}\dot{\hat{\phi}}(t,\mathbf{k}), \tag{2.42}$$

so that

$$\hat{\phi}(t,\mathbf{k}) = \frac{1}{\sqrt{2\omega(\mathbf{k})}}[a(t,\mathbf{k}) + a^*(t,-\mathbf{k})] \tag{2.43}$$

and equations (2.41) are equivalent to

$$\{a(t,\mathbf{k}), a(t,\mathbf{k}')\} = 0, \tag{2.44a}$$

$$\{a^*(t,\mathbf{k}), a^*(t,\mathbf{k}')\} = 0, \tag{2.44b}$$

$$\{a(t,\mathbf{k}), a^*(t,\mathbf{k}')\} = -i\delta(\mathbf{k}-\mathbf{k}'). \tag{2.44c}$$

The Hamiltonian takes the form

$$H = \int_{\mathbb{R}^3} \mathrm{d}^3k\, a^* a \omega, \tag{2.45}$$

which implies, using the equation of motion given by the Poisson bracket,

$$a(t,\mathbf{k}) = a(\mathbf{k})e^{-i\omega t}, \tag{2.46}$$

where $a(\mathbf{k}) := a(0,\mathbf{k})$. The field is then split into parts of positive and negative frequencies,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{d}^3k\, \frac{1}{\sqrt{2\omega(\mathbf{k})}}\left(a(\mathbf{k})e^{ik_L x^L} + a^*(\mathbf{k})e^{-ik_L x^L}\right), \tag{2.47}$$

where $k_0 = \omega$. The plane wave modes

$$f(x,k) = \frac{e^{ik_L x^L}}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{k})}}, \tag{2.48}$$

which lay outside of $\mathcal{S}_{\mathbb{C}}$, are orthonormal with respect to the KG inner product,

$$\langle f(x,k)|f(x,k')\rangle_{\mathrm{KG}} = \delta(\mathbf{k}-\mathbf{k}'), \tag{2.49}$$

and the coefficients $a(t,\mathbf{k})$ satisfy

$$a(t,\mathbf{k}) = \langle f(x,k)|\phi(t,\mathbf{x})\rangle_{\mathrm{KG}}\,. \tag{2.50}$$

Quantization adds extra structure to a field theory: quantum fields take values on the space of operators over some Hilbert space, which must be constructed. Naturally, for the fields in question, it should model a family[19] of harmonic oscillators, but there is

[18] Here π is the momentum conjugate to the Fourier transform of the field and not the Fourier transform of the momentum conjugate to the field. They are related by complex conjugation.

[19] Assumed here—pedagogically—to be a *countable* family of oscillators, in contrast with the implication of equation (2.45).

a caveat. If $\mathcal{H}$ is the Hilbert space of a 3-dimensional harmonic oscillator (one can take, e.g., $\mathcal{H} = L^2(\mathbb{R}^3)$), the tensor product of infinitely many copies of $\mathcal{H}$ is well defined and a natural but unsuitable choice for this type of QFT, since it is non-separable (see [41]). A better choice is the symmetric Fock space $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$, defined[20] as

$$\mathcal{F}_{\mathrm{s}}(\mathcal{H}) \coloneqq \bigoplus_{m=0}^{\infty} \bigotimes_{n=1}^{m}{}_{\mathrm{s}} \mathcal{H}, \tag{2.51}$$

where the empty tensor product is defined as $\mathbb{C}$.

An element $|\Psi\rangle$ of a Fock space $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ may be written as

$$|\Psi\rangle = (\psi, \psi^{a_1}, \psi^{a_1 a_2}, \dots), \tag{2.52}$$

where $\psi^{a_1 \cdots a_k} = \psi^{(a_1 \cdots a_k)}$ is tensor of type $(k, 0)$ over $\mathcal{H}$. For every $\xi^a \in \mathcal{H}$ there are operators $a(\xi^*)$ and $a^\dagger(\xi)$, called, respectively, the *annihilation* and *creation* operators associated to ξ^a, given by

$$a(\xi^*)\, |\Psi\rangle = (\xi^*_a \psi^a, \sqrt{2}\xi^*_a \psi^{a a_1}, \sqrt{3}\xi^*_a \psi^{a a_1 a_2}, \dots), \tag{2.53a}$$
$$a^\dagger(\xi)\, |\Psi\rangle = (0, \psi \xi^a, \sqrt{2}\xi^{(a}\psi^{a_1)}, \sqrt{3}\xi^{(a}\psi^{a_1 a_2)}, \dots). \tag{2.53b}$$

It is clear that $a(\xi^*)$ depends linearly on ξ^*_a (or antilinearly on ξ^a) and $a^\dagger(\xi)$ depends linearly on ξ^a. So long as the domains of the operators are defined appropriately (see [14]), $a^\dagger(\xi)$ is the adjoint of $a(\xi^*)$. It is a simple computation to verify that

$$[a(\xi^*)\,, a^\dagger(\xi')] = \xi^*_a \xi'^a \mathrm{I} = \langle \xi | \xi' \rangle_{\mathcal{H}}\, \mathrm{I}, \tag{2.54}$$

where $[\cdot\,,\cdot]$ is the commutator of two operators, I is the identity and $\langle \cdot\,|\,\cdot \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. Additionally, all annihilation operators commute amongst themselves, as do creation operators. A special element of $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ is the *vacuum state* $|0\rangle$, which can be written as

$$|0\rangle = (1, 0, 0, \dots) \tag{2.55}$$

and is in the kernel of all annihilation operators, i.e., for every $\xi \in \mathcal{H}$:

$$a(\xi^*)\, |0\rangle = 0. \tag{2.56}$$

The vacuum is a *cyclic vector* for the set of creation and annihilation operators associated to a basis of $\mathcal{H}$, i.e., the successive application of creation and annihilation operators on $|0\rangle$ spans a dense subspace of $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$.

Canonical quantization is a prescription mapping the dynamics of a classical system to those of a quantum system. This is accomplished by mapping the Poisson bracket of

[20] The definition given here applies to a countable family of Hilbert spaces, e.g., a quantum field on a compact manifold. Rigorous results on the properties of this space can be found in [42]. The results for a field in, e.g., Minkowski space, where the spectrum of the momentum operator is continuous, are simply adapted from these.

observables in the classical phase space into the commutator of operators acting on on the Hilbert space in question,

$$[\hat{F}\,,\hat{G}] = i\widehat{\{F\,,G\}}, \tag{2.57}$$

where hatted variables are the self-adjoint operators corresponding to the classical, unhatted observables. In the particular case of Darboux coordinates, dropping the hats (as will be done from this point onward), the fundamental Poisson brackets are mapped into the *canonical commutation relations* (CCR):

$$[\phi(t,\mathbf{x})\,,\phi(t,\mathbf{x}')] = 0, \tag{2.58a}$$

$$[\pi(t,\mathbf{x})\,,\pi(t,\mathbf{x}')] = 0, \tag{2.58b}$$

$$[\phi(t,\mathbf{x})\,,\pi(t,\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')\mathrm{I}. \tag{2.58c}$$

The time evolution is given by the analogue of equation (2.35),

$$i\frac{\mathrm{d}F}{\mathrm{d}t} = [F\,,H], \tag{2.59}$$

or in terms of the time evolution operator $U(t,t')$, defined by

$$F(t) = U^\dagger(t,t')F(t')U(t,t') \tag{2.60}$$

and satisfying the Schrödinger equation:

$$i\frac{\mathrm{d}U(t,t')}{\mathrm{d}t} = HU(t,t'). \tag{2.61}$$

This is the quantum analogue to the classical time flow on the phase space. It is then clear that

$$U(t,t') = e^{-iH(t-t')}. \tag{2.62}$$

One important property of U is that it is unitary, i.e.,

$$U(t,t')U^\dagger(t,t') = \mathrm{I}, \tag{2.63}$$

meaning that time evolution preserves probability amplitudes.

The quantization of the Fourier coefficients a and a^* can be achieved by mapping them into, respectively, annihilation and creation operators associated with the plane wave modes[21]

$$f(\mathbf{k}) = \frac{e^{ik_\ell x^\ell}}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{k})}}, \tag{2.64}$$

which satisfy a version of the CCR,

$$[a(t,\mathbf{k})\,,a(t,\mathbf{k}')] = 0, \tag{2.65a}$$

$$[a^\dagger(t,\mathbf{k})\,,a^\dagger(t,\mathbf{k}')] = 0, \tag{2.65b}$$

$$[a(t,\mathbf{k})\,,a^\dagger(t,\mathbf{k}')] = \delta(\mathbf{k}-\mathbf{k}')\mathrm{I}, \tag{2.65c}$$

[21] It must be emphasized that f and f^* do not lie in $\mathcal{H}$. Nonetheless, their associated creation and annihilation operators are well defined as operator-valued distributions.

where $a(t,\mathbf{k}) \coloneqq a(f^*(\mathbf{k}))(t)$ and $a^\dagger(t,\mathbf{k}) \coloneqq a(f(\mathbf{k}))(t)$. These correspond to equations (2.44). The dynamics of these operators are obtained from equations (2.59),

$$a(t,\mathbf{k}) = a(\mathbf{k})e^{-i\omega t}, \tag{2.66}$$

where, once again, $a(\mathbf{k}) \coloneqq a(0,\mathbf{k})$. The quantized field can then be written as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{d}^3k \, \frac{1}{\sqrt{2\omega(\mathbf{k})}} \Big(a(\mathbf{k})e^{ik_L x^L} + a^\dagger(\mathbf{k})e^{-ik_L x^L} \Big). \tag{2.67}$$

Since the creation and annihilation operators do not commute, care must be taken when quantizing the expression for Hamiltonian in equations (2.45). Any expression including the ordering $aa^\dagger$ of these operators gives an infinite value for the energy[22] of the Fock vacuum. To avoid problems of this kind, one must work with the normal ordering $:O:$ of a product O of creation and annihilation operators, obtained by changing the order of the product so that all annihilation operators are to the right of all creation operators. The normal ordered Hamiltonian is given by

$$:H: = \int_{\mathbb{R}^3} \mathrm{d}^3k \, a^\dagger a \omega, \tag{2.68}$$

and thus the energy of the vacuum state is zero, since

$$:H: |0\rangle = \int_{\mathbb{R}^3} \mathrm{d}^3k \, \omega a^\dagger a \, |0\rangle = 0. \tag{2.69}$$

2.2.2 Interacting Fields in Flat Spacetime

The developments above concern free or noninteracting fields. To describe *interacting* fields, a few extra concepts and results must be presented. A good introduction to interacting fields in flat spacetimes is [43].

The action S for an interacting field ϕ can be decomposed in two parts, $S = S_0 + S_\mathrm{I}$, where S_0 is the action for the free field and

$$S_\mathrm{I} = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \mathscr{L}_\mathrm{I} \tag{2.70}$$

is the action for the interaction, written in terms of the interaction Lagrangian density $\mathscr{L}_\mathrm{I}$. The Hamiltonian H may also be decomposed as $H = H_0 + H_\mathrm{I}$, where H_0 is the Hamiltonian of the free field and the interaction Hamiltonian H_I is given in terms of its density $\mathscr{H}_\mathrm{I}$, related to the interaction Lagrangian by

$$\mathscr{H}_\mathrm{I} = -\mathscr{L}_\mathrm{I}. \tag{2.71}$$

[22] In quantum systems, the value of an observable at some state of the system (represented by a ray in the Hilbert space) is the eigenvalue of the operator representing the observable associated to this state (which is clearly only defined if the state is an eigenstate of the operator).

Assuming $\mathscr{L}_{\mathrm{I}}$ does not depend on the derivatives of the field, the conjugate momentum π is the same as the free field conjugate momentum,

$$\pi \coloneqq \partial_{\dot{\phi}}\mathscr{L} = \partial_{\dot{\phi}}\mathscr{L}_0, \tag{2.72}$$

and the fundamental Poisson brackets remain unchanged, as do their quantum versions, the CCR.

Most results concerning interacting quantum fields are obtained with perturbation theory methods, where the dynamics of the free field are presumed to be an approximation of the dynamics of the interacting field and corrections are introduced with increasing order of complexity with respect to the *coupling constant*, a measure of the relative amplitude between the free and interacting dynamics. A helpful tool when dealing with perturbation theory is the *interaction picture* of time evolution (compare with the Heisenberg picture, used throughout this section), which introduces the temporal evolution of states via the interaction Hamiltonian and, therefore, delegates the temporal evolution of operators to the free Hamiltonian. Let $F^{(\mathrm{H})}$ and $|\psi\rangle^{(\mathrm{H})}$ denote an operator and a state in the Heisenberg picture and, similarly, $F^{(\mathrm{I})}$ and $|\psi\rangle^{(\mathrm{I})}$ an operator and a state in the interaction picture. The relationship between these entities is

$$F^{(\mathrm{I})}(t) = U_0^\dagger(t)U(t)F^{(\mathrm{H})}(t)U^\dagger(t)U_0(t), \tag{2.73}$$

$$|\psi(t)\rangle^{(\mathrm{I})} = U_0^\dagger(t)U(t)\,|\psi\rangle^{(\mathrm{H})}\,, \tag{2.74}$$

where $U(t) \coloneqq U(t,0)$ and U_0 is the time evolution operator associated to the free Hamiltonian. These two pictures are related at the time $t=0$, since states and operators coincide at it. The differential equations for the time flow are

$$i\frac{\mathrm{d}F^{(\mathrm{I})}}{\mathrm{d}t} = [F^{(\mathrm{I})}, H_0], \tag{2.75}$$

$$i\frac{\mathrm{d}|\psi\rangle^{(\mathrm{I})}}{\mathrm{d}t} = H_{\mathrm{I}}\,|\psi\rangle^{(\mathrm{I})}\,. \tag{2.76}$$

From now on, superscripts indicating pictures will be dropped.

The prototypical problem tackled with QFT methods is the scattering of fields. Solving this sort of problem essentially reduces to calculating probability amplitudes corresponding to the process, a goal achieved with the *S-matrix* formalism. Given initial and final states for the scattering process, denoted $|\psi\rangle$ and $|\psi'\rangle$, respectively, the probability amplitude of interest $\mathcal{A}(\psi \to \psi')$ is obtained by evolving the initial state to the time at which the final state is defined, t', and taking their inner product:

$$\mathcal{A}(\psi \to \psi') = \langle\psi'|\psi(t')\rangle\,. \tag{2.77}$$

Since the scattering states are assumed to be asymptotically free, the S-matrix may be formally defined as the limit of the time evolution operator (associated with the interaction

Hamiltonian) in the distant past and future,

$$S = \lim_{\substack{t' \to \infty \\ t \to -\infty}} U(t', t), \tag{2.78}$$

but a more accurate description of it is as a map between the spaces of initial and final states, so that

$$\mathcal{A}(\psi \to \psi') = \langle \psi' | S | \psi \rangle \,. \tag{2.79}$$

Frequently, only low order contributions to the S-matrix can be calculated explicitly. The expansion[23] up to first-order in the Hamiltonian may be written as

$$S = \mathrm{I} - i \int_{\mathbb{R}^4} \mathrm{d}^4 x \,{:}\mathscr{H}_\mathrm{I}{:} + \mathcal{O}\big(\mathscr{H}_\mathrm{I}^2\big) = \mathrm{I} + i \int_{\mathbb{R}^4} \mathrm{d}^4 x \,{:}\mathscr{L}_\mathrm{I}{:} + \mathcal{O}\big(\mathscr{L}_\mathrm{I}^2\big). \tag{2.80}$$

The expansion of the S-matrix into contributions of increasing order of complexity can be interpreted in terms of Feynman diagrams, with the help of Wick's theorem (see [43]). The probability $\mathcal{P}$ that the process takes place is given by the square of the absolute value of the amplitude,

$$\mathcal{P}(\psi \to \psi') = |\mathcal{A}(\psi \to \psi')|^2, \tag{2.81}$$

and its time derivative is called the rate Γ of the process:

$$\Gamma(\psi \to \psi') = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{P}(\psi \to \psi'). \tag{2.82}$$

One of the simplest examples of field interactions is given by a Klein-Gordon field and a two-level quantum mechanical system. Despite its simplicity, it is a model of extreme importance, since it allows for the description of the *particle interpretation* of QFT, an important tool to bridge the gap between theoretical and experimental developments and a source of counterintuitive results in QFT in curved spacetimes. This system may be described by the Hamiltonian $H = H_\mathrm{KG} + H_\mathrm{Q} + H_\mathrm{I}$, where H_KG is the KG Hamiltonian,

$$H_\mathrm{Q} = \sigma A^\dagger A \tag{2.83}$$

is the Hamiltonian of the two-level system and

$$H_\mathrm{I} = \varepsilon(t) \int_{\mathbb{R}^3} \mathrm{d}^3 x \, \phi(t, \mathbf{x}) \big(F(\mathbf{x}) e^{-i\sigma t} A + F^*(\mathbf{x}) e^{i\sigma t} A^\dagger \big) \tag{2.84}$$

is the interaction Hamiltonian. In the definitions above, A and $A^\dagger$ are ladder operators on the Hilbert space of the two-level system, spanned by the energy eigenstates $\{|\chi_0\rangle\,, |\chi_1\rangle\}$, i.e.,

$$A\,|\chi_0\rangle = 0, \tag{2.85a}$$
$$A\,|\chi_1\rangle = |\chi_0\rangle\,. \tag{2.85b}$$

[23] This expansion is valid even for time-dependent Hamiltonians, not addressed in this text.

These operators evolve according to $A(t) = Ae^{-i\sigma t}$. The mode F and the coupling constant ε belong to the spaces of smooth functions of compact support, i.e., $F \in C_0^\infty(\mathbb{R}^3)$ and $\varepsilon \in C_0^\infty(\mathbb{R})$. If the system is in an initial state $|\psi_\mathrm{i}\rangle = |\chi\rangle\,|n_\psi\rangle$, where $|\chi\rangle$ may be any state in the Hilbert space of the two-level system and $|n_\psi\rangle$ is a state in the Fock space of the KG field given by

$$|n_\psi\rangle = (0, \dots, 0, \psi^{a_1} \cdots \psi^{a_n}, 0, \dots), \tag{2.86}$$

the final state $|\psi_\mathrm{f}\rangle$ of the system, to first-order in ε, is given, according to equation (2.80), by

$$|\psi_\mathrm{f}\rangle = |\psi_\mathrm{i}\rangle - i\int_{-\infty}^{\infty} \mathrm{d}t\, H_\mathrm{I}\,|\psi_\mathrm{i}\rangle = |\psi_\mathrm{i}\rangle - i\int_{\mathbb{R}^4} \mathrm{d}^4x\, \varepsilon(t)\phi\big(Fe^{-i\sigma t}A + F^*e^{i\sigma t}A^\dagger\big)\,|\psi_\mathrm{i}\rangle\,. \tag{2.87}$$

Putting $f(x) = \varepsilon(t)e^{-i\sigma t}F(\mathbf{x})$, the integral above can be shown to yield (see [44] and subsection 2.2.3)

$$\int_{-\infty}^{\infty} \mathrm{d}t\, H_\mathrm{I} = \big(ia^\dagger\big(\lambda^{(+)}\big) - ia\big(\lambda^{(-)}\big)\big)A + \big(ia^\dagger\big(\lambda^{(-)*}\big) - ia\big(\lambda^{(+)*}\big)\big)A^\dagger, \tag{2.88}$$

where $\lambda^{(+)}$ and $\lambda^{(-)}$ are, respectively, the positive and negative frequency parts of λ, which is the retarded minus advanced solution of the KG equation with source f, i.e.,

$$\lambda(t, \mathbf{x}) = \int_{\mathbb{R}^4} \mathrm{d}^4x\, (G_R(x - x') - G_A(x - x'))f(x'), \tag{2.89}$$

where G_R and G_A are the Green functions of the KG equation. Assuming that $\varepsilon(t)$ oscillates much slower than $e^{-i\sigma t}$, the dominant contribution to f is its positive frequency part, which implies the same to λ. Therefore,

$$\begin{aligned} |\psi_\mathrm{f}\rangle &\approx \big(\mathrm{I} + a^\dagger\big(\lambda^{(+)}\big)A - a\big(\lambda^{(+)*}\big)A^\dagger\big)\,|\psi_\mathrm{i}\rangle \\ &\approx |\chi\rangle\,|n_\psi\rangle + \sqrt{n+1}\|\lambda^{(+)}\|_\mathcal{H} A\,|\chi\rangle\,|(n+1)'\rangle - \sqrt{n}\,\langle\lambda^{(+)}|\psi\rangle_\mathcal{H}\, A^\dagger\,|\chi\rangle\,|(n-1)'\rangle\,, \end{aligned} \tag{2.90}$$

where

$$|(n+1)'\rangle = a^\dagger\big(\lambda^{(+)}\big)\,|n_\psi\rangle = \frac{1}{\|\lambda^{(+)}\|_\mathcal{H}}(0, \dots, 0, \psi^{(a_1} \cdots \psi^{a_n}\lambda^{(+)a_{n+1})}, 0, \dots). \tag{2.91}$$

If the initial state of two-level system is $|\chi_0\rangle$, it is clear that the only allowed transitions are the one where no interaction occurs, $\psi_\mathrm{i} \to \psi_\mathrm{i}$, and the one to the state $|\chi_1\rangle\,|(n-1)_\psi\rangle$, whose probability is proportional to n. On the other hand, if the two-level system is initially in the state $|\chi_1\rangle$, the only possible transitions are the noninteracting one and the the one to the state $|\chi_0\rangle\,|(n+1)_\psi\rangle$, whose probability is proportional to $n+1$. This makes this system a good model for the interaction of a field and a particle detector, where the state $|\chi_0\rangle$ corresponds to the unexcited state of the detector, $|\chi_1\rangle$ to the excited one and $|(n-1)_\psi\rangle$, $|n_\psi\rangle$ and $|(n+1)'\rangle$ correspond, respectively, to $(n-1)$-, n- and $(n+1)$-particle states of the field. This allows for the interpretation of $N(\mathbf{k}) := a^\dagger(\mathbf{k})a(\mathbf{k})$ as the *number density* operator, counting the number of particles of the state with a certain wave vector $\mathbf{k}$, and of its integral over all wave vectors, N, as the *number* operator, counting the total number of particles of the state. The detection of a particle is in correspondence to the absorption of a particle from the field by the detector, which is accompanied by the excitation of the detector. The detector may also emit a particle if it is initially in the excited space, even if the field is in the vacuum state, illustrating the difference between a field in the vacuum state and the absence of a field.

