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Robustness of the quadratic partial eigenvalue assignment using spectrum sensitivities for state and derivative feedback designs

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Abstract

Based on the notions of spectrum sensitivities, proposed by us earlier, we develop a novel optimization approach to deal with robustness in the closed-loop eigenvalues for partial quadratic eigenvalue assignment problem arising in active vibration control. A distinguished feature of this new approach is that the objective function is composed of only the system and the closed-loop feedback matrices. It does not need an explicit knowledge of the eigenvalues and eigenvectors. Furthermore, the approach is applicable to both the state-feedback and derivative feedback designs. These features make the approach viable to design an active vibration controller for practical applications to large real-life structures. A comparative study with existing algorithms and a study on the transient response of a real-life system demonstrate the effectiveness, superiority, and competitiveness of the proposed approach.

Keywords

Partial eigenvalue assignment, active vibration control, robustness, spectrum sensitivity

Introduction

Vibrating structures, such as bridges, highways, automobiles, air and space crafts, and others, are usually modeled by a system of second-order differential equations generated by finite element discretization of the original distributed parameter systems. Such second-order system is known as the finite element model (FEM) in the vibration literature.[1–7] These structures sometimes experience dangerous vibrations caused by resonance when excited by external forces including earthquake, gusty winds, weights of human bodies that may result in partial or complete destruction of the structures. In practice, and very often, such vibrations are controlled by using passive damping forces. Besides being economic to apply it, such an approach has several practical drawbacks: it is ad hoc in nature and is able to control only localized vibrations. On the other hand, the technique of active vibration control (AVC) is scientifically based and can control vibrations globally in a structure if properly implemented.[8–13] The most important aspect of the AVC implementation is to effectively and efficiently compute the feedback forces needed to control the measured unwanted vibrations, caused by the resonant frequencies.

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Recently, a mathematically elegant approach that reassigns a few resonant eigenvalues to suitably chosen ones while keeping the other large number of them and the associated eigenvectors unchanged has been proposed. This latter approach is known to have the no-spill over property, and the problem of computing the feedback matrices to reassign the unwanted eigenvalues in this way is called partial quadratic eigenvalue problem (PQEVAP). The approach works exclusively in the second-order setting itself and is capable of taking advantages of computationally exploitable inherit structural properties of FEM, such as definiteness, sparsity, bandness, etc. which are assets in large-scale computational settings. Typically, the mass and stiffness matrices are symmetric, the mass matrix is positive diagonal, and the stiffness matrix is three-diagonal and positive definite or semidefinite. The most attractive feature of this approach is that the no-spill over property is guaranteed by means of a mathematical theory. This is in sharp contrast with the standard and obvious solution approach of the PQEVAP by transforming a second-order control system to a standard linear state-space. By doing so, one can clearly make use of the existing excellent numerical methods for eigenvalue assignment problems.[14] However, in this case one needs to deal with a system of dimensions twice that of the original model, which then becomes computationally prohibitive even with a moderate-size model. Note that the FEM models that arise from practical applications, especially in aerospace and space engineering, and power systems control, could be very large, possibly of multi-million degree of freedom, and computational methods for such large-scale matrix computations are not well developed.[15] More importantly, by transforming to a standard state-space linear system, all the exploitable properties of the FEM, as stated above, will be completely destroyed. By transforming it to a generalized state-space system,[15] the symmetry can be preserved but not the definiteness. Furthermore, such generalized transformations give rise to descriptor control problems, and the numerical methods for such control problems, especially for singular and nearly singular and large-scale systems, do not virtually exist.[14]