2.2.3 Field Theory in Curved Spacetimes

To be able to describe a field theory when the dynamics of the underlying spacetime are not trivial, a new set of tools must be introduced. This necessity is a consequence of the lack of global symmetries in generic spacetimes. A host of phenomena with no analogue in flat spacetime QFT[24] are described by this formalism, challenging notions that play prominent roles in many treatments of QFT in flat spacetimes.

While Minkowski spacetime has the structure of a metric vector space and a global symmetry group, the Poincaré group, a spacetime which is a solution of GR has, generically, the structure of a metric manifold but no global symmetry group. The lack of a global symmetry implies that there is no preferred notion of time and, therefore, no preferred Hamiltonian or vacuum state for a field. This leads to multiple, observer dependent notions of particles, as illustrated by the Unruh effect, described in subsection 2.2.4.

The canonical formalism requires a split of spacetime into space and time, implicitly used in subsections 2.2.1 and 2.2.2. Given a spacetime manifold M, this split can be achieved by specifying a foliation of M (see [22]) into a family of 3-dimensional submanifolds Σ_t parameterized by t, called *leaves*, which represent spatial slices, with the *transversals* related to the time evolution (see figure 2). The Lorentzian spacetime metric $g_{\mu\nu}$ induces a Riemannian spatial metric $h_{\mu\nu}$ on Σ_t via

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{2.92}$$

where n^μ is the unit normal vector field of Σ_t. The tangent vector field to the t-coordinate curves, t^μ, can be decomposed into its normal and tangential parts, respectively, the *lapse* N and the *shift* N^μ:

$$t^\mu = Nn^\mu + N^\mu. \tag{2.93}$$

The time derivative of a function f is then derivative in the direction of t^μ, i.e.,

$$\dot{f} := t^\mu \nabla_\mu f = Nn^\mu \nabla_\mu f + N^\mu \nabla_\mu f. \tag{2.94}$$

[24] Note, however, that some phenomena described by QFT in curved spacetimes concern fields in flat spacetimes, as show in subsection 2.2.4.

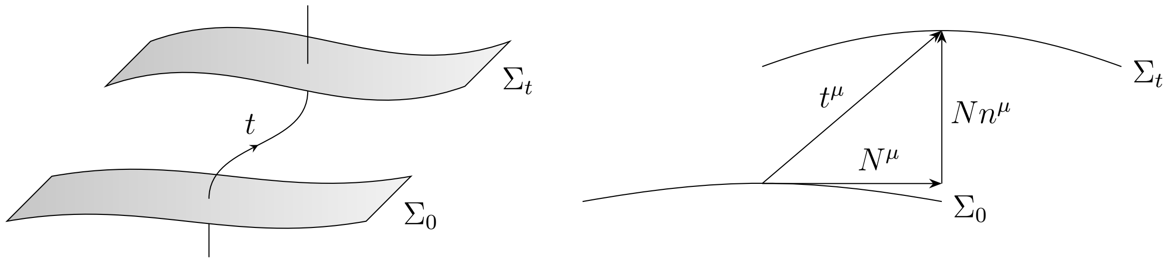

Figure 2 – Illustrations of a typical foliation and of the decomposition of the time direction vector.

Since the intention of this formalism is describing the temporal evolution of observables, the foliations of interest are the ones in which the transversals have the topology of $\mathbb{R}$, which implies that M has the topology of $\mathbb{R} \times \Sigma$ for some 3-dimensional manifold Σ, and are future directed timelike curves. There is a theorem (see [15, 45, 46]) that indicates sufficient conditions on M such that a foliation of this kind can be obtained, which uses the concept of a *globally hyperbolic* spacetime, i.e., a spacetime that admits a Cauchy surface. A *Cauchy surface* Σ for a time orientable spacetime M is a closed achronal set such that its *domain of dependence* $D(\Sigma)$, i.e., the set of all points p such that every past and future inextendible causal curve passing through p intersects Σ (see [15]), is equal to M.

Theorem 2.1. *A globally hyperbolic spacetime M with a Cauchy surface Σ is diffeomorphic to $\mathbb{R} \times \Sigma$ and admits a foliation by a one-parameter family of smooth Cauchy surfaces Σ_t.*

Globally hyperbolic spacetimes are natural choices for field theories, since an initial value formulation for hyperbolic second-order linear differential equations is available. A precise statement is given in the following theorem (see [15] for more details and [47] for its proof).

Theorem 2.2. *A globally hyperbolic spacetime $(M, g_{\mu\nu})$ with Cauchy surface Σ_0 and a connection ∇_μ admits a unique solution for any system of differential equations of n unknown functions $\phi^{\underline{A}}$ of the form*

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi^{\underline{A}} + \sum_{\underline{B}=1}^{n} A^{\underline{A}}{}_{\underline{B}}{}^{\mu}\nabla_\mu\phi^{\underline{B}} + \sum_{\underline{B}=1}^{n} B^{\underline{A}}{}_{\underline{B}}\phi^{\underline{B}} + C^{\underline{A}} = 0, \tag{2.95}$$

where the coefficients $g_{\mu\nu}$, $A^{\underline{A}}{}_{\underline{B}}{}^{\mu}$, $B^{\underline{A}}{}_{\underline{B}}$, $C^{\underline{A}}$ are smooth. The solutions $\phi^{\underline{A}} \in W^{k,\infty}(M)$ depend continuously on the initial data $(\phi_0^{\underline{A}}, n^\mu\nabla_\mu\phi_0^{\underline{A}}) \in H^{k+3}(\Sigma_0) \times H^{k+3}(\Sigma_0)$ on Σ_0 and the restriction of a solution $\phi^{\underline{A}}$ to $D(S)$, where S is a closed subset of Σ_0, depends only on the restriction to S of its initial data.

The dynamics of real free scalar fields on a curved spacetime $(M, g_{\mu\nu})$ are described by the KG Lagrangian density,

$$\mathscr{L} = -\frac{1}{2}\sqrt{-g}(\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2), \tag{2.96}$$

where ∇ is the Levi-Civita connection on M and g is the determinant of the metric, and its associated equation of motion,

$$(\nabla_\mu\nabla^\mu - m^2)\phi = 0, \tag{2.97}$$

which, according to theorem 2.2, has a well posed initial value formulation. The Hamiltonian density is computed with respect to a chosen foliation,

$$\mathscr{H} = N^\mu\nabla_\mu\phi\pi + \frac{1}{2}\frac{N}{\sqrt{h}}\pi^2 + \frac{1}{2}\sqrt{h}\,N\big(h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\big), \tag{2.98}$$

where h is the determinant of the spatial metric and the momentum π is given by

$$\pi = \partial_{\dot\phi}\mathscr{L} = \sqrt{h}\,n^\mu\nabla_\mu\phi. \tag{2.99}$$

As is the case with QFT in flat spacetimes, a symplectic form can be defined on the phase space of the system, but the choice of phase space is different from the one of the theory in flat spacetimes. Recall that $C_0^\infty(\Sigma_t)$ denotes the space of smooth functions of compact support on Σ_t and let $\Omega = C_0^\infty(\Sigma_t)\times C_0^\infty(\Sigma_t)$ be the phase space of initial data on Σ_t of compact support. A strong symplectic form ω on Ω may be defined by

$$\omega\big((\phi,\pi),(\phi',\pi')\big) = \int_{\Sigma_t}\mathrm{d}^3x\,(\phi\pi' - \phi'\pi), \tag{2.100}$$

and it induces a symplectic form on the space $\mathcal{S}$ of solutions of the KG equation with initial data in Ω:

$$\omega(\phi,\phi') = \int_{\Sigma_t}\mathrm{d}^3x\,\sqrt{h}\,n^\mu(\phi\nabla_\mu\phi' - \phi'\nabla_\mu\phi). \tag{2.101}$$

From the symplectic form, the Poisson bracket can be defined as

$$\{F\,,G\} \coloneqq \omega(X_F,X_G), \tag{2.102}$$

and the time flow is generated by the Poisson bracket with the Hamiltonian. Since the Fourier transform is available in flat spacetime, the fundamental Poisson brackets given in equations (2.36) are in a particularly useful form. In curved spacetimes, however, there are no preferred expansion modes and a better way of representing the fundamental brackets is working with functionals on Ω, for which the analogue of equation (1.36) holds:

$$\big\{\omega\big((\phi,\pi),\cdot\big)\,,\omega\big((\phi',\pi'),\cdot\big)\big\} = \omega\big((\phi,\pi),(\phi',\pi')\big). \tag{2.103}$$

Some solutions of the KG equation have properties of great usefulness in the definition of the field operators in the quantum theory. These are the advanced and retarded solutions to the KG equation with a source. It can be shown (see [48]) that there exist maps $A\colon C_0^\infty(M) \to C_0^\infty(M)$ and $R\colon C_0^\infty(M) \to C_0^\infty(M)$ taking a function $f \in C_0^\infty(M)$ to, respectively, the advanced Af and retarded Rf solutions of the KG equation with source f, i.e., putting $P := \nabla_\mu \nabla^\mu - m^2$,

$$PAf = (\nabla_\mu \nabla^\mu - m^2)Af = f, \tag{2.104a}$$

$$PRf = (\nabla_\mu \nabla^\mu - m^2)Rf = f, \tag{2.104b}$$

with the property that $\operatorname{supp}(Af) \subset J^+(\operatorname{supp}(f))$ and $\operatorname{supp}(Rf) \subset J^-(\operatorname{supp}(f))$, where supp denotes the support of a function. Additionally, Af and Rf are the only solutions of equations (2.104) such that the intersection of their support with $J^-(\operatorname{supp}(f))$ and $J^+(\operatorname{supp}(f))$, respectively, is compact. It is clear that Af and Rf are given by

$$Af = \int_M \mathrm{d}^4x \sqrt{-g}\, G_A(x, x') f(x'), \tag{2.105}$$

$$Rf = \int_M \mathrm{d}^4x \sqrt{-g}\, G_R(x, x') f(x'), \tag{2.106}$$

where G_A and G_R are, respectively, the advanced and retarded Green functions of the KG equation. The advanced minus retarded map may also be defined by $E := A - R$, and the image of a function f with respect to E is a solution of the homogeneous KG equation. Important properties of this map are summarized in the following theorem (see [14] for the proof).

Theorem 2.3. *The map $E\colon C_0^\infty(M) \to \mathcal{S}$ fulfills the following properties:*

1. *Every $\phi \in \mathcal{S}$ can be written as $\phi = Ef$ for some $f \in C_0^\infty(M)$, i.e., it is a surjective map.*

2. *The kernel of E is $\ker(E) = PC_0^\infty(M)$, i.e., $Ef = 0$ if and only if $f = Pf'$ for some $f' \in C_0^\infty(M)$. Therefore, $\mathcal{S} = EC_0^\infty(M) = C_0^\infty(M)/PC_0^\infty(M)$.*

3. *For every $\phi \in \mathcal{S}$ and $f \in C_0^\infty(M)$,*

$$\omega(\phi, Ef) = \int_M \mathrm{d}^4x \sqrt{-g}\, f\phi. \tag{2.107}$$

The computation of Poisson brackets may be simplified with the use of property 3 from theorem 2.3. Defining

$$E(f, f') := \int_M \mathrm{d}^4x \sqrt{-g}\, f E f' \tag{2.108}$$

for all $f, g \in C_0^\infty(M)$ and introducing the notation

$$\phi(f) := \int_M \mathrm{d}^4x \sqrt{-g}\, f\phi \tag{2.109}$$

for the smearing functional taking ϕ to its smearing by f, it is simple to check that

$$\{\phi(f)\,,\phi(f')\} = \{\omega(\cdot, Ef)\,,\omega(\cdot, Ef')\} = \omega(Ef, Ef') = -E(f, f'). \tag{2.110}$$

Note that the Poisson bracket computed here is defined for functionals on $\mathcal{S}$ rather than on Ω, since it is the one defined by the symplectic form on $\mathcal{S}$. Smearings also clarify the meaning of equations (2.36). By passing from the Poisson bracket on $\mathcal{S}$ to the one on Ω and denoting by $\phi_{\Sigma_t}(f)$ and $\pi_{\Sigma_t}(f')$ the functionals on Ω taking (ϕ, π) to the smearings of ϕ by $f \in C_0^\infty(\Sigma_t)$ and π by $h^{-1/2}f' \in C_0^\infty(\Sigma_t)$, respectively, the following Poisson bracket can be calculated:

$$\{\phi_{\Sigma_t}(f)\,,\pi_{\Sigma_t}(f')\} = \int_{\Sigma_t} \mathrm{d}^3x\,\sqrt{h}\,ff'. \tag{2.111}$$

But, utilizing the linearity of the Poisson bracket, it is expected that, if all appearing quantities are well defined,

$$\begin{aligned}\{\phi_{\Sigma_t}(f)\,,\pi_{\Sigma_t}(f')\} &= \left\{\int_{\Sigma_t} \mathrm{d}^3x\,\sqrt{h(\mathbf{x})}f(\mathbf{x})\phi(t,\mathbf{x})\,,\int_{\Sigma_t} \mathrm{d}^3x'\,f'(\mathbf{x}')\pi(t,\mathbf{x}')\right\}\\ &= \int_{\Sigma_t\times\Sigma_t} \mathrm{d}^3x\,\mathrm{d}^3x'\,\sqrt{h(\mathbf{x})}f(\mathbf{x})f'(\mathbf{x}')\{\phi(t,\mathbf{x})\,,\pi(t,\mathbf{x}')\}.\end{aligned} \tag{2.112}$$

If the Dirac delta distribution on the right-hand side of equation (2.36c) is interpreted as a densitized Dirac delta distribution, the right-hand sides of equations (2.111) and (2.112) are equal[25]. Similar interpretations can be given to equations (2.36a) and (2.36b).

The construction of the Hilbert space of the theory in flat spacetimes involves, even if indirectly, specifying the positive frequency part of solutions of the KG equation (see [14] for an explicit construction in flat spacetimes). Since no preferred notion of time exist in generic spacetimes, an analogue of this process must be found. One possible construction involves the specification of and inner product $\mu\colon \mathcal{S}\times\mathcal{S}\to\mathbb{R}$ which satisfies, for every $\psi\in\mathcal{S}$,

$$\mu(\psi,\psi) = \frac{1}{4}\sup_{\psi'\in\mathcal{S}-\{0\}}\frac{(\omega(\psi,\psi'))^2}{\mu(\psi',\psi')}. \tag{2.113}$$

This is shown to be equivalent to a Cauchy–Schwarz equality below. The space $\mathcal{S}$ equipped with μ has the structure of a pre-Hilbert space, and, thus, must be Cauchy completed to yield a Hilbert space $\mathcal{S}^{\mu'}$, with inner product $\mu' = 2\mu$. A complex structure J on $\mathcal{S}^{\mu'}$ can then be specified by the inverse process of the one described in subsection 2.2.1, i.e., by

$$\omega(\psi, J\psi') = \mu'(\psi,\psi'), \tag{2.114}$$

since $J^\dagger = -J$ by the antisymmetry of ω and $JJ^\dagger = \mathrm{I}$ by equation (2.113). The complexification of $\mathcal{S}^{\mu'}$, $\mathcal{S}^{\mu'}_{\mathbb{C}}$, is endowed with an antisymmetric form ω, a symmetric form μ' and a

25 The Dirac delta distribution on a manifold is, in general, not translation invariant, so a more appropriate notation would be $\delta(\mathbf{x},\mathbf{x}')$.

complex structure J, the complex linear extensions of the corresponding operators acting on $\mathcal{S}^{\mu'}$, and an inner product for this space can be defined via

$$\langle\psi|\psi'\rangle_{\mathcal{S}^{\mu'}_{\mathbb{C}}} := \mu'(\psi^*, \psi'). \tag{2.115}$$

The Hilbert space $\mathcal{H}$ from which the Fock space for the theory is to be obtained (the "single particle" Hilbert space) is defined as one the eigensubspaces of J. More specifically, since iJ is self-adjoint, J has two eigenvalues, i and $-i$, and the two orthogonal eigensubspaces associated to these eigenvalues span $\mathcal{S}^{\mu'}_{\mathbb{C}}$. The space $\mathcal{H}$ is then defined as the eigensubspace associated to i. It can be shown that $\mathcal{H}$ is a Hilbert space with inner product

$$\langle\psi|\psi'\rangle_{\mathcal{H}} = \langle\psi|\psi'\rangle_{\mathcal{S}^{\mu'}_{\mathbb{C}}}, \tag{2.116}$$

and that $\mathcal{S}^{\mu'}_{\mathbb{C}} = \mathcal{H} \oplus \overline{\mathcal{H}}$, i.e., the eigensubspace of J associated to $-i$ is $\overline{\mathcal{H}}$. The Hilbert space for the theory in curved spacetimes is then taken to be $\mathcal{F}_s(\mathcal{H})$.

A map $K\colon \mathcal{S} \to \mathcal{H}$ with dense range provides an analogue of the specification of a positive frequency subspace of $\mathcal{S}$. Such a map can be defined by the restriction to $\mathcal{S}$ of the orthogonal projection from $\mathcal{S}^{\mu'}_{\mathbb{C}}$ to $\mathcal{H}$ and it satisfies

$$\langle K\psi|K\psi'\rangle_{\mathcal{H}} = \mu(\psi, \psi') + \frac{i}{2}\omega(\psi, \psi'). \tag{2.117}$$

The Cauchy–Schwarz inequality for the inner product on $\mathcal{H}$ is equivalent to

$$\mu(\psi, \psi)\mu(\psi', \psi') \geq \frac{1}{4}(\omega(\psi, \psi'))^2, \tag{2.118}$$

which justifies the assumption of equation (2.113). If that condition is relaxed to allow for any μ satisfying the inequality, rather than the equality, similar results may be obtained (see [49]). In flat spacetimes, the preferred choice for K is given by the Fourier transform of $a(\mathbf{k})$ (see equations (2.42) and (2.47)).

Quantization of the theory is achieved with the implementation of the CCR corresponding to equation (2.110),

$$[\phi(f)\,, \phi(f')] = -iE(f, f')\mathrm{I}. \tag{2.119}$$

Defining the operator-valued functional

$$\phi(f) := ia((KEf)^*) - ia^\dagger(KEf), \tag{2.120}$$

it can be verified that the CCR are satisfied, i.e.,

$$\begin{aligned}
[\phi(f)\,, \phi(f')] &= [a((KEf)^*)\,, a^\dagger(KEf')] - [a((KEf')^*)\,, a^\dagger(KEf)] \\
&= (\langle KEf|KEf'\rangle_{\mathcal{H}} - \langle KEf'|KEf\rangle_{\mathcal{H}})\mathrm{I} = 2i\,\mathrm{Im}(\langle KEf|KEf'\rangle_{\mathcal{H}})\mathrm{I} \\
&= i\omega(Ef, Ef')\mathrm{I} = -iE(f, f')\mathrm{I}.
\end{aligned} \tag{2.121}$$

An important quantity for the quantum theory is the *Wightman function* (also called the *two-point function*) of a field, given by the vacuum expectation value of the product of two fields at different points in spacetime. The smeared version of this function has a particularly nice form,

$$\langle 0|\phi(f)\phi(f')|0\rangle = \langle KEf|KEf'\rangle_{\mathcal{H}} = \mu(Ef, Ef') + \frac{i}{2}\omega(Ef, Ef'). \tag{2.122}$$

This function, along with its generalizations, the n-point functions, play a key role in constructive QFT, since it can be show that any QFT in flat spacetimes can be reconstructed using only the information contained in these functions (see [41] for the statement and proof of the reconstruction theorem).

The construction presented above can always be achieved in any globally hyperbolic spacetime (see [14]), but multiple choices can be made for the inner products on $\mathcal{S}$, so it is natural to question whether different choices yield different physical predictions. Two QFTs are said to be unitarily equivalent if there exists a unitary map $U\colon \mathcal{F}_1 \to \mathcal{F}_2$ between the Hilbert spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ of each theory, here taken to be the Fock spaces $\mathcal{F}_1 = \mathcal{F}_s(\mathcal{H}_1)$ and $\mathcal{F}_1 = \mathcal{F}_s(\mathcal{H}_2)$, such that the corresponding fields are related via

$$U\phi_1(f)U^{-1} = \phi_2(f). \tag{2.123}$$

If two theories are unitarily equivalent, all physical predictions obtained from them are equal.

A necessary condition for unitary equivalence is the strong equivalence of the inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$. Two inner products, μ_1 on $\mathcal{H}_1$ and μ_2 on $\mathcal{H}_2$, are said to be strongly equivalent if there exist constants $C, C' > 0$ such that, for every $\phi \in \mathcal{S}$,

$$C\mu_1(\phi,\phi) \leq \mu_2(\phi,\phi) \leq C'\mu_1(\phi,\phi). \tag{2.124}$$

Strong equivalence of inner products implies strong metric equivalence and, in particular, that every sequence which converges in the norm induced by μ_1 also converges in the norm induced by μ_2, and vice-versa. Unitary equivalence relies on this convergence property and, therefore, is impossible for Hilbert spaces with inequivalent inner products. If the two Hilbert spaces in question are equipped with equivalent inner products, the Cauchy completions of $\mathcal{S}$ in the norms induced by μ_1' and μ_2', defined as above, are equal (due to strong metric equivalence) and denoted $\mathcal{S}^{\mu'}$. It is then possible, in light of the constructions above, to see $\mathcal{H}_1$ and $\mathcal{H}_2$ as subspaces of $\mathcal{S}^{\mu'} = \mathcal{H}_1 \oplus \overline{\mathcal{H}}_1 = \mathcal{H}_2 \oplus \overline{\mathcal{H}}_2$. Comparing the orthogonal projection maps $K_k\colon \mathcal{S}^{\mu'}_{\mathbb{C}} \to \mathcal{H}_k$ and $K_k^*\colon \mathcal{S}^{\mu'}_{\mathbb{C}} \to \overline{\mathcal{H}}_k$, with $k \in \{1,2\}$ and which satisfy $K_k + K_k^* = \mathrm{I}$, can be done by analyzing the restriction of these maps to the Hilbert spaces of the other theory. Defining $A\colon \mathcal{H}_2 \to \mathcal{H}_1$ to be the restriction of K_1 to $\mathcal{H}_2$, $B\colon \mathcal{H}_2 \to \overline{\mathcal{H}}_1$ the restriction of K_1^* to $\mathcal{H}_2$, $C\colon \mathcal{H}_1 \to \mathcal{H}_2$ the restriction of K_2 to $\mathcal{H}_1$ and $D\colon \mathcal{H}_1 \to \overline{\mathcal{H}}_2$ the restriction of K_2^* to $\mathcal{H}_1$, it can be seen that the strong equivalence for the inner products implies that these operators are bounded and that, if $\psi, \chi \in \mathcal{H}_2$,

$$\begin{aligned}\langle\psi|\chi\rangle_{\mathcal{H}_2} &= i\omega(\psi^*, \chi) = i\omega([(K_1 + K_1^*)\psi]^*, (K_1 + K_1^*)\chi) \\ &= \omega((K_1\psi)^*, J_1K_1\chi) - \omega((K_1^*\psi)^*, J_1K_1^*\chi) \\ &= \langle A\psi|A\chi\rangle_{\mathcal{H}_1} - \langle B\psi|B\chi\rangle_{\overline{\mathcal{H}}_1},\end{aligned} \tag{2.125}$$

since $\mathcal{H}_1$ is orthogonal to $\overline{\mathcal{H}}_2$, which also gives, if $\psi \in \overline{\mathcal{H}}_2$ and $\chi \in \mathcal{H}_2$,

$$\begin{aligned}\langle\psi|\chi\rangle_{\mathcal{S}_{\mathbb{C}}^{\mu'}} &= i\omega(\psi^*, \chi) = i\omega([(K_1 + K_1^*)\psi]^*, (K_1 + K_1^*)\chi) \\ &= \omega((K_1^*\psi^*), J_1K_1\chi) - \omega((K_1\psi^*), J_1K_1^*\chi) \\ &= \langle B^*\psi|A\chi\rangle_{\mathcal{H}_1} - \langle A^*\psi|B\chi\rangle_{\overline{\mathcal{H}}_1} = 0.\end{aligned} \tag{2.126}$$

Similar relations also hold for C and D. Finally, for $\psi \in \mathcal{H}_1$ and $\chi \in \mathcal{H}_2$,

$$\begin{aligned}\langle\psi|A\chi\rangle_{\mathcal{H}_1} &= i\omega(\psi^*, K_1\chi) = i\omega(\psi^*, (K_1 + K_1^*)\chi) = i\omega(\psi^*, \chi) \\ &= i\omega([(K_2 + K_2^*)\psi]^*, \chi) = i\omega((K_2\psi)^*, \chi) = \langle C\psi|\chi\rangle_{\mathcal{H}_2}.\end{aligned} \tag{2.127}$$

These relations give

$$A^\dagger A - B^\dagger B = \mathrm{I}, \tag{2.128a}$$
$$A^\dagger B^* = B^\dagger A^*, \tag{2.128b}$$
$$C^\dagger C - D^\dagger D = \mathrm{I}, \tag{2.128c}$$
$$C^\dagger D^* = D^\dagger C^*, \tag{2.128d}$$
$$A^\dagger = C, \tag{2.128e}$$
$$B^\dagger = -D^*. \tag{2.128f}$$

A unitary transformation satisfying equation (2.123), which can be reduced to

$$Ua_1(\chi^*)U^{-1} = a_2((C\chi)^*) - a_2^\dagger((D\chi)^*) \tag{2.129}$$

for any $\chi \in \mathcal{H}_1$, together with the operators A, B, C and D satisfying equations (2.128) is called a *Bogoliubov transformation*. The conditions for the existence of a Bogoliubov transformation can be found by considering the action of U on the vacuum state of $\mathcal{F}_1$,

$$\Psi = U\,|0\rangle_1 = c(1, \psi^a, \psi^{ab}, \psi^{abc}, \ldots), \tag{2.130}$$

where c is a constant, and which, in conjunction with equation (2.129), gives

$$(a_2(\xi^*) - a_2^\dagger(E\xi^*))\Psi = 0, \tag{2.131}$$

for every $\xi \in \mathcal{H}_2$, where $\xi = C\chi$ for some $\chi \in \mathcal{H}_1$ and $E = D^*C^{*-1}$. Expanding Ψ in terms of its Fock space components, an infinite set of coupled equations is obtained, the first four of which are:

$$\xi_a^*\psi^a = 0, \tag{2.132a}$$
$$\sqrt{2}\xi_a^*\psi^{ab} = \xi^{*a'}E^b{}_{a'}, \tag{2.132b}$$
$$\sqrt{3}\xi_a^*\psi^{abc} = \sqrt{2}\xi^{*a'}E^{(b}{}_{a'}\psi^{c)}, \tag{2.132c}$$
$$\sqrt{4}\xi_a^*\psi^{abcd} = \sqrt{3}\xi^{*a'}E^{(b}{}_{a'}\psi^{cd)}. \tag{2.132d}$$

The solution to this system of equations is given by

$$\Psi = c(1, 0, \sqrt{\frac{1!!}{2!!}}E^{ab}, 0, \sqrt{\frac{3!!}{4!!}}E^{(ab}E^{cd)}, \dots), \tag{2.133}$$

where $n!!$ denotes the double factorial of n. This state is only well defined if E is symmetric, which follows from equation (2.128d), and is in $\mathcal{H}_2 \otimes \mathcal{H}_2$, which requires that[26] $\mathrm{tr}(E^\dagger E) < \infty$ and is equivalent[27] to either $\mathrm{tr}(B^\dagger B) < \infty$ or $\mathrm{tr}(D^\dagger D) < \infty$. Alternatively, this second condition can be described by the statement that the map $Q\colon \mathcal{S}^{\mu'} \to \mathcal{S}^{\mu'}$, defined via

$$\mu_1(\phi, Q\phi') = \mu_1(\phi, \phi') - \mu_2(\phi, \phi'), \tag{2.134}$$

fulfills $\mathrm{tr}(Q) < \infty$. The action of U on $|0\rangle_1$ can be extended to the whole of $\mathcal{F}_1$, since $|0\rangle_1$ is cyclic, and U can be shown to be unitary if the constant c is defined by normalizing Ψ.

An important note is that all the conditions delineated above are satisfied in the finite dimensional case, meaning that all representations of finite families of harmonic oscillators are unitarily equivalent, in accordance with the celebrated Stone–von Neumann theorem (see [50]). There are simple examples in infinite dimensions where unitarilly inequivalent constructions can be found (see [14]). A physical application of the construction of the map U is when the spacetime in question is asymptotically stationary (see subsection 2.2.4) in the future and the past, with $U\colon \mathcal{F}_s(\mathcal{H}_{\text{in}}) \to \mathcal{F}_s(\mathcal{H}_{\text{out}})$ corresponding to the S-matrix mapping the Hilbert space of in-going states, $\mathcal{F}_s(\mathcal{H}_{\text{in}})$, to the one of out-going states, $\mathcal{F}_s(\mathcal{H}_{\text{out}})$.

2.2.4 The Unruh Effect

One of the main predictions of QFT in curved spacetimes also applies to QFTs in flat spacetimes: the *Fulling–Davies–Unruh effect* or, simply, *Unruh effect*. It states that a uniformly accelerating observer in Minkowski spacetime perceives the state of a field determined to be the vacuum state by inertial observers as a thermal state, i.e., a thermal bath of particles. In this subsection, the effect is derived for a more general class of spacetimes, *stationary spacetimes*. Emerging questions about the nature of particles may

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26 This is known as the *Hilbert–Schmidt condition*, and is equivalent to the statement that the operator is of finite norm with respect to the inner product $\langle A|B\rangle = \mathrm{tr}(A^\dagger B)$.

27 Equations (2.128a) and (2.128c) imply that A^{-1} and C^{-1} exist and are bounded.

be answered by careful analysis of the domain of validity of this concept. In chapter 3, the consequences of the Unruh effect for particle decay will be presented. Historical references on the Unruh effect are [7–9]. A comprehensive review can be found in [51] and the derivation presented here is based on [14].

A uniformly accelerating trajectory[28] in Minkowski spacetime is one with constant *proper acceleration*. To define the proper acceleration (in a generic spacetime), the 4-acceleration a^μ must be defined,

$$a^\mu := u^\nu \nabla_\nu u^\mu, \tag{2.135}$$

where u^μ is the 4-velocity of the trajectory, i.e., its tangent vector in the proper time parametrization, satisfying $u_\mu u^\mu = -1$. The acceleration measures the failure of the trajectory to be geodesic and indicates that some non-gravitational force $f^\mu = m a^\mu$ is present. The magnitude $a = \sqrt{a_\mu a^\mu}$ of the 4-acceleration is called the proper acceleration. Returning to Minkowski spacetime and introducing an inertial coordinate system (t, x, y, z) with a suitable choice for the origin and such that the x-direction is the direction of motion of the trajectories of interest, it is evident that the 4-velocity of a family of uniformly accelerating trajectories indexed by a is given by

$$u^I = a[x(\partial_t)^I + t(\partial_x)^I] \tag{2.136}$$

and its integral curves are determined by $u_I u^I = -1$, which gives

$$x^2 - t^2 = \frac{1}{a^2}. \tag{2.137}$$

The trajectories are timelike curves only in the region defined by $|x| > |t|$ and future directed only in the region defined by $x > |t|$. Figure 3 illustrates the behaviour of these objects.

The vector field u^K introduced above is a *Killing vector field* for the Minkowski metric η_{KL}, i.e., the flow generated by it is metric preserving (an *isometry*). This means that the Lie derivative of the metric with respect to u^K vanishes:

$$\mathcal{L}_u \eta_{IJ} = u^K \partial_K \eta_{IJ} + \eta_{KJ} \partial_I u^K + \eta_{IK} \partial_J u^K = 2\,\partial_{(I} u_{J)} = 0. \tag{2.138}$$

The concept of a Killing field can be generalized to a generic spacetime $(M, g_{\mu\nu})$ by imposing the condition above and noting that the Lie derivative in the direction of the Killing field χ^μ is now taken with respect to the Levi-Civita connection:

$$\mathcal{L}_\chi g_{\mu\nu} = 2\nabla_{(\mu}\chi_{\nu)} = 0. \tag{2.139}$$

Isometries have the special property that their action on a connected manifold M may be completely recovered from their action on a point p and of their pushforward on T_pM.

[28] A trajectory is understood to be a timelike curve in spacetime.

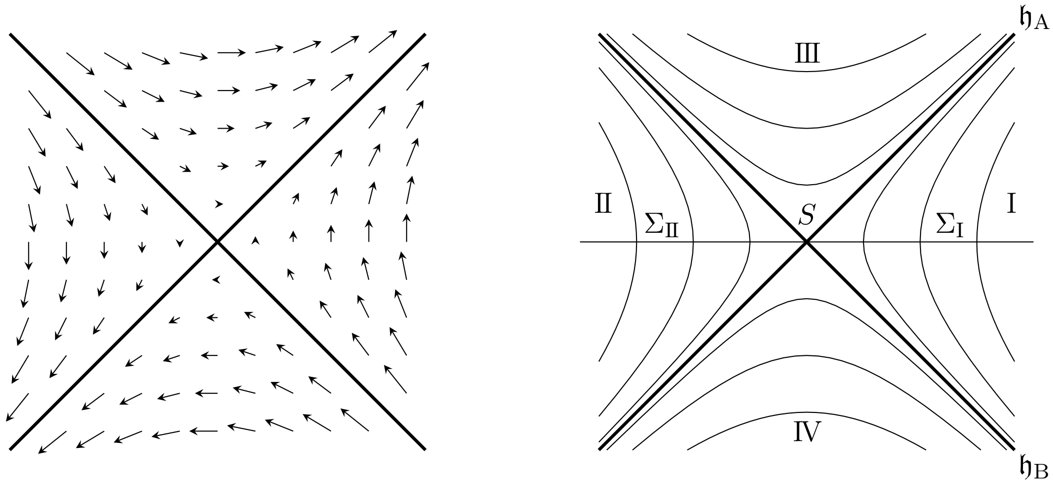

Figure 3 – Minkowski diagrams illustrating the vector field u^μ and its integral curves (adapted from [14]). Labeled are the Cauchy surfaces, Σ_{I} and Σ_{II}, the Killing horizons, $\mathfrak{h}_{\mathrm{A}}$ and $\mathfrak{h}_{\mathrm{B}}$ (which divide the diagram into regions I–IV), and their intersection S.

Extending this property to Killing fields allows for the characterization of a one-parameter group of isometries by its behaviour around a 2-dimensional spacelike surface S whenever the Killing field vanishes on S (see [14]), which leads to the conclusion that the behaviour of the field is locally well represented by figure 3. Two null surfaces orthogonal to S, $\mathfrak{h}_{\mathrm{A}}$ and $\mathfrak{h}_{\mathrm{B}}$, arise in this construction and constitute a *bifurcate Killing horizon*, separating the spacetime (at least locally) into four regions, called *wedges*, enumerated I–IV as in figure 3. Assuming global hyperbolicity for the spacetime, with a Cauchy surface intersecting S, these regions are given by

$$\text{wedge I} = I^-(\mathfrak{h}_{\mathrm{A}}) \cap I^+(\mathfrak{h}_{\mathrm{B}}), \tag{2.140a}$$

$$\text{wedge II} = I^+(\mathfrak{h}_{\mathrm{A}}) \cap I^-(\mathfrak{h}_{\mathrm{B}}), \tag{2.140b}$$

$$\text{wedge III} = J^+(S), \tag{2.140c}$$

$$\text{wedge IV} = J^-(S). \tag{2.140d}$$

It can be seen that χ^μ is normal to the Killing horizon and, since the horizon is comprised of null surfaces, that $\chi_\mu \chi^\mu = 0$ on it, which implies that $\nabla_\mu(\chi_\nu\chi^\nu)$ is also normal to the Killing horizon. The function relating these two vector fields on the horizon is called the *surface gravity* κ of the horizon, given by

$$\nabla_\mu(\chi_\nu\chi^\nu) = -2\kappa\chi_\mu. \tag{2.141}$$

An explicit expression for κ can be found,

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu\chi_\nu)(\nabla^\mu\chi^\nu), \tag{2.142}$$

which can be rewritten as limit approaching the horizon,

$$\kappa^2 = -\lim \frac{(\chi^\nu\nabla_\nu\chi^\mu)(\chi^\rho\nabla_\rho\chi_\mu)}{\chi_\sigma\chi^\sigma}. \tag{2.143}$$

The acceleration of the Killing field,

$$a^\mu = \frac{\chi^\nu \nabla_\nu \chi^\mu}{\chi_\sigma \chi^\sigma}, \tag{2.144}$$

can be recognized in the expression above, yielding

$$\kappa = \lim a\chi, \tag{2.145}$$

where $a = \sqrt{a_\mu a^\mu}$ and $\chi = \sqrt{\chi_\mu \chi^\mu}$ may be identified with a sort of "gravitational redshift factor" in the case of an asymptotically flat spacetime (see [15]), giving κ the interpretation of the limit approaching the horizon of the redshifted proper acceleration. It can be shown that κ is constant on the whole bifurcate[29] Killing horizon.

A spacetime with a one-parameter group of isometries α_t with timelike orbits is called *stationary*. A special QFT can be constructed in globally hyperbolic spacetimes of this kind, with the "single particle" Hilbert space corresponding to the space of positive frequency solutions of the KG equation with respect to the *Killing time* t corresponding to the family of isometries (an affine parameter for its orbits). It involves specifying an inner product on $\mathcal{S}_\mathbb{C}$ with the use of the KG stress-energy tensor,

$$T_{\mu\nu}(\phi, \phi') = \nabla_{(\mu}\phi^* \nabla_{\nu)}\phi' - \frac{1}{2} g_{\mu\nu}(\nabla^\rho \phi^* \nabla_\rho \phi' + m^2 \phi^* \phi'). \tag{2.146}$$

Defining the inner product $\langle \cdot \,|\, \cdot \rangle_T : \mathcal{S}_\mathbb{C} \times \mathcal{S}_\mathbb{C} \to \mathbb{C}$ by

$$\langle \phi | \phi' \rangle_T \coloneqq \int_\Sigma \mathrm{d}^3x \sqrt{h}\, T_{\mu\nu}(\phi, \phi') \chi^\mu n^\nu, \tag{2.147}$$

where χ^μ is the Killing field corresponding to the family of isometries, Σ is a Cauchy surface and n^ν is normal to Σ, it can be seen that it is independent of the choice of Σ and invariant under Killing time translations, since

$$\nabla_\mu (T^{\mu\nu}\chi_\nu) = \nabla_\mu (T^{(\mu\nu)}\chi_\nu) = \nabla_{(\mu}(T^{\mu\nu}\chi_{\nu)}) = 0 \tag{2.148}$$

and the Killing time translation $\tau_t \colon \mathcal{S}_\mathbb{C} \to \mathcal{S}_\mathbb{C}$ of $\phi \in \mathcal{S}_\mathbb{C}$ is defined by $\tau_t \phi \coloneqq \phi \circ \alpha_{-t}$, which implies that $\tau_t \phi$ evaluated on Σ is equal to ϕ evaluated on $\alpha_{-t}\Sigma$. The Cauchy completion of $\mathcal{S}_\mathbb{C}$ in the norm induced by this inner product is denoted $\tilde{\mathcal{H}}$ and the Killing time translation can be extended via continuity in the strong operator topology to $\tilde{\mathcal{H}}$. It clearly forms a one-parameter unitary group and, by Stone's theorem (see [50]),

$$\tau_t = e^{-i\tilde{H}t}, \tag{2.149}$$

where $\tilde{H}$ is the self-adjoint operator generating the time translations on $\tilde{\mathcal{H}}$, i.e.,

$$\mathcal{L}_\chi \phi = \tilde{H}\phi. \tag{2.150}$$

29 This result can be extended to non-bifurcate Killing horizons (see [14]).

Under suitable conditions, it can be shown that the spectrum of $\tilde{H}$ has a mass gap and that $\tilde{H}^{-1}$ is bounded. Putting $\tilde{\mathcal{H}}^{+}$ as the subspace of $\tilde{\mathcal{H}}$ spanned by the eigenstates of $\tilde{H}$ with positive eigenvalues, the projection map $K\colon \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^{+}$ may be defined alongside with an inner product $\mu\colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$, given by

$$\mu(\phi, \phi') \coloneqq 2\,\mathrm{Re}(\langle K\phi|\tilde{H}^{-1}K\phi'\rangle_T). \tag{2.151}$$

This inner product can be used to repeat the process introduced in subsection 2.2.3 and the QFT obtained has as "single particle" Hilbert space the Cauchy completion of $\tilde{\mathcal{H}}^{+}$ in the norm induced by the KG inner product, which may be interpreted as the space of "positive frequency" solutions with respect to t. There is, thus, a notion of particles corresponding to the Killing time t arising from the particle detector construction introduced in subsection 2.2.2.