A basic solution of the original PQEVAP that meets with the above practical requirements is originally proposed by Datta et al.,[16] in the single input case and then subsequently generalized to the multi-input cases by Datta et al.[17] and Ram and Elhay.[18] For practical effectiveness, it is not enough just to compute a pair of feedback matrices, but they should be computed in such a way that they have norms as small as possible and the closed-loop eigenvalues are as insensitive as possible to small perturbations to the data.[19–22] Minimization of feedback norms leads to economic design while the minimization of the closed-loop eigenvalue sensitivity ensures numerical robustness in the control design. While the former is straightforward,[19,20] the later poses a difficult computational task, because the major part of the closed-loop eigenvector matrix consists of the large number of eigenvalues and eigenvectors which are not known to the users. In order to overcome these computational difficulties, we propose a novel minimization approach to deal with the robustness issue. It is based on the notions of spectrum sensitivity, introduced earlier by the authors. The distinguished feature of this approach is that the objective function is formulated in such a way that it does not require explicit knowledge of the eigenvalues and eigenvectors. It is composed of only the closed-loop feedback matrices. The gradient formulas are then computed also in terms of the feedback matrices. This feature makes it possible to design robust controllers in a practical way. In this paper, we propose a new optimization algorithm for RPQEVAP for which the objective function is formulated in terms of the closed-loop mass, stiffness, and damping matrices, thus computation of this function and of the associated gradient formula can be performed without explicitly knowing the closed-loop eigenvectors. This objective function depends upon several spectrum sensitivity results which exhibit these eigenvalue sensitivity relations with the closed-loop feedback matrices. The required gradient formulas are derived in the paper in terms of the closed-loop feedback matrices. These new optimization algorithms are obtained for both cases of the state feedback and state-derivative feedback. Numerical examples, both with small and large order matrices, are performed and an experiment to study the transient response of a real-life system is carried out to demonstrate the effectiveness of the proposed approach in both cases of state and velocity feedback. A comparative study with other existing algorithms shows that the robustness achieved by the proposed algorithms is comparable to, and, in some cases, is better than that achieved by algorithms which aim at minimizing the condition number of the closed-loop eigenvector matrix.

Preliminary concepts on second-order systems and the partial quadratic eigenvalue assignment problems (PQEVAPs)

A vibrating structure modeled by a system matrix second-order differential equations has the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are, respectively, the mass, damping, and stiffness matrices, each of them of order n, and $\mathbf{x}(t)$ is the displacement vector. Since this model is often generated by using finite element techniques, it is known as the FEM. The matrices often have special structures

$$\mathbf{M} = \mathbf{M}^T \succ 0, \quad \mathbf{C} = \mathbf{C}^T, \quad \mathbf{K} = \mathbf{K}^T \succcurlyeq 0 \tag{2}$$

in which the superscript T denotes the matrix transpose operator. Dynamics of such a system are governed by the eigenvalues and eigenvectors of the associated quadratic matrix eigenvalue problem

$$\mathbf{Q}(\lambda_k)\mathbf{y}_k = 0, \quad k = 1, 2 \ldots, 2n \tag{3}$$

with the pencil

$$\mathbf{Q}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} \tag{4}$$

The details on the quadratic eigenvalue problem can be found in Datta.[15] Suppose a control force of the form

$$\mathbf{f}(t) = \mathbf{B}\mathbf{u}(t) \tag{5}$$

where $\mathbf{B}$ is an $n \times m$ control matrix and $\mathbf{u}(t)$ is a control vector of order m, applied to the model to control the unwanted vibrations caused by resonances. Thus, we have the control model

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t) \tag{6}$$

Assuming that the state and the velocity vectors $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ are known, let us take

$$\mathbf{u}(t) = \mathbf{F}_s\dot{\mathbf{x}}(t) + \mathbf{G}_s\mathbf{x}(t) \tag{7}$$

where $\mathbf{F}_s$ and $\mathbf{G}_s$ are two unknown velocity and state feedback matrices. Many times the state vector $\mathbf{x}(t)$ is not explicitly known, but the velocity vector $\dot{\mathbf{x}}(t)$ and the acceleration vector $\ddot{\mathbf{x}}(t)$ can rather be estimated. In such case, it is more practical to assume that

$$\mathbf{u}(t) = \mathbf{F}_d\dot{\mathbf{x}}(t) + \mathbf{G}_d\ddot{\mathbf{x}}(t) \tag{8}$$

Here, the unknown feedback matrices $\mathbf{F_d}$ and $\mathbf{G_d}$ are named, respectively, velocity and acceleration feedback matrices. The control laws defined by equations (7) and (8) are, respectively, called the state feedback and the derivative feedback laws. Given then these expressions of the control inputs, the respective closed-loop systems can be written as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + (\mathbf{C} - \mathbf{B}\mathbf{F}_s)\dot{\mathbf{x}}(t) + (\mathbf{K} - \mathbf{B}\mathbf{G}_s)\mathbf{x}(t) = 0 \tag{9}$$

$$(\mathbf{M} - \mathbf{B}\mathbf{G}_d)\ddot{\mathbf{x}}(t) + (\mathbf{C} - \mathbf{B}\mathbf{F}_d)\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = 0 \tag{10}$$

PQEVAP and robustness

The PQEVAP is to assign a few eigenvalues of $Q(\lambda)$, say, $\lambda_1, \ldots, \lambda_p$; $p \ll 2n$, which are believed to cause resonances in a vibrating structure, to suitably chosen numbers $\mu_1, \ldots, \mu_p$ by computing the two feedback matrices $\mathbf{F}_s$ and $\mathbf{G}_s$ for the state feedback case, and $\mathbf{F}_d$ and $\mathbf{G}_d$ for the derivative feedback case, while leaving the other eigenvalues and the associated eigenvectors unchanged.