To derive the Unruh effect in the context of stationary spacetimes, the concepts of accelerated and inertial trajectories must be delineated and the QFTs corresponding to the choices of proper time along these trajectories must be constructed. Assuming the existence of some Killing field χ^μ with a bifurcate Killing horizon on a globally hyperbolic spacetime, it can be seen that wedges I and II, as defined above, are themselves globally hyperbolic spacetimes with Cauchy surfaces $\Sigma_{\mathrm{I}} = \Sigma \cap \text{wedge I}$ and $\Sigma_{\mathrm{II}} = \Sigma \cap \text{wedge II}$ if Σ is a Cauchy surface intersecting S, the surface where the Killing field vanishes. Choosing χ^μ as the Killing field (assumed, furthermore, to be globally timelike on wedge I) for the QFT on wedge I (seen as a stationary spacetime) and $-\chi^\mu$ for the QFT on wedge II (since χ^μ is past-directed on wedge II), a QFT on wedge I $\cup$ wedge II can be constructed with "single particle" Hilbert space given by $\mathcal{H} = \mathcal{H}_{\mathrm{I}} \oplus \mathcal{H}_{\mathrm{II}}$, where $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ are the "single particle" Hilbert spaces for the QFTs on wedges I and II, respectively. The states in the Fock space of this QFT, $\mathcal{F}_s(\mathcal{H}) = \mathcal{F}_s(\mathcal{H}_{\mathrm{I}}) \otimes \mathcal{F}_s(\mathcal{H}_{\mathrm{II}})$, are the ones described by observers on the accelerated trajectories, i.e., the orbits of the Killing fields. An extension of the idea of inertial trajectories can be found by considering the null geodesics tangent to the Killing field generating one of the Killing horizons. Equation (2.141) implies

$$\chi^\nu \nabla_\nu(\chi^\mu) = \kappa\chi^\mu \tag{2.152}$$

because χ^ν is a Killing field, i.e., $\nabla_{(\nu}\chi_{\mu)} = 0$, but this equation is also the geodesic equation for a nonaffinely parameterized curve with χ^μ as its tangent field. To find an affine parameter V (also called *affine time*) for this geodesic (which is analogous to the proper time for timelike geodesics), one must determine the change of parameter under which this equation takes the form of the geodesic equation (2.4). A straightforward calculation yields that

$$V = e^{\kappa v}, \tag{2.153}$$

where v is the *Killing time* (the parameter for which the curve satisfies equation (2.152)), is a valid choice for $V > 0$. The relation between the affine and Killing times can then be extended via

$$v = \frac{1}{\kappa} \ln|V|. \tag{2.154}$$

This choice is clearly adequate for the geodesics in the Killing horizon[30] $\mathfrak{h}_{\mathrm{A}}$, while for the ones on $\mathfrak{h}_{\mathrm{B}}$ it is a parameter U such that

$$u = -\frac{1}{\kappa} \ln|U|, \tag{2.155}$$

where u is, again, the Killing time on $\mathfrak{h}_{\mathrm{B}}$. The specification of trajectories only on the horizon is enough for the purposes of this derivation because every solution of the KG equation is uniquely determined by its value on $\mathfrak{h}_{\mathrm{A}} \cup \mathfrak{h}_{\mathrm{B}}$, at least for most spacetimes of interest (see [49] for more details).

Computing the S-matrix between QFTs with Killing time and affine time is sufficient to demonstrate the existence of the Unruh effect. The first step of the process assuming that there exists a QFT with a map $K\colon \mathcal{S} \to \mathcal{H}_0$, where $\mathcal{H}_0$ is the "single particle" Hilbert space, such that $K\phi$ restricted to the bifurcate horizon is a (nontrivial) solution of the KG equation with positive frequency with respect to V on $\mathfrak{h}_{\mathrm{A}}$ and with respect to U on $\mathfrak{h}_{\mathrm{B}}$.[31] The second step is to add the hypothesis that there exists an isometry ι which reflects points in wedge I to those on wedge II (and vice versa) and across the Cauchy surface Σ, leaving Σ invariant, with the additional condition that ι commutes with the family of isometries induced by the Killing fields in question.[32] To compute the S-matrix, a family $\{\psi_{\mathrm{I}\omega}\}$ of "plane wave" solutions of the KG equation, which oscillate with (positive) frequency ω with respect to the Killing time on wedge I and characterized by its vanishing on wedge II, is considered. This family spans $\mathcal{H}_{\mathrm{I}}$, though it is not contained in it. Restricting $\psi_{\mathrm{I}\omega}$ to the horizon $\mathfrak{h}_{\mathrm{A}}$ gives the function

$$f_{\mathrm{I}\omega}(V, \boldsymbol{\xi}) = h(\boldsymbol{\xi}) e^{-i\omega v} \theta(V), \tag{2.156}$$

where $\boldsymbol{\xi} \in \Sigma$, h is some function on $\mathfrak{h}_{\mathrm{A}}$ and θ is the unit step function. The matrices A, B, C, D and E for the Bogoliubov transformation may be obtained by performing Fourier analysis on f,

$$\hat{f}_{\mathrm{I}\omega}(\sigma, \boldsymbol{\xi}) = \frac{h(\boldsymbol{\xi})}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}V \, e^{i(\sigma V - \omega/\kappa \ln(V))}, \tag{2.157}$$

since it can be shown that any solution of the KG equation is uniquely determined by its restriction on the bifurcate horizon. Rotating the integration contour on the complex plane and using the relation between branches of the logarithm, the following relation can be obtained for $\sigma > 0$:

$$\hat{f}_{\mathrm{I}\omega}(-\sigma, \boldsymbol{\xi}) = -e^{-\pi\omega/\kappa} \hat{f}_{\mathrm{I}\omega}(\sigma, \boldsymbol{\xi}). \tag{2.158}$$

[30] This horizon is split into two parts, one laying in the causal past of S and one in the causal future of S. The Killing time v takes values on the whole of $\mathbb{R}$ in each part, which explains why the relation is not bijective. The same applies to the horizon $\mathfrak{h}_{\mathrm{B}}$ and to the parameters u and U.

[31] This is strongly motivated by results on *Hadamard states*, which can be found in [49].

[32] This hypothesis turns out to be somewhat tame, as discussed in [49].

The isometry ι induces a mapping between $\{\psi_{\mathrm{I}\omega}\}$ and a similar family of harmonically oscillating solutions with frequency ω and vanishing on wedge I, denoted $\{\psi^*_{\mathrm{I}\omega}\}$, which spans $\overline{\mathcal{H}}_{\mathrm{I}}$. The restriction of one of these functions to $\mathfrak{h}_{\mathrm{A}}$ is given by

$$f^*_{\mathrm{II}\omega}(V,\boldsymbol{\xi}) = h(\boldsymbol{\xi})e^{-i\omega v}\theta(-V), \tag{2.159}$$

and its Fourier transform is given by

$$\hat{f}^*_{\mathrm{I}\omega}(\sigma,\boldsymbol{\xi}) = \hat{f}_{\mathrm{I}\omega}(-\sigma,\boldsymbol{\xi}), \tag{2.160}$$

since ι commutes with the isometries generated by the Killing fields. This implies that the function

$$F_\omega(V,\boldsymbol{\xi}) \coloneqq f_{\mathrm{I}\omega}(V,\boldsymbol{\xi}) + e^{-\pi\omega/\kappa} f^*_{\mathrm{II}\omega}(V,\boldsymbol{\xi}) \tag{2.161}$$

oscillates with purely positive frequency with respect to V. The same procedure can be applied to functions on $\mathfrak{h}_{\mathrm{B}}$, from where it can be concluded that

$$F'_\omega(V,\boldsymbol{\xi}) \coloneqq f_{\mathrm{II}\omega}(V,\boldsymbol{\xi}) + e^{-\pi\omega/\kappa} f^*_{\mathrm{I}\omega}(V,\boldsymbol{\xi}) \tag{2.162}$$

oscillates with purely positive frequency with respect to U. If F_ω and F'_ω are the restrictions of solutions Ψ_ω and Ψ'_ω of the KG equation, the action of the matrices C, D and E on Ψ_ω may be obtained, even if only formally[33]:

$$C\Psi_\omega = \psi_{\mathrm{I}\omega}, \tag{2.163a}$$
$$C\Psi'_\omega = \psi_{\mathrm{II}\omega}, \tag{2.163b}$$
$$D\Psi_\omega = e^{-\pi\omega/\kappa}\psi^*_{\mathrm{I}\omega}, \tag{2.163c}$$
$$D\Psi'_\omega = e^{-\pi\omega/\kappa}\psi^*_{\mathrm{II}\omega}, \tag{2.163d}$$
$$E\psi_{\mathrm{I}\omega} = e^{-\pi\omega/\kappa}\psi^*_{\mathrm{I}\omega}, \tag{2.163e}$$
$$E\psi_{\mathrm{II}\omega} = e^{-\pi\omega/\kappa}\psi^*_{\mathrm{II}\omega}. \tag{2.163f}$$

The action of the S-matrix U on the vacuum state of the QFT defined by the affine time, $|0_M\rangle$, is then given purely in terms of E^{ab} and may be written as:

$$U\,|0_M\rangle = \prod_\omega \sum_{n_\omega=0}^{\infty} e^{-\pi n_\omega \omega/\kappa}\,|n_{\mathrm{I}\omega}\rangle \otimes |n_{\mathrm{II}\omega}\rangle\,, \tag{2.164}$$

where $|n_{\mathrm{I}\omega}\rangle \in \mathcal{F}_s(\mathcal{H}_{\mathrm{I}})$ and $|n_{\mathrm{II}\omega}\rangle \in \mathcal{F}_s(\mathcal{H}_{\mathrm{II}})$ are the "n-particle" states corresponding to the modes $\psi_{\mathrm{I}\omega}$ and $\psi_{\mathrm{II}\omega}$, respectively. The restriction of this state to wedge I is represented by the density matrix $\rho = \mathrm{tr}(U|0_M\rangle\langle 0_M|U^\dagger)$, where the trace is taken over the states in wedge II:

$$\rho = \prod_\omega \sum_{n_\omega=0}^{\infty} e^{-2\pi n_\omega \omega/\kappa}|n_{\mathrm{I}\omega}\rangle\langle n_{\mathrm{I}\omega}|. \tag{2.165}$$

[33] This action can be made rigorous by considering suitable wavepackets constructed from the "plane wave" modes instead of those modes.

This state is easily recognized to be a bosonic thermal state with temperature

$$T = \frac{\kappa}{2\pi}, \tag{2.166}$$

called the *Unruh temperature*, if it is "normalized" by the partition function, yielding

$$\rho = \prod_{\omega}(1 - e^{-2\pi\omega/\kappa}) \sum_{n_\omega=0}^{\infty} e^{-2\pi n_\omega \omega/\kappa} |n_{\mathrm{I}\omega}\rangle\langle n_{\mathrm{I}\omega}|. \tag{2.167}$$

This indicates that accelerated observers in wedge I perceive themselves to be immersed in a thermal bath of particles when the field is in the vacuum state as described by the inertial observers. The temperature of this thermal bath is measured by the accelerated observers to be

$$T = \frac{\kappa}{2\pi\chi}, \tag{2.168}$$

where χ is the gravitational redshift factor, and, having equation (2.145) in mind, it can be seen that, near the horizon, the measured temperature is actually

$$T \approx \frac{a}{2\pi}, \tag{2.169}$$

where a is the acceleration of their orbits. In particular, when the spacetime is taken to be the Minkowski spacetime coordinatized by (t, x, y, z), χ^μ is taken to be the 4-velocity u^I of a uniformly accelerated observer moving in the x-direction and the isometry ι is taken to be the mapping $(t, x, y, z) \mapsto (-t, -x, y, z)$, the original statement of the Unruh effect is recovered, with the temperature of the thermal state being exactly

$$T = \frac{a}{2\pi}. \tag{2.170}$$

A few remarks on the physical interpretation of the Unruh effects must be made. Firstly, the effect should be detectable only on extremely high acceleration scales, as can be seen by restoring the SI units to equation (2.170):

$$T = \frac{\hbar a}{2\pi k_{\mathrm{B}} c} \approx a \times 4.055 \times 10^{-21}\,\frac{\mathrm{K\,s^2}}{\mathrm{m}}. \tag{2.171}$$

The prediction that different observers disagree on the particle content of spacetime may also be worrying if QFT is seen as a theory of *quantum particles*, but, as its name suggests, QFT describes *quantum fields*. This prediction just leads to the conclusion that a global particle number operator is not a Dirac observable, although proper detection rates for particle detectors are (see [51]). Particles are not a fundamental concept in the theory but a (very useful) frame dependent interpretation of special states on Fock space, and no mention of this interpretation must be made in order to introduce the formalism. References to particles made above were either explicit about how this interpretation arises or used as shorthand for ideas that rely only on the concept of fields. There is also no disagreement between observations made, e.g., by inertial and accelerated observers in Minkowski spacetime: if the accelerated observer absorbs a particle from the thermal bath, the inertial observer sees that a particle is emitted by the accelerated observer (see [44] for details), which illustrates that any interaction between the field and the accelerated observer is perceived to happen by both observers (although their interpretations might differ). Finally, the correlations between the causally disconnected regions wedge I and wedge II appearing in equation (2.164) are not alarming since the Wightman function of the KG field in Minkowski spacetime, which measures correlations in the vacuum state, is nonvanishing at spacelike separations.

Some problems of mathematical nature are present in this derivation, the main one being that the state given in the right-hand side equation (2.164) is not a state in $\mathcal{F}_s(\mathcal{H}) = \mathcal{F}_s(\mathcal{H}_{\mathrm{I}}) \otimes \mathcal{F}_s(\mathcal{H}_{\mathrm{II}})$, as it is clearly non-normalizable. This implies that the computation of the S-matrix is only a formal one, since the two QFTs are unitarily inequivalent. Though the Unruh effect (in Minkowski spacetime) can be derived rigorously in the context of constructive QFT (see [52]), the algebraic approach is the most commonly used tool to investigate phenomena like the Unruh effect. The central object in this approach is the algebra of physical observables, rather than the Hilbert space, which facilitates dealing with unitarily inequivalent constructions and clarifies some of the questions that may arise from the derivation of the Unruh effect. A detailed reference on the algebraic approach is [53]; for its application to QFT in curved spacetimes, see [14].

3 CP Violation

In this chapter are presented the main results of this dissertation, and one of the phenomena that motivated them, CP *violation*, is discussed. The observation that the weak interaction violates the CP symmetry forces the introduction of some modifications to the usual descriptions of the CP-violating processes, briefly presented in section 3.1. The impact of the Unruh effect on the decay of accelerated particles—in particular, of CP-violating species—is derived, and its impact on a very interesting physical phenomenon related to CP violation, the *matter-antimatter asymmetry*, is discussed in section 3.2.

3.1 CP Violation and Kaons

Fundamental results on the CP symmetry are described in this section. The first evidence for CP violation by the weak interaction was the detection of the decay of two neutral kaons of a certain species into two pions, forbidden by the CP symmetry, in the Fitch–Cronin experiment (see [12]). Decay channels of this kind will be of interest due to the wealth of experimental results available, so a study of them is presented. A complete reference on CP violation is [54], while details on the weak interaction can be found in [55].

3.1.1 The CP Symmetry

A symmetry of a system is, in general, a transformation of the system that does not affect its description. For a QFT, symmetries may be implemented by operators acting on a Fock space in such a way that probabilities are conserved. Therefore, it is clear that symmetries must correspond to unitary or antiunitary operators, i.e., if O is the operator corresponding to some symmetry,

$$|\langle\psi|O^\dagger O|\psi'\rangle|^2 = |\langle\psi|\psi'\rangle|^2, \tag{3.1}$$

which implies that either $O^\dagger O = \mathrm{I}$ or $O^\dagger O = K^\dagger K$, where $K\colon \mathcal{F}(\mathcal{H}) \to \overline{\mathcal{F}}(\mathcal{H})$ is the complex conjugation map between the Fock space $\mathcal{F}(\mathcal{H})$ of the QFT and its complex conjugate space. Since a field ϕ is operator valued, the transformed field ϕ' has the form

$$\phi' = O\phi O^{-1}. \tag{3.2}$$

For a QFT to be invariant with respect to O, its vacuum, its action and the CCR must be invariant. The condition over the action, in particular, implies that O is a constant of motion.

The QFT of interest in this work is the one describing pseudoscalar fields in Minkowski spacetime, which is very similar to the QFT constructed as in subsection 2.2.1, describing scalar fields. All developments presented here will be based on the latter and may be generalized to the former (see, e.g., [54]). The symmetries of interest are the *charge conjugation* symmetry C, the *parity* symmetry P and *time reversal* symmetry T. For the charge conjugation symmetry to act nontrivially it is required that the fields in question be able to describe charged species. A real scalar field cannot describe a charged species, so complex fields must be considered. Fortunately, generalizing the construction of subsection 2.2.1 to complex fields is a simple task. All that has to be done is to substitute the binomial expressions on the field in the Lagrangian given in equation (2.28) by products of the corresponding expressions for the field and its complex conjugate, and to take both fields as independent fields. This leads to the conclusion that the (quantum) field ϕ can be represented as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{d}^3k \, \frac{1}{\sqrt{2\omega(\mathbf{k})}} \Big(a(\mathbf{k}) e^{ik_L x^L} + b^\dagger(\mathbf{k}) e^{-ik_L x^L} \Big), \tag{3.3}$$

where $b(\mathbf{k})$ is an annihilation operator defined very similarly to $a(\mathbf{k})$,

$$b(t, \mathbf{k}) := \sqrt{\frac{\omega(\mathbf{k})}{2}} \hat{\phi}^\dagger(t, \mathbf{k}) + i \sqrt{\frac{1}{2\omega(\mathbf{k})}} \dot{\hat{\phi}}^\dagger(t, \mathbf{k}), \tag{3.4}$$

where $\hat{\phi}^\dagger$ is the Fourier transform of the adjoint field[34]. It is clear that this operator must satisfy the CCR, i.e.,

$$[b(\mathbf{k}) \, , b^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}')\mathrm{I}. \tag{3.5}$$

A natural way to compute the action of C on the fields is to consider the KG norm[35] of ϕ multiplied by a coupling parameter q for some interaction (which gives the scale of the charge) as the charge Q of the field, i.e,

$$Q(\phi) = iq \int_{\mathbb{R}^3} \mathrm{d}^3x \, (\phi^\dagger \, \partial_t \phi - \phi \, \partial_t \phi^\dagger) = q(N_a - N_b), \tag{3.6}$$

where N_a and N_b are the number operators[36] corresponding to the annihilation operators a and b, respectively. It is then simple to obtain

$$C\phi(t, \mathbf{x})C^{-1} = \phi^\dagger(t, \mathbf{x}). \tag{3.7}$$

Invariance of the vacuum can be investigated by noting that the expressions for the induced transformation of the annihilation operators are given, if postulated that C is unitary, by

$$Ca(\mathbf{k})C^{-1} = b(\mathbf{k}), \tag{3.8}$$

[34] Note that this differs from the adjoint of the the Fourier transform of the field. They are related by a change of sign of $\mathbf{k}$ in their arguments.

[35] This quantity is clearly a constant of motion, since the symplectic form is preserved by the Hamiltonian flow.

[36] The number operators appear in this expression because Q is normally ordered.

since $C^2 = \mathrm{I}$. Then it is clear that the vacuum state of the theory is invariant, given that it is the unique state annihilated by all annihilation operators. Invariance of the action is easily obtained if the appropriate transformation for the current of the field is verified. Since the the current density J^I is the 4-vector with the density of Q as its time component, it is natural to put

$$J^I(\phi) = iq(\phi\,\partial^I\phi^* - \phi^*\,\partial^I\phi). \tag{3.9}$$

The current transforms via

$$CJ^IC^{-1} = -J^I, \tag{3.10}$$

which implies that any action coupling some field that changes sign when acted upon by C to ϕ by the current is C invariant, as is the case with Yang–Mills couplings (see [54]). The CCR are invariant only if C is unitary, which justifies the assumption made above. Their invariance come from the fact that the momentum of the adjoint field is the adjoint of the momentum of the field and the CCR for the adjoint field are equivalent to the CCR for the field. The requirement of unitarity appears because the CCR establish that the commutator of the field with its momentum is proportional to $i\mathrm{I}$.

The parity transformation on Minkowski spacetime is a reflection through the origin, inducing the following transformation on the fields:

$$P\phi(t,\mathbf{x})P^{-1} = \phi(t,-\mathbf{x}). \tag{3.11}$$

Vacuum invariance stems from the transformation rules for annihilation operators which may, again, be obtained from:

$$\begin{aligned} P\phi(t,\mathbf{x})P^{-1} &= \phi(t,-\mathbf{x}) \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{\sqrt{2\omega(\mathbf{k})}}\Big(a(\mathbf{k})e^{-i(k_\ell x^\ell+\omega t)} + b^\dagger(\mathbf{k})e^{i(k_\ell x^\ell+\omega t)}\Big) \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{\sqrt{2\omega(\mathbf{k})}}\Big(a(-\mathbf{k})e^{ik_L x^L} + b^\dagger(-\mathbf{k})e^{-ik_L x^L}\Big). \end{aligned} \tag{3.12}$$

Thus, if P is unitary,

$$Pa(\mathbf{k})P^{-1} = a(-\mathbf{k}), \tag{3.13}$$

$$Pb(\mathbf{k})P^{-1} = b(-\mathbf{k}), \tag{3.14}$$

which establishes the vacuum invariance. Verifying the invariance of the action is trivial, since $P\,\partial_{\underline{I}}P^{-1} = -\,\partial^{\underline{I}}$. So is verifying the invariance of the CCR, since the Dirac delta distribution is parity invariant. Once again, invariance of the CCR implies unitarity for P.

Time reversal is also a symmetry of Minkowski spacetime and, thus, induces a transformation on the fields:

$$T\phi(t,\mathbf{x})T^{-1} = \phi(-t,\mathbf{x}). \tag{3.15}$$

Checking invariance of the vacuum again reduces to obtaining the expressions for the transformations of the annihilation operators from

$$\begin{aligned} T\phi(t,\mathbf{x})T^{-1} &= \phi(-t,\mathbf{x}) \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{\sqrt{2\omega(\mathbf{k})}}\Big(a(\mathbf{k})e^{i(k_\ell x^\ell+\omega t)}+b^\dagger(\mathbf{k})e^{-i(k_\ell x^\ell+\omega t)}\Big) \\ &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{\sqrt{2\omega(\mathbf{k})}}\Big(a(-\mathbf{k})e^{-ik_L x^L}+b^\dagger(-\mathbf{k})e^{ik_L x^L}\Big), \end{aligned} \tag{3.16}$$

which yields different results whether T is taken to be unitary of antiunitary. Assuming the latter to be true,

$$Ta(\mathbf{k})T^{-1} = a(-\mathbf{k}), \tag{3.17}$$

$$Tb(\mathbf{k})T^{-1} = b(-\mathbf{k}), \tag{3.18}$$

proving the invariance of the vacuum. Invariance of the action is once again trivial, since $T\,\partial_{\underline{I}}T^{-1} = \partial^{\underline{I}}$. Invariance of the CCR now implies that T is antiunitary, since the momentum π transforms via $T\pi(t,\mathbf{x})T^{-1} = -\pi(t,\mathbf{x})$.

By combining C and P transformations, one obtains the CP transformation, acting on fields as

$$CP\phi(t,\mathbf{x})(CP)^{-1} = \phi^\dagger(t,-\mathbf{x}). \tag{3.19}$$

The physical interpretation of this transformation is that the application of P induces a parity transformation on space and C transforms particles into antiparticles. The interpretation for C is clear from the creation operator transformation law. A very important result relating the three symmetries mentioned above is the CPT *theorem*, which states the transformation CPT is always a symmetry of any QFT[37]. In particular, it implies that any violation of one these symmetries must be accompanied by the violation of the combination of the two others. If CP violation is to be considered, T violation must also be taken into account. CPT invariance is very well tested experimentally (see [56]).

3.1.2 The Kaon to Pion Decay Channels

The kaon to pion decay channels are of interest not only because they were the first detected source of CP violation, but also because a great deal of experimental data about them is available. A model for a system with kaons is presented below, which enlightens the connection between eigenstates of the strong and weak interactions and the experimental results on CP violation. More details on this model can be found in [55].

Kaons comprise a family of strange pseudoscalar *mesons*, i.e., quark-antiquark pairs with nonzero strangeness. The four kaon eigenstates of the strong force are: $K^+ = (u\bar{s})$,

[37] "Any QFT" here means any QFT satisfying the *Wightman axioms* (see [41]), which essentially encode that the theory is relativistic and satisfies microcausality.

$K^- = (s\bar{u})$, $K^0 = (d\bar{s})$ and $\overline{K}^0 = (s\bar{d})$, where u, d and s are, respectively, the *up*, *down* and *strange* quarks, and $\bar{u}$, $\bar{d}$ and $\bar{s}$ are their antiparticles. Note that K^- is the antiparticle of K^+ and $\overline{K}^0$ is the antiparticle of K^0. The CP-violating decay channel is the one where a certain kind of neutral kaon decays into two pions, denoted $K \to \pi\pi$. Since they involve neutral kaons, charged kaons will not be discussed. Pions also comprise a family of pseudoscalar mesons and the strong eigenstates are: $\pi^+ = (u\bar{d})$, $\pi^- = (d\bar{u})$ and $\pi^0 = 1/\sqrt{2}(u\bar{u} - d\bar{d})$. Note that while π^- is the antiparticle of π^+, π^0 is its own antiparticle.

To understand the decay of a neutral kaon into a two pions, a basis of CP eigenstates for the neutral kaon space must be found, since

$$CP\,|\pi^+\pi^-\rangle = (-1)^2 C\,|\pi^-\pi^+\rangle = |\pi^+\pi^-\rangle\,, \tag{3.20a}$$

$$CP\,|\pi^0\pi^0\rangle = (-1)^2 C\,|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle\,. \tag{3.20b}$$

The neutral kaons K^0 and $\overline{K}^0$, however, are not CP eigenstates, since

$$CP\,|K^0\rangle = -C\,|K^0\rangle = -\,|\overline{K}^0\rangle\,, \tag{3.21a}$$

$$CP\,|\overline{K}^0\rangle = -C\,|\overline{K}^0\rangle = -\,|K^0\rangle\,. \tag{3.21b}$$

Taking linear combinations of theses states yields the CP eigenstates,

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle), \tag{3.22a}$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle), \tag{3.22b}$$

which span the neutral kaon space and satisfy $CP\,|K_1\rangle = |K_1\rangle$ and $CP\,|K_2\rangle = -\,|K_2\rangle$. This implies that the kaon K_2 cannot decay into two pions, and that the transition $K_1 \to \pi\pi$ contains all the information on the decay channel.

Strangeness is conserved by the strong and electromagnetic interactions, so the decay of a kaon into pions cannot be mediated by them. The weak interaction, however, violates conservation of strangeness. The dynamics of a neutral kaon system may then be described by a quantum mechanical system with Hamiltonian $H = H_0 + H_{\mathrm{W}}$, where H_0 describes the dynamics of the free field and the strong and electromagnetic interactions, and H_{W} describes the weak interaction. Results in perturbation theory imply that the restriction of the time flow to the subspace[38] $\mathcal{H}_K$ of neutral kaons is generated by the effective Hamiltonian

$$H_{\mathrm{eff}} = H_{\mathrm{W}} + \sum_{|\psi\rangle \notin \mathcal{H}_K} \frac{H_{\mathrm{W}}|\psi\rangle\langle\psi|H_{\mathrm{W}}}{m_{K^0} - E_\psi}, \tag{3.23}$$

here expanded up to second order. The neutral kaon mass[39] is denoted m_{K^0} and E_ψ is the energy eigenvalue of $|\psi\rangle$. The effective Hamiltonian may be decomposed as

$$H_{\mathrm{eff}} = M - \frac{i}{2}\Gamma, \tag{3.24}$$

[38] The total space is spanned by all states that can result from the time evolution of the neutral kaons, e.g., states with pions, neutrinos, etc.

[39] That is, the eigenvalue of H_0 associated to $|K^0\rangle$ and $|\overline{K}^0\rangle$.

where M and Γ are self-adjoint and correspond, respectively, to the time flows resulting in oscillations and decays. CPT symmetry imposes constraints on M and Γ:

$$\begin{aligned} M_{11} &\coloneqq \langle K^0|M|K^0\rangle = \langle K^0|(CPT)^\dagger (CPT)^{-\dagger} M (CPT)^{-1}(CPT)|K^0\rangle \\ &= \langle \overline{K}^0|K^\dagger K (CPT) M (CPT)^{-1}|\overline{K}^0\rangle = \langle \overline{K}^0|M|\overline{K}^0\rangle^* = \langle \overline{K}^0|M|\overline{K}^0\rangle \eqqcolon M_{22}, \end{aligned} \tag{3.25a}$$

$$\begin{aligned} \Gamma_{11} &\coloneqq \langle K^0|\Gamma|K^0\rangle = \langle K^0|(CPT)^\dagger (CPT)^{-\dagger} \Gamma (CPT)^{-1}(CPT)|K^0\rangle \\ &= \langle \overline{K}^0|K^\dagger K (CPT) \Gamma (CPT)^{-1}|\overline{K}^0\rangle = \langle \overline{K}^0|\Gamma|\overline{K}^0\rangle^* = \langle \overline{K}^0|\Gamma|\overline{K}^0\rangle \eqqcolon \Gamma_{22}. \end{aligned} \tag{3.25b}$$

If the theory is assumed to be CP invariant or, equivalently, T invariant, more constraints can be obtained:

$$\begin{aligned} M_{12} &\coloneqq \langle K^0|M|\overline{K}^0\rangle = \langle K^0|(CP)^\dagger (CP)^{-\dagger} M (CP)^{-1}(CP)|\overline{K}^0\rangle \\ &= \langle \overline{K}^0|(CP) M (CP)^{-1}|K^0\rangle = \langle \overline{K}^0|M|K^0\rangle \eqqcolon M_{21}, \end{aligned} \tag{3.26a}$$

$$\begin{aligned} \Gamma_{12} &\coloneqq \langle K^0|\Gamma|\overline{K}^0\rangle = \langle K^0|(CP)^\dagger (CP)^{-\dagger} \Gamma (CP)^{-1}(CP)|\overline{K}^0\rangle \\ &= \langle \overline{K}^0|(CP) \Gamma (CP)^{-1}|K^0\rangle = \langle \overline{K}^0|\Gamma|K^0\rangle \eqqcolon \Gamma_{21}. \end{aligned} \tag{3.26b}$$

These constraints imply that the CP eigenstates, $|K_1\rangle$ and $|K_2\rangle$, diagonalize H_{eff}, with

$$H_{\text{eff}}\,|K_1\rangle = \left[(M_0 - \tilde{M}) - \frac{i}{2}(\Gamma_0 - \tilde{\Gamma})\right]|K_1\rangle\,, \tag{3.27a}$$

$$H_{\text{eff}}\,|K_2\rangle = \left[(M_0 + \tilde{M}) - \frac{i}{2}(\Gamma_0 + \tilde{\Gamma})\right]|K_2\rangle\,, \tag{3.27b}$$

where $M_0 \coloneqq M_{11}$, $\tilde{M} \coloneqq M_{12}$, $\Gamma_0 \coloneqq \Gamma_{11}$ and $\tilde{\Gamma} \coloneqq \Gamma_{12}$. This implies that the species K_1 and K_2 have different masses and decay rates, with $M_1 \coloneqq M_0 - \tilde{M} < M_0 + \tilde{M} \eqqcolon M_2$ and[40] $\Gamma_1 \coloneqq \Gamma_0 - \tilde{\Gamma} > \Gamma_0 + \tilde{\Gamma} \eqqcolon \Gamma_2$. As shown above, only one of the weak eigenstates, $|K_1\rangle$, can decay into two pions (which partially explains the difference in the decay rates) if CP is assumed to be a symmetry of the system.

Experimentally, the decay of two weak eigenstates of neutral kaons into two pions is detected. Calling the higher mass eigenstate $|K^0_{\text{L}}\rangle$ and the lower mass eigenstate $|K^0_{\text{S}}\rangle$, the ratios between measured decay amplitudes are

$$\eta_{00} \coloneqq \frac{\mathcal{A}(K^0_{\text{L}} \to \pi^0\pi^0)}{\mathcal{A}(K^0_{\text{S}} \to \pi^0\pi^0)} = (2.220 \pm 0.011) \times 10^{-3} e^{i(43.52\,\pm\,0.05)^\circ}, \tag{3.28a}$$

$$\eta_{+-} \coloneqq \frac{\mathcal{A}(K^0_{\text{L}} \to \pi^+\pi^-)}{\mathcal{A}(K^0_{\text{S}} \to \pi^+\pi^-)} = (2.232 \pm 0.011) \times 10^{-3} e^{i(43.51\,\pm\,0.05)^\circ}, \tag{3.28b}$$

according to [1]. This indicates that there is violation of the CP symmetry in the decays and, consequently, the eigenstates of the weak interaction are not $|K_1\rangle$ and $|K_2\rangle$ but rather the "K-long" (or "K-large") kaon state $|K^0_{\text{L}}\rangle$ and the "K-short" (or "K-small") kaon

40 While M_0, $\tilde{M}$ and Γ_0 are positive, $\tilde{\Gamma}$ can be show to be negative (see [55]).

state $|K^0_S\rangle$, given by

$$|K^0_L\rangle = \frac{1}{\sqrt{1+|q|^2}}(|K^0\rangle + q\,|\overline{K}^0\rangle), \tag{3.29a}$$

$$|K^0_S\rangle = \frac{1}{\sqrt{1+|q|^2}}(|K^0\rangle - q\,|\overline{K}^0\rangle), \tag{3.29b}$$

where q is some complex mixing parameter, to be written in terms of the components of H_{eff}. The names for these species come from the difference in the decay rates and masses. Experimental results on the mass difference $m_{K^0_L} - m_{K^0_S}$,

$$m_{K^0_L} - m_{K^0_S} = (3.484 \pm 0.006) \times 10^{-12}\,\frac{\text{MeV}}{c^2}, \tag{3.30}$$

can be found in [1]. This difference is clearly very small, given that the mass of the neutral strong eigenstate is $m_{K^0} = (497.611 \pm 0.013)\,\text{MeV}/c^2$ according to [1].

By dropping the requirement of CP invariance and diagonalizing H_{eff}, the following expression for q is obtained:

$$q = \sqrt{\frac{M^*_{12} - \frac{i}{2}\Gamma^*_{12}}{M_{12} - \frac{i}{2}\Gamma_{12}}}. \tag{3.31}$$

The decay amplitudes can be obtained by the projection into $|K\rangle_1$,

$$\eta_{00} = \frac{\langle\pi^0\pi^0|S(|K_1\rangle\langle K_1| + |K_2\rangle\langle K_2|)|K^0_L\rangle}{\langle\pi^0\pi^0|S(|K_1\rangle\langle K_1| + |K_2\rangle\langle K_2|)|K^0_S\rangle} = \frac{\langle\pi^0\pi^0|S|K_1\rangle\langle K_1|K^0_L\rangle}{\langle\pi^0\pi^0|S|K_1\rangle\langle K_1|K^0_S\rangle} = \frac{\langle K_1|K^0_L\rangle}{\langle K_1|K^0_S\rangle}, \tag{3.32a}$$

$$\eta_{+-} = \frac{\langle\pi^+\pi^-|S(|K_1\rangle\langle K_1| + |K_2\rangle\langle K_2|)|K^0_L\rangle}{\langle\pi^+\pi^-|S(|K_1\rangle\langle K_1| + |K_2\rangle\langle K_2|)|K^0_S\rangle} = \frac{\langle K_1|K^0_L\rangle}{\langle K_1|K^0_S\rangle}, \tag{3.32b}$$

since $\mathrm{I} = |K_1\rangle\langle K_1|+|K_2\rangle\langle K_2|$ in the current approximation and since $|K_2\rangle$ cannot transition to a 2-pion state. Computing the projection is simple:

$$\langle K_1|K^0_L\rangle = \frac{1}{\sqrt{2(1+|q|^2)}}(\langle K^0| - \langle\overline{K}^0|)(|K^0\rangle + q\,|\overline{K}^0\rangle) = \frac{1-q}{\sqrt{2(1+|q|^2)}}, \tag{3.33a}$$

$$\langle K_1|K^0_S\rangle = \frac{1}{\sqrt{2(1+|q|^2)}}(\langle K^0| - \langle\overline{K}^0|)(|K^0\rangle - q\,|\overline{K}^0\rangle) = \frac{1+q}{\sqrt{2(1+|q|^2)}}. \tag{3.33b}$$

Then, putting $\eta \coloneqq \eta_{00} = \eta_{+-}$,

$$\eta = \frac{1-q}{1+q}. \tag{3.34}$$

It must be noted that the experimental data suggests a difference between η_{00} and η_{+-}. In fact, putting $\eta_{+-} \eqqcolon \epsilon + \epsilon'$ and $\eta_{+-} \eqqcolon \epsilon - 2\epsilon'$, the value of ϵ' is nonzero but three orders of magnitude smaller then the value of ϵ. This can be accounted for theoretically by considering certain isospin transitions neglected here (see [54, 55] for greater details). Inverting equation (3.34) allows for the estimation of the value of q from the experimental data,

$$q = \frac{1-\eta}{1+\eta} = 0.99677 \pm 0.00001 - i(0.00306 \pm 0.00001), \tag{3.35}$$

using $\eta \approx \epsilon = (\eta_{00} + 2\eta_{+-})/3$.

A better description of kaon mixing is achieved by introducing the *Cabibbo–Kobayashi–Maskawa (CKM) matrix*, coupling to the SM (see [54]). It describes quark mixing in a way that relates the states that couple to the weak interaction with the strong eigenstates. The approximation for kaon mixing introduced in equations (3.29) is valid when the fields are sufficiently localized (as will be the case with the applications presented in section 3.2). Outside this domain, certain problems arise, as pointed out in [57, 58].

There are similar systems presenting some evidence of CP violation, namely, the neutral B, B_s and D mesons (see [1] and [59]). The description of the mechanism responsible for CP violation in the meson species is very similar to the one given above, but they are not discussed in more detail in this text, since the best established experimental results are for the kaon decays.

3.2 CP Violation and Accelerated Particles

Taking the Unruh effect into account when describing particle decay leads to a very interesting conclusion: proper decay rates increase when a particle is accelerated. This was first described by Muller in [10], which models the scalar case, and further investigated by Matsas and Vanzella in [11, 60–62], which model the spinorial case. Adapting the model presented in [10] allows for the description of $K \to \pi\pi$ decays when the kaon is accelerated. The consequences of this effect on the behaviour of the CP violation parameter η is investigated and some considerations on its impact on the matter–antimatter asymmetry are presented.

3.2.1 Decay of Accelerated Particles

The Unruh effect states that accelerated observers perceive the inertial vacuum as a thermal bath of particles. It seems, then, that accelerated particles should have increased proper decay rates, as they would if immersed in a thermal bath. A model for the decay of an accelerated scalar particle is presented below, justifying this assertion.

Consider the QFT describing scalar fields in Minkowski spacetime (with inertial coordinates (t, x, y, z)) as constructed in subsection 2.2.1. To model the decay of a massive field into two other massive fields, e.g., the $K \to \pi\pi$ decays, an interaction Lagrangian must be introduced. Its form is taken to be[41]

$$\mathscr{L}_{\mathrm{I}}(x) = G_{\Gamma}\Phi(x)\phi_1(x)\phi_2(x), \tag{3.36}$$

[41] This action is nonrenormalizable but this should not be an alarming source of concern for the predictions of interest.

where Φ is a real scalar field of mass M, ϕ_1 and ϕ_2 are complex scalar fields of mass m and G_Γ is the coupling parameter of the interaction. Although kaons and pions are described by pseudoscalar fields, it is assumed that scalar fields provide approximate descriptions of their behaviour. The decay rate may then be obtained from the decay amplitude, i.e., the transition amplitude for the process $\Phi \to \phi_1\phi_2$, given, up to first order in G_Γ, by

$$\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2) = \langle \mathbf{k}_1,\mathbf{k}_2| \otimes \langle 0|\, S\, |i\rangle \otimes |0\rangle = G_\Gamma \int_{\mathbb{R}^4} \mathrm{d}^4x\ \langle 0|\Phi(x)|i\rangle \prod_{j=1}^{2} \langle \mathbf{k}_j|\phi_j(x)|0\rangle\,, \tag{3.37}$$

where the final state consists of two particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. The decay probability can be computed from the amplitude,

$$\begin{aligned}\mathcal{P} &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}^3k_1\,\mathrm{d}^3k_2\,|\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2)|^2\\ &= G_\Gamma^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} \mathrm{d}^4x\,\mathrm{d}^4x'\ \langle 0|\Phi(x)|i\rangle\,\langle i|\Phi(x')|0\rangle \prod_{j=1}^{2}\langle 0|\phi_j^\dagger(x) \int_{\mathbb{R}^3}\mathrm{d}^3k_j\,|\mathbf{k}_j\rangle\langle\mathbf{k}_j|\phi_j(x')|0\rangle\\ &= G_\Gamma^2 \int_{\mathbb{R}^4\times\mathbb{R}^4} \mathrm{d}^4x\,\mathrm{d}^4x'\ f^*(x)f(x')\prod_{j=1}^{2}\langle 0|\phi_j^\dagger(x)\phi_j(x')|0\rangle\,,\end{aligned} \tag{3.38}$$

where $f(x)$ is the mode associated to the initial state $|i\rangle$. The Wightman function of a scalar field appears above and is given, for a scalar field ϕ, by

$$\langle 0|\phi^\dagger(x)\phi(x')|0\rangle = i\frac{m}{8\pi}\frac{H_1^{(2)}(m\Delta s)}{\Delta s}, \tag{3.39}$$

where $H_1^{(2)}$ is a Hankel function of the second kind and Δs is the proper time interval of the timelike separated events x^I and x'^I. The computation of this function and the definition of Δs are presented in appendix A.

Assuming that f is peaked over a trajectory $x(\tau)$ parameterized by its proper time τ, i.e., over the trajectory of a particle, it can be written as

$$f(x) = h(\mathbf{x}(\tau))e^{-iM\tau} \tag{3.40}$$

in the instantaneous rest frame. Assuming, furthermore, that the decay products do not deviate much from this trajectory, the decay probability can be written as

$$\begin{aligned}\mathcal{P} &= G_\Gamma^2\kappa \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{d}\tau'\,e^{iM(\tau-\tau')}\prod_{j=1}^{2}\langle 0|\phi_j^\dagger(t(\tau),\mathbf{x}(\tau))\phi_j(t'(\tau'),\mathbf{x}'(\tau'))|0\rangle\\ &= -\frac{G_\Gamma^2\kappa}{64\pi^2}m^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{d}\tau'\,e^{iM(\tau-\tau')}\frac{\left[H_1^{(2)}(m\Delta s)\right]^2}{|\Delta s^2|},\end{aligned} \tag{3.41}$$

where κ is given by

$$\kappa = \left|\int_{\mathbb{R}^3}\mathrm{d}^3x\,h(\mathbf{x})\right|^2. \tag{3.42}$$

A change of variables of the form $v \coloneqq \tau - \tau'$ leads to

$$\mathcal{P} = -\frac{G_\Gamma^2 \kappa}{64\pi^2} m^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{d}\tau\, e^{iMv} \frac{\left[H_1^{(2)}(m\Delta s)\right]^2}{|\Delta s^2|}. \tag{3.43}$$

Since the Wightman function depends only on the difference $x^I - x'^I$, the integral over τ is trivial (and infinite). "Dividing out" this integral gives the following expression for the proper decay rate,

$$\Gamma = -\frac{G_\Gamma^2 \kappa}{64\pi^2} m^2 \int_{-\infty}^{\infty} \mathrm{d}v\, e^{iMv} \frac{\left[H_1^{(2)}(m\Delta s)\right]^2}{|\Delta s^2|}, \tag{3.44}$$

which is a Dirac observable (q.v. the discussion at the end of subsection 2.2.4).

A uniformly accelerated particle's trajectory satisfies the constraint imposed by equation (2.137) and may be parameterized in terms of the proper time as

$$t(\tau) = \frac{1}{a}\sinh(a\tau), \tag{3.45a}$$
$$x(\tau) = \frac{1}{a}\cosh(a\tau), \tag{3.45b}$$
$$y(\tau) = 0, \tag{3.45c}$$
$$z(\tau) = 0, \tag{3.45d}$$

where a is the magnitude of the proper acceleration. The squared spacetime interval Δs^2 at a certain values τ and τ' of the proper times is given by

$$\begin{aligned}
\Delta s^2 &= -(t(\tau) - t'(\tau'))^2 + (x(\tau) - x'(\tau'))^2 \\
&= \frac{1}{a^2}(-\sinh^2(a\tau) - \sinh^2(a\tau') + 2\sinh(a\tau)\sinh(a\tau') \\
&\qquad + \cosh^2(a\tau) + \cosh^2(a\tau') - 2\cosh(a\tau)\cosh(a\tau')) \\
&= -\frac{2}{a^2}[\cosh(a(\tau - \tau')) - 1] = -\frac{4}{a^2}\sinh^2\Big(\frac{a}{2}(\tau - \tau')\Big),
\end{aligned} \tag{3.46}$$

which implies that $\Delta s = 2/a \sinh(a/2(\tau - \tau'))$. Introducing the variable $u \coloneqq a(\tau - \tau')/2 = av/2$ and inserting the expression for Δs^2 in equation (3.44) gives the decay rate for uniformly accelerated scalar particles:

$$\Gamma = -\frac{G_\Gamma^2 \kappa}{128\pi^2} m^2 a \int_{-\infty}^{\infty} \mathrm{d}u\, e^{i2Mu/a} \frac{\left[H_1^{(2)}\left(\frac{2m}{a}\sinh(u)\right)\right]^2}{\sinh^2(u)}. \tag{3.47}$$

3.2.2 Acceleration and CP- Violating Decays

The expression for the proper decay rate Γ of an accelerated massive particle given in equation (3.47) depends on the value of the acceleration a. In fact, the conclusion presented in [10] is that, for very similar models, Γ increases with a. Since Γ also depends on the masses of the particles and K_L^0 and K_S^0 have different masses, it may be expected that the decay rates for the processes $K_\mathrm{L}^0 \to \pi\pi$ and $K_\mathrm{S}^0 \to \pi\pi$ increase with a at different rates, which, in light of the result presented in the previous subsection, could imply that the amplitude of CP violation varies with a. Investigating this proposal is the goal of this subsection. A similar examination for the case of oscillating neutrinos has been presented in [57].

The cases to be considered are the ones that model the processes $K^0_\mathrm{L} \to \pi^0\pi^0$ and $K^0_\mathrm{S} \to \pi^0\pi^0$, in which $M = m_{K^0} = 497.611\,\mathrm{MeV}/c^2$ (since $m_{K^0_\mathrm{L}} \approx m_{K^0} \approx m_{K^0_\mathrm{S}}$) and $m = m_{\pi^0} = 134.9770\,\mathrm{MeV}/c^2$, and $K^0_\mathrm{L} \to \pi^+\pi^-$ and $K^0_\mathrm{S} \to \pi^+\pi^-$, in which $M = m_{K^0} = 497.611\,\mathrm{MeV}/c^2$ and $m = m_{\pi^\pm} = 139.57061\,\mathrm{MeV}/c^2$ (cf. [1]). Additionally, the units to be used are the ones in which $\Gamma_0 \coloneqq G_\Gamma^2\kappa/(128\pi^2) = 1$ and $m_{K^0} = 1$, which implies that $m_{\pi^0} = 0.271250$, $m_{\pi^\pm} = 0.280481$ and $m_{K^0_\mathrm{L}} - m_{K^0_\mathrm{S}} = 7.001 \times 10^{-15}$. The computation of the integral appearing in equation (3.47) presents several challenges and is detailed in appendix B. Its results are summarized in figure 4.

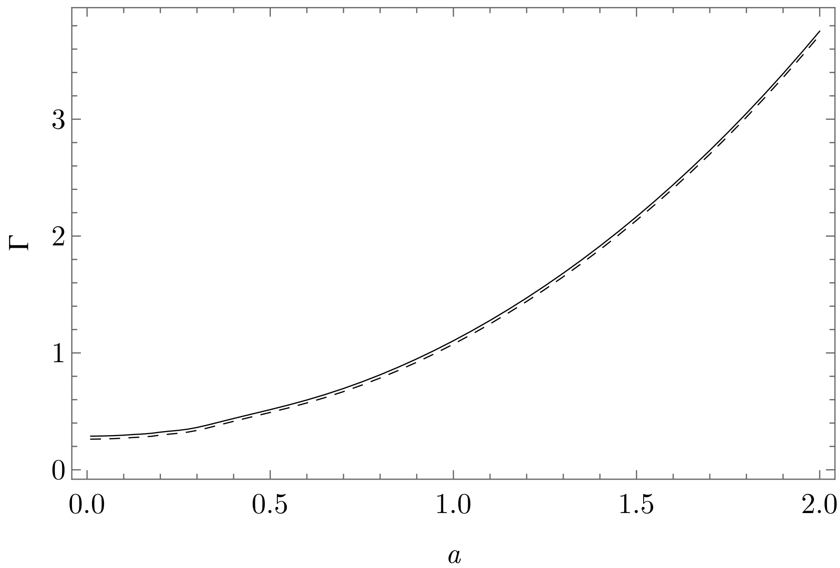

Figure 4 – The plot of Γ (in units of Γ_0) as a function of a (in units of m_{K^0}) for the case where $M = m_{K^0}$ and $m = m_{\pi^0}$ (solid line) or $m = m_{\pi^\pm}$ (dashed line).

The figure indicates that there is an increase in the rates of both processes, but also that it could be (directly) detected only at very high values for the acceleration. In the units used, $a = 1$ corresponds to $a \approx 2 \times 10^{32}\,\mathrm{m/s^2}$ in SI units, or $a \approx 2 \times 10^{31} g$, where g is the standard gravity. By comparison, the Texas Petawatt Laser is able to accelerate electrons up to energies of $2\,\mathrm{GeV}$ over distances of just $1\,\mathrm{cm}$–$2\,\mathrm{cm}$ (see [63]), which, at best, gives

$$a_\mathrm{TPL} \approx \frac{E}{m_e \Delta x} \approx 3.5 \times 10^{22}\,\mathrm{m/s^2}, \tag{3.48}$$

where E is the energy of the electrons and $\Delta x = 1\,\mathrm{cm}$ is the distance traveled by them. For any increase in the decay rate to be experimentally detectable with currently available setups, an increase of at least 0.5% should be expected (the uncertainty in the values for the decay rates of K^0_{L} given in [1] fluctuate around 0.5% and 0.7%), which would require accelerations greater than $10^{30}\,\mathrm{m/s^2}$.

To evaluate the impact of the increase of the decay rate on CP violation, the behaviour of η must be investigated. Since η is the ratio of the decay amplitudes,

$$|\eta|^2 = \frac{|\mathcal{A}(K^0_{\mathrm{L}} \to \pi\pi)|^2}{|\mathcal{A}(K^0_{\mathrm{S}} \to \pi\pi)|^2} = \frac{\Gamma(K^0_{\mathrm{L}} \to \pi\pi)}{\Gamma(K^0_{\mathrm{S}} \to \pi\pi)}. \tag{3.49}$$

Care must be taken, however, as $G_\Gamma(K^0_{\mathrm{L}} \to \pi\pi)$ and $G_\Gamma(K^0_{\mathrm{S}} \to \pi\pi)$ must be different to yield different results for $\Gamma(K^0_{\mathrm{L}} \to \pi\pi)(a = 0)$ and $\Gamma(K^0_{\mathrm{L}} \to \pi\pi)(a = 0)$. In fact, the information about isospin transitions, responsible for the difference in the decay rates (see [54]), must be contained in the coupling parameter. Thus, η must be rewritten as

$$\eta = \beta\eta', \tag{3.50}$$

where $\beta = G_\Gamma(K^0_{\mathrm{L}} \to \pi\pi)/G_\Gamma(K^0_{\mathrm{S}} \to \pi\pi)$, so that β contains the information coming from experimental sources and η' encodes the behaviour predicted by the theoretical model. Since there are now two different masses for the initial particles involved in the calculation of η, the labels M_1 and M_2 for the numerator initial mass and the denominator initial mass, respectively, are adopted.

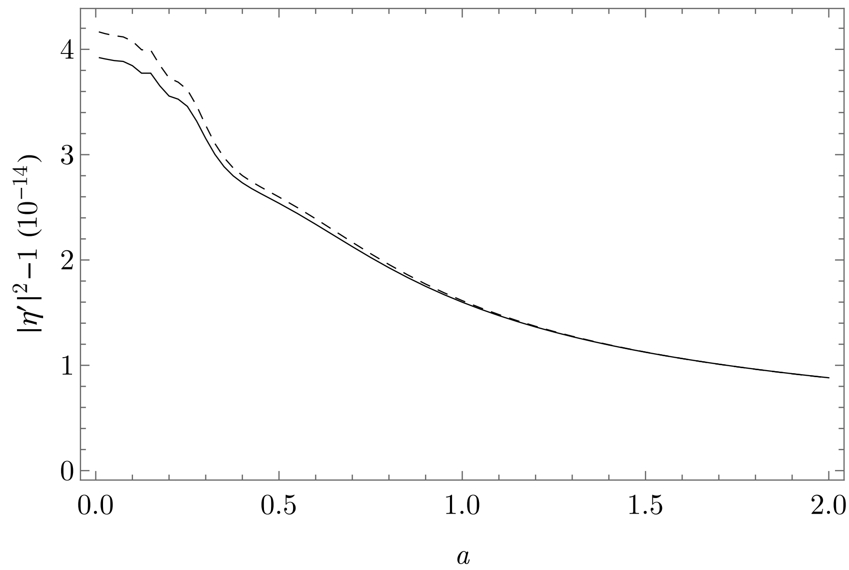

Figure 5 – The plots of $|\eta'|^2 - 1$ as a function of a (in units of m_{K^0}) for the cases where $M_1 = m_{K^0} + (m_{K^0_{\mathrm{L}}} - m_{K^0_{\mathrm{S}}})$, $M_2 = m_{K^0}$ and $m = m_{\pi^0}$ (solid line) or $m = m_{\pi^\pm}$ (dashed line).

Figure 5 implies that $|\eta|$ decreases with a (since $|\eta'|^2$ also decreases), although very slightly (note that the plot displayed in figure 5 gives the values for $|\eta'|^2 - 1$ as a function of a, *not* $|\eta'|^2$). For the decrease in $|\eta|$ to be experimentally detectable, a difference of, again, at least 0.5% in $|\eta'|$ would be needed (the uncertainty for the values of η_{00} and η_{+-} is about 0.5%). This is impossible even if incredibly high accelerations were in reach, since the dominant part (see below) of $|\eta'|$ tends to 1 in the $a \to \infty$ limit, constraining the oscillation amplitude of $|\eta'|^2$ to less than 5×10^{-14}.

The dependency of η on the mass difference of the two kaons is particularly interesting. By approximating $|\eta'|^2$ by the ratio of the dominant parts[42] of Γ (stemming from the integral $\mathcal{I}'_0$, defined in appendix B) and computing it for different values of the mass difference $M_1 - M_2$, it can be seen that, although the behaviour of $|\eta|$ with respect to a does not change appreciably, the oscillation amplitude increases (somewhat linearly) with an increase on the mass difference. It follows that the detection of CP violation in a system similar to the one investigated, but with a greater mass difference, would facilitate experimental detection. The results for $m = m_{\pi^0}$ are shown in figure 6.

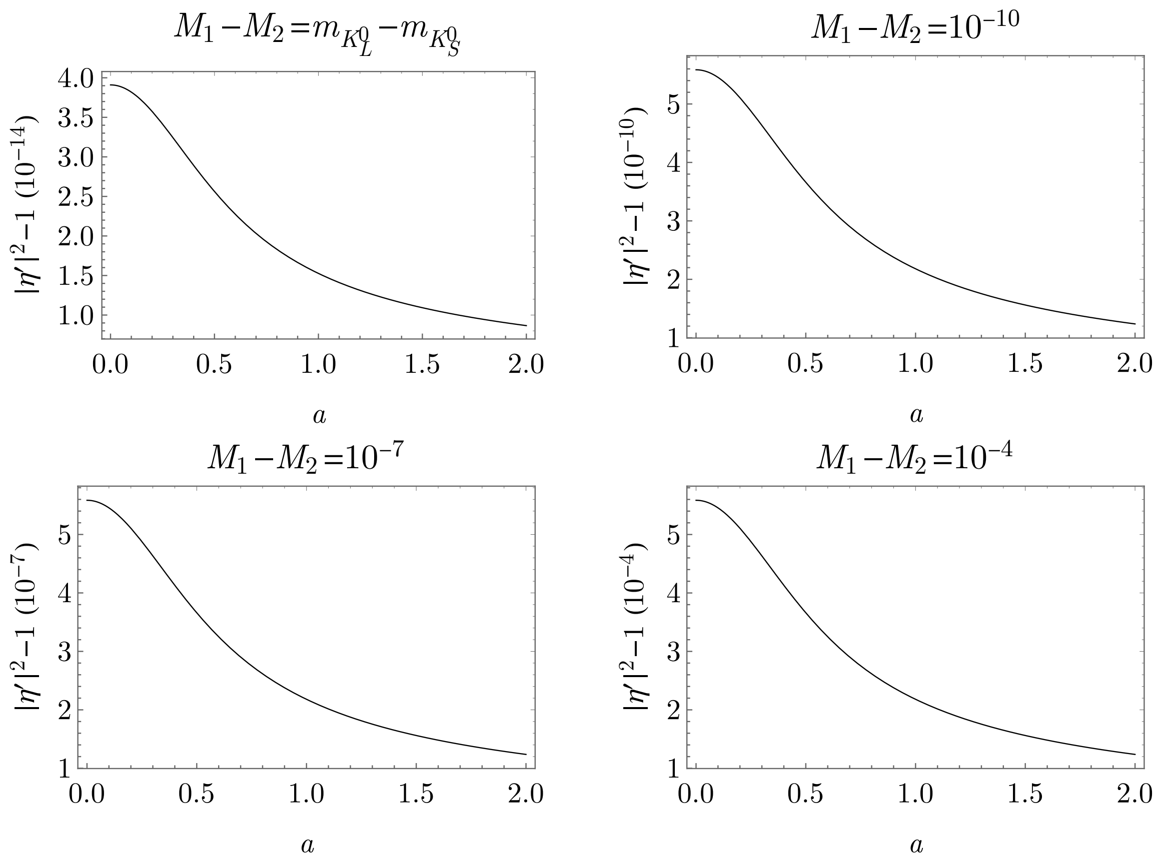

Figure 6 – The plots of an approximation of $|\eta'|^2 - 1$ as a function of a (in units of m_{K^0}) and with $m = m_{\pi^0}$ for different values of the mass difference $M_1 - M_2$ (also in units of m_{K^0}).

Even if the required accelerations could be obtained in a laboratory setting and the mass difference was high enough for the detection of the effect, traditional techniques depend on the particle subject to the acceleration being electrically charged, while the kaons in question are neutral. Furthermore, the other known sources of CP violation are

[42] This was done because the dominant parts are the fastest to compute.

also neutral particle systems. The detection of CP violation in charged particle species would be a possible solution this problem.

There is one more avenue that may be pursued if experimental tests are sought. Some proposals concerning the Unruh effect have been presented (see [51] for a review and [64]) and, if any of them are executed, could lend some credence to the predictions presented, given the intimate connection between them and the Unruh effect.

3.2.3 Matter–Antimatter Asymmetry

One of the biggest unsolved problems in physics has a profound relationship with CP violation: that of the *matter–antimatter asymmetry*. It arises from the simple observation that, although antimatter can be produced in laboratory conditions, matter dominates the contents of the Universe. A classic analysis presented in [13] concludes that CP violation is a fundamental ingredient in explaining the observed asymmetry, the basic idea being that the difference in the behaviour of matter and antimatter would lead to their concentrations being different. The impact of the prediction presented above—that the amplitude of CP violation decreases with increasing acceleration—on this problem is investigated here. For a reference on the cosmological aspects, see [65].

The best cosmological models available are based on the Friedmann–Lemaître–Robertson–Walker (FLRW) solutions of the Einstein equation. They describe expanding universes and the Big Bang, the origin of the Universe as a singularity in the *scale factor*, which, in turn, measures the relative size of the Universe. The Friedmann equations (the set of constraints on the family of FLRW solutions that come from the Einstein equation) impose certain relations between the quantities that characterize the matter content of the Universe and the scale factor. Of chief interest is the temperature of matter, which can be shown to be given, close to the Big Bang and under appropriate assumptions (see [65]), by

$$T = \frac{A}{S}, \tag{3.51}$$

where A is some constant and S is the scale factor. The dependence of S on the cosmic time t is given by

$$S = A\left(\frac{4\pi^2}{45}\right)^{1/4} t^{1/2}, \tag{3.52}$$

which implies that

$$T = \left(\frac{45}{4\pi^2}\right)^{1/4} t^{-1/2}. \tag{3.53}$$

Thus, the temperature increases as one approaches the Big Bang.

At very high temperatures, the matter content of the Universe may be described as a “plasma” of particles and radiation in (local) thermal equilibrium, with matter–antimatter annihilation occurring abundantly. As it cools down, particles species start to decouple from it, as the typical time scale of interactions dips below the time scale of expansion, and the number of particles freezes. Decoupling is fundamental for the formation of stable structures such as nucleons and atomic nuclei (a process know as *baryogenesis*), and it is at this stage that the matter–antimatter asymmetry is expected to manifest itself, especially for baryons. What the analysis in [13] concludes is that all of the following must be satisfied if baryonic matter is to dominate over baryonic antimatter:

1. There is some interaction that violates the conservation of baryon number.
2. There is some interaction that violates the C and CP symmetries.
3. There was an epoch when the contents of the Universe departed from thermal equilibrium.

As described in [66], conditions 2 and 3 are essentially guaranteed by cosmological considerations and the SM, but condition 1 presents some difficulties if no "new physics" is assumed. In the perturbative regime, all interactions of the SM conserve baryon number (and lepton number), but nonperturbative effects in the electroweak sector present a source of violation, namely, the *sphaleron processes* (see [67] for a review). The effectiveness of these processes in baryogenesis, however, is often put into question, given that their temperature scale T_{sph} is much higher than the CP violation scale determined by the Jarlskog determinant D, the lowest order CP noninvariant combination of the parameters of the CKM matrix (see [54, 68]). Specifically,

$$\frac{D}{T_{\mathrm{sph}}^{12}} \approx 10^{-20} \ll 10^{-10} \approx \eta_{b\gamma}, \tag{3.54}$$

where $\eta_{b\gamma}$ is the baryon to photon ratio. This reveals the impact that the amplitude of CP violation has on baryogenesis.

A connection between the temperature of matter in the Universe and QFT in curved spacetimes is drawn by Parker in [69]. Based on earlier work predicting particle creating in expanding universes (see [6, 36]), Parker concludes that the matter created due to the expansion of the Universe is in a thermal state with temperature

$$T \approx \frac{1}{4\pi}\frac{S_1}{S}, \tag{3.55}$$

where S_1 is the past asymptotic value of the scale factor[43]. This temperature is clearly consistent with the thermodynamical considerations implied by the Friedmann equations (see equation (3.51)). Given that the model presented in this work is based on the Unruh effect, an analogy can be traced between the two effects. Since an accelerated observer in Minkowski spacetime perceives the inertial vacuum as a thermal state and the analysis

[43] This value is nonzero because the cosmological model used consists of a FLRW universe smoothed out near the Big Bang.

in question predicts that the matter created is in a thermal state, an acceleration can be associated to each value of the scale factor by combining equations (2.169) and (3.55) into

$$a \approx \frac{1}{2}\frac{S_1}{S}. \tag{3.56}$$

It must be made clear that this connection stems from an analogy between two very distinct effects, the Unruh effect in Minkowski spacetime and the effect of particle creation in expanding universes. The acceleration in equation (3.56) has no physical interpretation other than as the acceleration at which the Unruh temperature is equal to the temperature of matter in the universe in question, but it may be expected that the results obtained for the change in decay rates with varying acceleration can be translated to results on the change of decay rates with varying temperatures. This situation is very similar to other analogies between effects of QFT in curved spacetimes, such as the one between the Unruh effect and the Hawking effect. The Hawking effect predicts that the vacuum state in the asymptotic past of a spacetime with a black hole evolves to a thermal state with temperature

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}, \tag{3.57}$$

where κ is the *surface gravity* of the black hole, in the asymptotic future (see [14]). The analogy between the two phenomena arises because an observer that maintains themselves near the event horizon of the black hole must be accelerating at a rate $a = \kappa$, i.e., an accelerating observer very close to the black hole measures the same temperature as an observer in the asymptotic future. It is clear that, however different the effects (see, once again, [14]), reasonably accurate results can be obtained using this kind of analogy.

To examine the behaviour of the amplitude of CP violation in the early Universe, the temperature at which massive scalar particles (the ones that are well described by the model presented) decouple from the "plasma" must be determined. Since most massive scalar particles are mesons, if should suffice to determine at which temperature that happens to quarks. Calculations in the framework of lattice QCD (see [70]) returns the value of $T_{\mathrm{QCD}} \approx 155\,\mathrm{MeV}$ for this process, called the *QCD crossover*. The corresponding Unruh acceleration can be found using equation (2.170) (equation (3.56) is, fortunately, not needed here). A simple computation yields $a(T_{\mathrm{QCD}}) \approx 4 \times 10^{32}\,\mathrm{m\,s^{-2}}$. Equation (3.54) places a stringent constraint on the effect of CP violation on baryogenesis which is not appreciably affected by the phenomenon predicted by the model introduced here. Furthermore, the temperature scale at which baryogenesis is expected to take place is that of the EW crossover, $T_{\mathrm{EW}} \approx 159\,\mathrm{GeV}$ (see [71]). This investigation, however, is not fruitless, since the early Universe may serve as a substitute for very high energy experiments and some signature of the effect could be found in astronomical observations. As can be inferred from figure 6, the only cases where the decrease is significant is those where the mass difference is not much smaller than the initial masses. The plot for the case of a particle species with mass equal to the neutral kaon's (utilizing the same approximation used for the plots in figure 6) but with a much larger mass difference can be found in figure 7 and shows that $|\eta|^2$ decreases by about a third in the range presented, which implies a decrease of almost 18% in $|\eta|$. This could be enough to cause some noticeable effect if some particle species with a naturally high value for η were to exist.

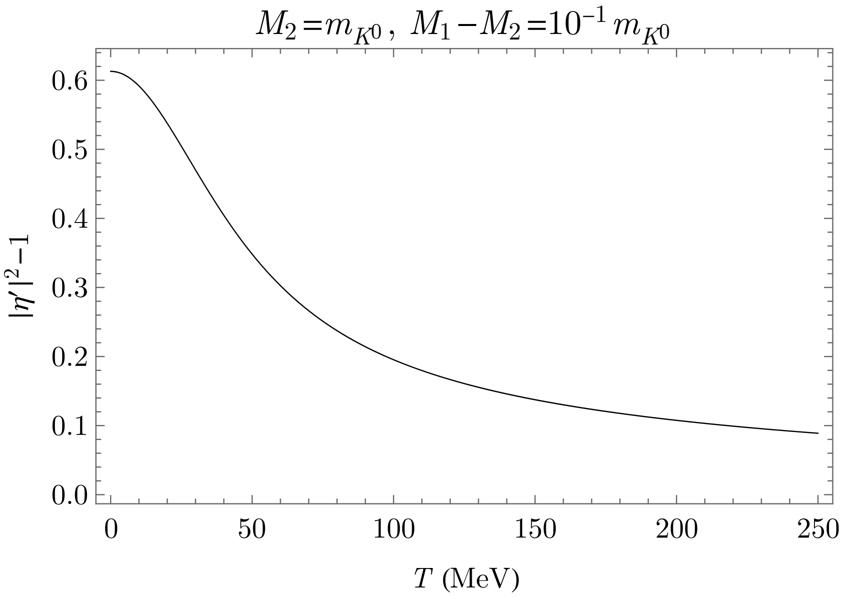

Figure 7 – The plot of an approximation of $|\eta'|^2$ as a function of the temperature T, with $m = m_{\pi^0}$.

The discussion above illustrates that two conditions are needed for the prediction of decrease in η with increasing acceleration to yield a noticeable effect in the cosmological context:

1. The mass of the particle species with CP-violating decay channels must be lower than $a(T_{\text{QCD}}) = 2\pi T_{\text{QCD}}$.
2. The scale of the mass difference between states analogous to K^0_{L} and K^0_{S} must be near the scale of their masses.

It seems unreasonable, at the present time, to expect this, since a species satisfying condition 2 would be too dissimilar to species which are sources of CP violation. Additionally, condition 1 implies that contributions to CP violation stemming from higher mass particles (like, say, the B_s meson) are not affected as much as the ones coming from kaons. This can be seen by analyzing the plot in figure 8, which presents the results for the case of a particles species similar to B_s.

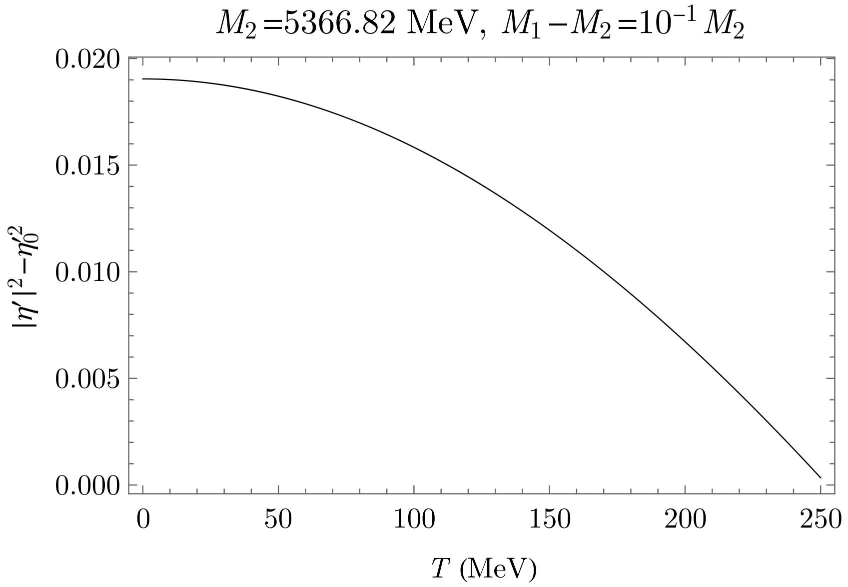

Figure 8 – The plot of an approximation of $|\eta'|^2$ as a function of the temperature T, with $m = m_{\pi^0}$, $M_2 = m_{B_s} = 5366.82\,\mathrm{MeV}/c^2$ and $\eta_0'^2 = 1.315$.

Conclusion

The developments presented above establish a connection between the Unruh effect—a prediction of QFT in curved spacetimes that illustrates, very clearly, the importance of the notion of Dirac observables—and the phenomenon of CP violation—a necessary ingredient in explaining the matter content of the Universe. The mathematical structures introduced in chapter 1 play a crucial role in the task of understanding the fundamental aspects of GR, QFT and their relationship, the focus of chapter 2. General covariance and its consequences, discussed in subsection 2.1.2, have a great impact on the process of adapting techniques of QFT in flat spacetimes to the kinds of spacetime described by GR. The study of CP violation, particularly in neutral kaon systems, reveals important properties of the weak interaction, as is shown in chapter 3.

Quantitative results on CP violation in non-inertial settings are obtained by adapting the model for the interaction mediating the decay of massive scalar, presented in [10], for the case of one massive scalar field decaying into two other massive scalar fields and computing the proper decay rate, given in equation (3.47). This is done in subsection 3.2.1. An explicit computation of the decay rate requires analytic manipulations and the execution of numerical computations, both described in appendix B. The results for the case reflecting the decay of the kaon species K^0_{L} and K^0_{S} into two pions are found in figure 4 and indicate a clear increase in the rate of these processes. Direct experimental tests seem out of reach due to the incredibly high accelerations required. Since CP violation implies a nonzero rate for the decay channels involving K^0_{L}, this could be taken as a sign that CP violation increases in amplitude for accelerating systems. Equation (3.35) indicates that this is not enough. The dependence of the CP violation parameter q on η—with vanishing η implying $q = 1$, i.e., no CP violation, and unbounded η implying $q = -1$, i.e., inversion of the CP eigenstates—favours the argument that the increase in the decay rate of K^0_{S} must also be taken into account. This is accomplished by analyzing the behaviour of the ratio η of the decay rates in these conditions.

Since the increase in decay rates is sensible to the mass of the decaying particle, η must present some variation with increasing acceleration. This is one of the results obtained with the model, presented in figure 5. The amplitude of oscillation of η in the range examined is, however, very small and obtaining experimental evidence for this phenomenon would require an enormous increase in precision. Figure 6 indicates that the mass difference between the two kaon species essentially determines the oscillation amplitude of η, implying that the effect is much more pronounced when the two species have a large difference in their masses.

CP violation plays a crucial role in understanding the puzzle of matter–antimatter asymmetry via the mechanism of baryogenesis. Since this process takes place in the early stages of the Universe, a brief exposition of its cosmological aspects is presented in subsection 3.2.3. Utilizing an analogy between the Unruh effect and the effect of particle creation in curved spacetimes, the behaviour of CP-violating systems as their temperature changes is extrapolated. Figures 7 and 8 support the existence of two necessary conditions imposed on the particle species for any effect on η to be noticeable, which seem unrealistic given the properties of known CP-violating species.

Further investigations of this kind of effect in the context of quark mixing (via the CKM matrix) are warranted and would probably yield results comparable to the ones on neutrino mixing presented in [57]. Comparing the results obtained here with models describing the impact of thermal effects on CP violation could bring insights on the validity of the model for accelerated particle decay and on the analogy between the Unruh effect and particle creating on curved spacetimes. The detection of new CP-violating species could create scenarios in which the predictions presented here would be easier to observe with currently available technology, but this would require rather remarkable differences on the properties of these new species when compared to the known ones. There is also some indirect evidence against the existence of a fourth quark generation (see the review on the number of light neutrinos in [1]), which could limit (within the framework of the SM) the number of species yet to be discovered.

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Appendices

APPENDIX A – The Wightman Function of a Massive Scalar Field

The Wightman function $\langle 0|\phi(x)^\dagger\phi(x')|0\rangle$ of a complex scalar field ϕ with mass m in Minkowski spacetime is computed for timelike separations of x^L and x'^L in this appendix. Choosing a frame where $\mathbf{x} = \mathbf{x}'$ and assuming $t > t'$, it follows from the definition of the Wightman function that

$$\begin{aligned}
\langle 0|\phi^\dagger(x)\phi(x')|0\rangle &= \frac{1}{(2\pi)^3}\langle 0|\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{\sqrt{2\omega(\mathbf{k})}}\Big(b(\mathbf{k})e^{ik_Lx^L}+a^\dagger(\mathbf{k})e^{-ik_Lx^L}\Big)\\
&\qquad\int_{\mathbb{R}^3}\mathrm{d}^3k'\,\frac{1}{\sqrt{2\omega(\mathbf{k}')}}\Big(a(\mathbf{k}')e^{ik'_Lx'^L}+b^\dagger(\mathbf{k}')e^{-ik'_Lx'^L}\Big)|0\rangle\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}^3k\,\mathrm{d}^3k'\,\frac{1}{\sqrt{2\omega(\mathbf{k})2\omega(\mathbf{k}')}}\,\langle 0|b(\mathbf{k})b^\dagger(\mathbf{k}')|0\rangle\,e^{i(k_Lx^L-k'_Lx'^L)}\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}^3k\,\mathrm{d}^3k'\,\frac{1}{\sqrt{2\omega(\mathbf{k})2\omega(\mathbf{k}')}}\\
&\qquad\langle 0|(b^\dagger(\mathbf{k}')b(\mathbf{k})+\delta(\mathbf{k}-\mathbf{k}')\mathrm{I})|0\rangle\,e^{i(k_Lx^L-k'_Lx'^L)}\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{2\omega(\mathbf{k})}e^{ik_\ell(x^\ell-x'^\ell)-i\omega(t-t')}\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3k\,\frac{1}{2\omega(\mathbf{k})}e^{-i\omega(t-t')} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}\Omega\,\mathrm{d}k\,\frac{k^2}{2\omega(k)}e^{-i\omega(t-t')}\\
&= \frac{4\pi}{(2\pi)^3}\int_0^\infty\mathrm{d}k\,\frac{k^2}{2\omega(k)}e^{-i\omega(t-t')} = \frac{1}{(2\pi)^2}\int_m^\infty\mathrm{d}\omega\,\sqrt{\omega^2-m^2}e^{-i\omega(t-t')}\\
&= \frac{1}{(2\pi)^2}\int_m^\infty\mathrm{d}\omega\,\sqrt{\omega^2-m^2}[\cos(\omega(t-t'))-i\sin(\omega(t-t'))]\\
&= \frac{\sqrt{\pi}}{2(2\pi)^2}\frac{2m}{(t-t')}\Gamma\left(\frac{3}{2}\right)[-Y_{-1}(m(t-t'))-iJ_{-1}(m(t-t'))]\\
&= -i\frac{m\pi}{2(2\pi)^2}\frac{H^{(2)}_{-1}(m(t-t'))}{(t-t')} = i\frac{m}{8\pi}\frac{H^{(2)}_1(m(t-t'))}{(t-t')},
\end{aligned}\tag{A.1}$$

where J_{-1} is the Bessel function of first kind of order -1, Y_{-1} is the Bessel function of second kind of order -1 and $H^{(2)}_1$ is the Hankel function of second kind of order 1. Integrals 3.771–7 and 3.771–9 of [72] were used above. Since $t - t' = \sqrt{-\Delta s^2}$ in this frame, where Δs^2 is the squared spacetime interval, and the Wightman function is Lorentz invariant[44], this expression takes the form

$$\langle 0|\phi^\dagger(x)\phi(x')|0\rangle = i\frac{m}{8\pi}\frac{H^{(2)}_1(m\sqrt{-\Delta s^2})}{\sqrt{-\Delta s^2}} \tag{A.2}$$

[44] It is clear that the field expansion is Lorentz invariant, since it is an integral with Lorentz invariant measure of a Lorentz invariant expression. The vacuum state is also clearly Lorentz invariant.

in a frame where $\mathbf{x} = \mathbf{x}'$ does not hold. If it is assumed that $t' > t$, then

$$
\begin{aligned}
\langle 0|\phi^\dagger(x)\phi(x')|0\rangle &= \frac{1}{(2\pi)^2}\int_m^\infty \mathrm{d}\omega\,\sqrt{\omega^2-m^2}[\cos(\omega(t-t')) - i\sin(\omega(t-t'))] \\
&= \frac{1}{(2\pi)^2}\int_m^\infty \mathrm{d}\omega\,\sqrt{\omega^2-m^2}[\cos(\omega(t'-t)) + i\sin(\omega(t'-t))] \\
&= \frac{\sqrt{\pi}}{2(2\pi)^2}\frac{2m}{(t'-t)}\Gamma\left(\frac{3}{2}\right)[-Y_{-1}(m(t'-t)) + iJ_{-1}(m(t'-t))] \qquad \text{(A.3)} \\
&= -i\frac{m\pi}{2(2\pi)^2}\frac{H_{-1}^{(1)}(m(t'-t))}{(t-t')} = -i\frac{m}{8\pi}\frac{H_{-1}^{(2)}(m(t-t'))}{(t-t')} \\
&= i\frac{m}{8\pi}\frac{H_1^{(2)}(m(t-t'))}{(t-t')}.
\end{aligned}
$$

In this case, however, $t' - t = \sqrt{-\Delta s^2}$, which implies that, for frames where $\mathbf{x} = \mathbf{x}'$ does not hold,

$$
\langle 0|\phi(x)^\dagger\phi(x')|0\rangle = i\frac{m}{8\pi}\frac{H_1^{(2)}(-m\sqrt{-\Delta s^2})}{(-\sqrt{-\Delta s^2})}. \qquad \text{(A.4)}
$$

To combine equations (A.2) and (A.4) into a single equation, valid whether $t' < t$ or $t' > t$, the proper time interval is defined as[45] $\Delta s \coloneqq \mathrm{sgn}(x^0 - x'^0)\sqrt{-\Delta s^2}$. Thus,

$$
\langle 0|\phi^\dagger(x)\phi(x')|0\rangle = i\frac{m}{8\pi}\frac{H_1^{(2)}(m\Delta s)}{\Delta s}. \qquad \text{(A.5)}
$$

[45] The proper time interval Δs should not be confused with the square root of the symmetric of the squared spacetime interval Δs^2, which is equal to $|\Delta s|$.

APPENDIX B – Numerical Treatment of the Decay Rate

In this appendix, the numerical treatment[46] of the decay rate obtained in subsection 3.2.1 is presented. Recall the expression for this decay rate:

$$\Gamma = -\frac{G_\Gamma^2 \kappa}{128\pi^2} m^2 a \int_{-\infty}^{\infty} \mathrm{d}u \, e^{i2Mu/a} \frac{\left[H_1^{(2)}\left(\frac{2m}{a}\sinh(u)\right)\right]^2}{\sinh^2(u)}. \tag{B.1}$$

Call the integral appearing in this expression $\mathcal{I}$, i.e.,

$$\mathcal{I} = \int_{-\infty}^{\infty} \mathrm{d}u \, e^{i2Mu/a} \frac{\left[H_1^{(2)}\left(\frac{2m}{a}\sinh(u)\right)\right]^2}{\sinh^2(u)}. \tag{B.2}$$

It is difficult to tackle this expression analytically, so the numerical approach is favored. Problems in the implementation of the numerical methods appear due to singularities in this expression, which means that the singular parts of the integral must be separated into an integral $\mathcal{I}_0$ and treated analytically, so that $\mathcal{I}$ can be computed as

$$\mathcal{I} = \underbrace{(\mathcal{I} - \mathcal{I}_0)}_{\substack{\text{treated}\\\text{numerically}}} + \underbrace{\mathcal{I}_0}_{\substack{\text{treated}\\\text{analytically}}}, \tag{B.3}$$

The singular part of $\mathcal{I}$ can be determined using the power series of the Bessel functions of first and second kind, J_1 and Y_1, given in equations 10.2.2 and 10.8.1 of [73], up to third order to determine the singular parts of the Hankel function of second kind $H_1^{(2)}$:

$$\begin{aligned}
H_1^{(2)}(z) &= J_1(z) - iY_1(z) \\
&= \frac{z}{2}\left(\frac{1}{\Gamma(2)} - \frac{z^2}{4\Gamma(3)}\right) - i\left\{ -\frac{2}{z\pi} + \frac{2}{\pi}\ln\left(\frac{z}{2}\right)\frac{z}{2}\left(\frac{1}{\Gamma(2)} - \frac{z^2}{4\Gamma(3)}\right)\right. \\
&\qquad \left. - \frac{z}{2\pi}\left[-\gamma - \gamma + 1 - \left(-\gamma + 1 - \gamma + 1 + \frac{1}{2}\right)\frac{z^2}{8}\right]\right\} + \mathcal{O}(z^4) \\
&= \left(\frac{z}{2} - \frac{z^3}{16}\right) - \frac{i}{\pi}\left\{-\frac{2}{z} + \frac{3z^3}{32} + \left(z - \frac{z^3}{8}\right)\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\} + \mathcal{O}(z^4) \\
&= \frac{2i}{\pi z} - \frac{3iz^3}{32\pi} + \left(z - \frac{z^3}{8}\right)\left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\} + \mathcal{O}(z^4),
\end{aligned} \tag{B.4}$$

where γ is the *Euler–Mascheroni* constant. Since this function appears squared in the

[46] Nearly all the developments presented here were introduced in [10].

expression for the decay rate, the square of the equation above is obtained below:

$$
\begin{aligned}
\left[H_1^{(2)}(z)\right]^2 &= \left[\frac{2i}{\pi z} - \frac{3iz^3}{32\pi} + \left(z - \frac{z^3}{8}\right)\left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\}\right]^2 + \mathcal{O}(z^4) \\
&= -\frac{4}{\pi^2 z^2} + \frac{3z^2}{8\pi^2} + \frac{4i}{\pi z}\left(z - \frac{z^3}{8}\right)\left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\} \\
&\quad + \left(z - \frac{z^3}{8}\right)^2 \left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\}^2 + \mathcal{O}(z^4) \\
&= -\frac{4}{\pi^2 z^2} + \frac{3z^2}{8\pi^2} + \frac{4i}{\pi}\left(1 - \frac{z^2}{8}\right)\left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\} \\
&\quad + z^2\left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{z}{2}\right) - \frac{1}{2} + \gamma\right]\right\}^2 + \mathcal{O}(z^4).
\end{aligned} \tag{B.5}
$$

The expression also involves a division by a second order polynomial on the argument of function, $2m\sinh(u)/a$. Therefore, the following expression is computed:

$$
\begin{aligned}
\frac{\left[H_1^{(2)}\left(\frac{2m}{a}z\right)\right]^2}{z^2} &= -\frac{a^2}{\pi^2 m^2 z^4} + \frac{3m^2}{2\pi^2 a^2} + \frac{2i}{\pi}\left(\frac{1}{z^2} - \frac{m^2}{2a^2}\right) \\
&\quad + \frac{4}{\pi^2}\left(\frac{1}{z^2} - \frac{m^2}{2a^2}\right)\left[\ln\left(\frac{m}{a}z\right) - \frac{1}{2} + \gamma\right] \\
&\quad - \frac{4m^2}{a^2}\left\{\frac{1}{2} - \frac{i}{\pi}\left[\ln\left(\frac{m}{a}z\right) - \frac{1}{2} + \gamma\right]\right\}^2 + \mathcal{O}(z^2) \\
&= -\frac{a^2}{\pi^2 m^2 z^4} + \frac{3m^2}{2\pi^2 a^2} + \frac{2i}{\pi}\left(\frac{1}{z^2} - \frac{m^2}{2a^2}\right) \\
&\quad + \frac{4}{\pi^2}\left(\frac{1}{z^2} - \frac{m^2}{2a^2}\right)\left[\ln\left(\frac{m}{a}z\right) - \frac{1}{2} + \gamma\right] \\
&\quad + \frac{m^2}{a^2} - \frac{4i}{\pi}\frac{m^2}{a^2}\left[\ln\left(\frac{m}{a}z\right) - \frac{1}{2} + \gamma\right] \\
&\quad - \frac{4m^2}{\pi^2 a^2}\left[\ln^2\left(\frac{m}{a}z\right) + 2\left(\gamma - \frac{1}{2}\right)\ln\left(\frac{m}{a}z\right) + \left(\gamma - \frac{1}{2}\right)^2\right] + \mathcal{O}(z^2) \\
&= -\frac{a^2}{\pi^2 m^2}\frac{1}{z^4} + 2\left[\frac{i}{\pi} + \frac{2}{\pi^2}\left(\gamma - \frac{1}{2}\right)\right]\frac{1}{z^2} + \frac{4}{\pi^2}\frac{1}{z^2}\ln\left(\frac{m}{a}z\right) \\
&\quad + \frac{2m^2}{\pi^2 a^2}(1 - 4\gamma - 2i\pi)\ln\left(\frac{m}{a}z\right) - \frac{4m^2}{\pi^2 a^2}\ln^2\left(\frac{m}{a}z\right) + \mathcal{O}(1).
\end{aligned} \tag{B.6}
$$

The integral $\mathcal{I}_0$ of the singular parts is then given by

$$
\mathcal{I}_0 = \sum_{j=1}^{5} \mathcal{I}_j, \tag{B.7}
$$

where

$$\mathcal{I}_1 \coloneqq -\frac{a^2}{\pi^2 m^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^4(u)}, \tag{B.8a}$$

$$\mathcal{I}_2 \coloneqq 2\left[\frac{i}{\pi}+\frac{2}{\pi^2}\left(\gamma-\frac{1}{2}\right)\right]\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^2(u)}, \tag{B.8b}$$

$$\mathcal{I}_3 \coloneqq \frac{4}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^2(u)}\ln\!\left(\frac{m}{a}\sinh(u)\right), \tag{B.8c}$$

$$\mathcal{I}_4 \coloneqq \frac{2m^2}{\pi^2 a^2}(1-4\gamma-2i\pi)\int_{-\infty}^{\infty}\mathrm{d}u\,e^{i2Mu/a}\ln\!\left(\frac{m}{a}\sinh(u)\right), \tag{B.8d}$$

$$\mathcal{I}_5 \coloneqq -\frac{4m^2}{\pi^2 a^2}\int_{-\infty}^{\infty}\mathrm{d}u\,e^{i2Mu/a}\ln^2\!\left(\frac{m}{a}\sinh(u)\right). \tag{B.8e}$$

The integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ can be calculated using the residue theorem,

$$\begin{aligned}
\mathcal{I}_1 \propto{}& \int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^4(u)} = \frac{1-e^{-2\pi M/a}}{1-e^{-2\pi M/a}}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2M(u)/a}}{\sinh^4(u)}\\
&\propto \lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{1}{1-e^{-2\pi M/a}}\left[\int_{-R}^{R}\mathrm{d}u\,\frac{e^{i2M(u-i\varepsilon)/a}}{\sinh^4(u-i\varepsilon)}+\int_{R}^{-R}\mathrm{d}u\,\frac{e^{i2M(u-i\varepsilon+i\pi)/a}}{\sinh^4(u-i\varepsilon+i\pi)}\right.\\
&\qquad\left.+\,i\int_{-\varepsilon}^{\pi-\varepsilon}\mathrm{d}u\,\frac{e^{-2M(u-iR)/a}}{\sinh^4(iu+R)}+i\int_{\pi-\varepsilon}^{-\varepsilon}\mathrm{d}u\,\frac{e^{-2M(u+iR)/a}}{\sinh^4(iu-R)}\right]\\
&\propto \lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{1}{1-e^{-2\pi M/a}}\int_C\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^4(u)} = \lim_{u\to 0}\frac{2\pi i}{3!}\frac{1}{1-e^{-2\pi M/a}}\frac{\mathrm{d}^3}{\mathrm{d}u^3}\left[u^4\frac{e^{i2Mu/a}}{\sinh^4(u)}\right]\\
&\propto \frac{8\pi M}{3a^3}\frac{a^2+M^2}{1-e^{-2\pi M/a}},
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
\mathcal{I}_2 \propto{}& \int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^2(u)} = \frac{1-e^{-2\pi M/a}}{1-e^{-2\pi M/a}}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^2(u)}\\
&\propto \lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{1}{1-e^{-2\pi M/a}}\left[\int_{-R}^{R}\mathrm{d}u\,\frac{e^{i2M(u-i\varepsilon)/a}}{\sinh^2(u-i\varepsilon)}+\int_{R}^{-R}\mathrm{d}u\,\frac{e^{i2M(u-i\varepsilon+i\pi)/a}}{\sinh^2(u-i\varepsilon+i\pi)}\right.\\
&\qquad\left.+\,i\int_{0}^{\pi}\mathrm{d}u\,\frac{e^{-2M(u-iR)/a}}{\sinh^2(iu+R)}+i\int_{\pi}^{0}\mathrm{d}u\,\frac{e^{-2M(u+iR)/a}}{\sinh^2(iu-R)}\right]\\
&\propto \lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{1}{1-e^{-2\pi M/a}}\int_C\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^2(u)} = \lim_{u\to 0}2\pi i\frac{1}{1-e^{-2\pi M/a}}\frac{\mathrm{d}}{\mathrm{d}u}\left[u^2\frac{e^{i2Mu/a}}{\sinh^2(u)}\right]\\
&\propto -\frac{4\pi M}{a}\frac{1}{1-e^{-2\pi M/a}},
\end{aligned} \tag{B.10}$$

if the integration contour C is taken to be the one pictured in figure 9, given that $1/\sinh(u)$ has singularities at $ik\pi$ for every $k\in\mathbb{Z}$. Of note is that the contributions of $\mathcal{I}_1$ and $\mathcal{I}_2$ to the decay rate correspond to the decay rates of the processes $\Psi\to\psi_1\psi_2$ and $\Psi\to\psi_1$ respectively, where Ψ is a massive scalar field and ψ_1 and ψ_2 are massless scalar fields, since the Wightman function of a massless scalar field over an accelerated trajectory is proportional to $1/\sinh^2(u)$ (see [10] for an explicit calculation involving massless fields).

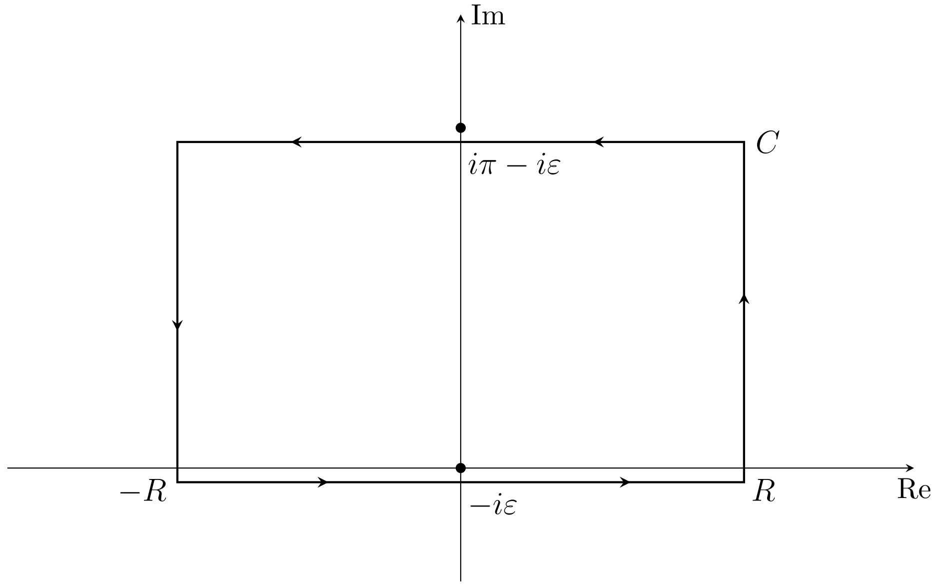

Figure 9 – The integration contour C for the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$. The singularities of $1/\sinh(u)$ are indicated by points.

The integrals $\mathcal{I}_4$ and $\mathcal{I}_5$ can be approximated by integrals $\mathcal{I}_4'$ and $\mathcal{I}_5'$ with similar singularity structures,

$$\begin{aligned}
\mathcal{I}_4' &\coloneqq \frac{2m^2}{\pi^2 a^2}(1-4\gamma-2i\pi)\int_{-\infty}^{\infty}\mathrm{d}u\, e^{-2M|u|/a}\ln\Big(\frac{m}{a}|u|\Big) \\
&= \frac{4m^2}{\pi^2 a^2}(1-4\gamma-2i\pi)\int_{0}^{\infty}\mathrm{d}u\, e^{-2Mu/a}\ln\Big(\frac{m}{a}u\Big) \\
&= \frac{4m^2}{\pi^2 a^2}(1-4\gamma-2i\pi)\left[-\frac{a}{m}\frac{m}{2M}\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)\right] \\
&= -\frac{2m^2}{\pi^2 aM}(1-4\gamma-2i\pi)\left(\gamma+\ln\left(\frac{2M}{m}\right)\right),
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
\mathcal{I}_5' &\coloneqq -\frac{4m^2}{\pi^2 a^2}\int_{-\infty}^{\infty}\mathrm{d}u\, e^{-2M|u|/a}\left[\ln^2\Big(\frac{m}{a}|u|\Big)+2i\pi\theta(-u)\ln\Big(\frac{m}{a}|u|\Big)\right] \\
&= -\frac{4m^2}{\pi^2 a^2}\int_{0}^{\infty}\mathrm{d}u\, e^{-2Mu/a}\left[2\ln^2\Big(\frac{m}{a}|u|\Big)+2i\pi\ln\Big(\frac{m}{a}|u|\Big)\right] \\
&= -\frac{8m^2}{\pi^2 a^2}\frac{a}{m}\frac{m}{2M}\left[\left[\frac{\pi^2}{6}+\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)^2\right]-i\pi\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)\right] \\
&= -\frac{4m^2}{\pi^2 aM}\left[\left[\frac{\pi^2}{6}+\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)^2\right]-i\pi\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)\right],
\end{aligned} \tag{B.12}$$

since $\sinh(x) = x + \mathcal{O}(x^3)$ and the singular parts of $\ln(u) = \ln(|u|) + i\pi\theta(-u)$ and its square $\ln^2(u) = \ln^2(|u|) + 2i\pi\theta(-u)\ln(|u|) - \pi^2\theta(-u)$, for $u \in \mathbb{R} - \{0\}$, are $\ln(|z|)$ and $\ln^2(|z|) + 2i\pi\theta(-u)\ln(|u|)$, respectively. Solving these integrals makes use of the parity of the integrands and equations 4.331–1 and 4.335–1 of [72].

Computing $\mathcal{I}_3$ is more cumbersome, but can be achieved using the following identity for the logarithm,

$$\ln(z) = \frac{\mathrm{d}}{\mathrm{d}w} z^w|_{w=0} = \lim_{w\to 0} \frac{z^w - 1}{w}, \tag{B.13}$$

and the series expansion of $1/\sinh^2(u)$,

$$\frac{1}{\sinh^2(u)} = \frac{1}{u^2} - \frac{1}{3} + \mathcal{O}(u). \tag{B.14}$$

Then,

$$\mathcal{I}_3 = \frac{4}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\, e^{i2Mu/a}\ln\Big(\frac{m}{a}\sinh(u)\Big)\Big(\frac{1}{u^2} - \frac{1}{3} + \mathcal{O}(u)\Big). \tag{B.15}$$

This implies that $\mathcal{I}_3$ can be approximated by two other integrals, as in equations (B.11) and (B.12),

$$\mathcal{I}_3' := \frac{4}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{u^2}\ln\Big(\frac{m}{a}u\Big) \tag{B.16}$$

$$\mathcal{I}_3'' := -\frac{4}{3\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\, e^{-2M|u|/a}\ln\Big(\frac{m}{a}u\Big) \tag{B.17}$$

The solution for the integral $\mathcal{I}_3''$ follows directly from equation (B.11),

$$\mathcal{I}_3'' = \frac{4a}{3\pi^2 M}\Big(\gamma + \ln\Big(\frac{2M}{m}\Big)\Big), \tag{B.18}$$

while the one for $\mathcal{I}_3'$ requires the use of equation (B.13):

$$\begin{aligned}\mathcal{I}_3' &:= \frac{4}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{u^2}\ln\Big(\frac{m}{a}u\Big) \propto \lim_{w\to 0}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{u^2}\frac{\big(\frac{m}{a}u\big)^w - 1}{w}\\ &\propto \lim_{w\to 0}\frac{(-i)^{w-2}}{w}\Big(\frac{m}{a}\Big)^w\int_{-\infty}^{\infty}\mathrm{d}u\, e^{i2Mu/a}(iu)^{w-2} - \lim_{w\to 0}\frac{(-i)^{-2}}{w}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{(iu)^2}.\end{aligned} \tag{B.19}$$

The solutions to the integrals appearing above can be found in equations 3.382–6 and 3.382–7 of [72]. Therefore,

$$\begin{aligned}\mathcal{I}_3' &\propto -\lim_{w\to 0}\frac{1}{w}\Big(-\frac{im}{a}\Big)^w\Big(\frac{2M}{a}\Big)^{1-w}\frac{2\pi}{\Gamma(2-w)} + \lim_{w\to 0}\frac{1}{w}\frac{2M}{a}\frac{2\pi}{\Gamma(2)}\\ &\propto -\frac{4\pi M}{a}\lim_{w\to 0}\frac{1}{w}\Big(-\frac{im}{2M}\Big)^w(1+(1-\gamma)w) + \lim_{w\to 0}\frac{1}{w}\frac{4\pi M}{a}\\ &\propto -\frac{4\pi M}{a}\lim_{w\to 0}\Big(\frac{1}{w}\Big[\Big(-\frac{im}{2M}\Big)^w - 1\Big] + \Big(\frac{-im}{2M}\Big)^w(1-\gamma)\Big)\\ &\propto -\frac{4\pi M}{a}\Big(\ln\Big(-\frac{im}{2M}\Big) + 1 - \gamma\Big) = -\frac{4\pi M}{a}\Big(\ln\Big(\frac{m}{2M}\Big) - \frac{i\pi}{2} + 1 - \gamma\Big),\end{aligned} \tag{B.20}$$

where it was used that $1/\Gamma(2-w) = 1 + (1-\gamma)w$

The strategy introduced at the beginning of this appendix can now be executed, but the singular integral $\mathcal{I}_0$ must be substituted by

$$\mathcal{I}_0' = \sum_{j=1}^{2}\mathcal{I}_j + \sum_{j=3}^{5}\mathcal{I}_j' + \mathcal{I}_3'', \tag{B.21}$$

where

$$\mathcal{I}_1 = -\frac{a^2}{\pi^2 m^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^4(u)} = -\frac{8M}{3\pi a m^2}\frac{a^2+M^2}{1-e^{-2\pi M/a}}, \tag{B.22a}$$

$$\begin{aligned}\mathcal{I}_2 &= 2\left[\frac{i}{\pi}+\frac{2}{\pi^2}\left(\gamma-\frac{1}{2}\right)\right]\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{\sinh^2(u)} \\ &= -\frac{8\pi M}{a}\left[\frac{i}{\pi}+\frac{2}{\pi^2}\left(\gamma-\frac{1}{2}\right)\right]\frac{1}{1-e^{-2\pi M/a}},\end{aligned} \tag{B.22b}$$

$$\mathcal{I}_3' = \frac{4}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{i2Mu/a}}{u^2}\ln\left(\frac{m}{a}u\right) = -\frac{16M}{\pi a}\left(\ln\left(\frac{m}{2M}\right)-\frac{i\pi}{2}+1-\gamma\right), \tag{B.22c}$$

$$\begin{aligned}\mathcal{I}_4' &= \frac{2m^2}{\pi^2 a^2}(1-4\gamma-2i\pi)\int_{-\infty}^{\infty}\mathrm{d}u\,e^{-2M|u|/a}\ln\left(\frac{m}{a}|u|\right) \\ &= -\frac{2m^2}{\pi^2 aM}(1-4\gamma-2i\pi)\left(\gamma+\ln\left(\frac{2M}{m}\right)\right),\end{aligned} \tag{B.22d}$$

$$\begin{aligned}\mathcal{I}_5' &= -\frac{4m^2}{\pi^2 a^2}\int_{-\infty}^{\infty}\mathrm{d}u\,e^{-2M|u|/a}\left[\ln^2\left(\frac{m}{a}|u|\right)+2i\pi\theta(-u)\ln\left(\frac{m}{a}|u|\right)\right] \\ &= -\frac{4m^2}{\pi^2 aM}\left[\left[\frac{\pi^2}{6}+\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)^2\right]-i\pi\left(\gamma+\ln\left(\frac{2M}{m}\right)\right)\right]\end{aligned} \tag{B.22e}$$

$$\mathcal{I}_3'' = -\frac{4}{3\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,e^{-2M|u|/a}\ln\left(\frac{m}{a}u\right) = \frac{4a}{3\pi^2 M}\left(\gamma+\ln\left(\frac{2M}{m}\right)\right) \tag{B.22f}$$

so that $\mathcal{I}$ can be computed as

$$\mathcal{I} = \underbrace{(\mathcal{I}-\mathcal{I}_0')}_{\substack{\text{treated}\\\text{numerically}}} + \underbrace{\mathcal{I}_0'}_{\substack{\text{treated}\\\text{analytically}}}. \tag{B.23}$$

The integral $(\mathcal{I}-\mathcal{I}_0')$ is highly oscillatory with a rapidly decaying integrand, requiring special techniques to be used in its computation, done in Wolfram Mathematica 11.2. The dominating contribution to Γ comes from $\mathcal{I}_0'$, especially for high values of a (see equation (B.6)). The results also present very small imaginary parts (at their largest, 2 orders of magnitude smaller than their real parts) that can be attributed to numerical errors. They are presented in subsection 3.2.2.