In the multi-input case, if there exists a feedback pair, then there are infinitely many.

The problems of choosing the feedback matrices with the property that they have minimum feedback norms and that the closed-loop eigenvector matrix is well conditioned are, respectively, called minimum norm and robust PQEVAPs, denoted by MNPQEVAP and RPQEVAP.

Notations

In order to state the solutions of these problems in the next section, let us introduce the following notations:

- $\mathbf{\Lambda}_1 = diag\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_l & \beta_l \\ -\beta_l & \alpha_l \end{bmatrix}, \lambda_{2l+1}, \ldots, \lambda_p\right)$, in which $\lambda_k = conj(\lambda_{k+1}) = \alpha_k + i\beta_k,\ \ k = 1, \ldots, 2l$ and $\lambda_{2l+1}, \ldots, \lambda_p \in \mathbb{R}$. It is a real representation of the eigenvalues that must be reassigned.
- $\bar{\mathbf{\Lambda}}_1 = diag\left(\begin{bmatrix} \bar{\alpha}_1 & \bar{\beta}_1 \\ -\bar{\beta}_1 & \bar{\alpha}_1 \end{bmatrix}, \ldots, \begin{bmatrix} \bar{\alpha}_l & \bar{\beta}_l \\ -\bar{\beta}_l & \bar{\alpha}_l \end{bmatrix}, \mu_{2\bar{l}+1}, \ldots, \mu_p\right)$, in which $\mu_k = conj(\mu_{k+1}) = \bar{\alpha}_k + i\bar{\beta}_k,\ \ k = 1, \ldots, 2\bar{l}$ and $\mu_{2\bar{l}+1}, \ldots, \mu_p \in \mathbb{R}$. It is a real representation of the new eigenvalues.
- $\mathbf{Y}_1 = \left[\Re e(\mathbf{y}_1) \quad \Im m(\mathbf{y}_1) \quad \cdots \quad \Re e(\mathbf{y}_l) \quad \Im m(\mathbf{y}_l) \quad \mathbf{y}_{2l+1} \cdots \mathbf{y}_p\right]$. It is a real representation of the eigenvectors that must be reassigned.

Note that l and $\bar{l}$ are not necessarily equal, that is the cardinality of the complex eigenvalues of the spectrum part to be assigned does not need to equal that of the reassigned part.

The PQEVAP and robustness solutions

In this section, we first state a known parametric solution to PQEVAP and then propose a new optimization approach to study its robustness.

A solution to PQEVAP

- Construction of $\mathbf{F}_s$ and $\mathbf{G}_s$: Let arbitrary $\mathbf{\Gamma}_s \in \mathrm{R}^{m\times p}$ and $\mathbf{Z}_s \in \mathrm{R}^{p\times p}$ be the solution of the Sylvester equation [23]

$$\mathbf{\Lambda}_1^T\mathbf{Z}_s^T - \mathbf{Z}_s^T\bar{\mathbf{\Lambda}}_1 = -\mathbf{Y}_1^T\mathbf{B}\mathbf{\Gamma}_s \tag{11}$$

If $\mathbf{Z}_s$ is invertible, and

$$\mathbf{\Phi}_s = \mathbf{\Gamma}_s\mathbf{Z}_s^{-T} \tag{12}$$

then it has been shown in Cai et al.[22] that

$$\mathbf{F}_s = \mathbf{\Phi}_s\mathbf{Y}_1^T\mathbf{M}, \quad \mathbf{G}_s = \mathbf{\Phi}_s(\mathbf{\Lambda}_1\mathbf{Y}_1^T\mathbf{M} + \mathbf{Y}_1^T\mathbf{C}) \tag{13}$$

- Construction of $\mathbf{F}_d$ and $\mathbf{G}_d$: Assume that $0 \notin spec(\mathbf{\Lambda}_1)$, and let $\mathbf{\Gamma}_d \in \mathrm{R}^{m\times p}$ and $\mathbf{Z}_d$ be the solution of the Sylvester equation

$$\mathbf{\Lambda}_1^T\mathbf{Z}_d^T - \mathbf{Z}_d^T\bar{\mathbf{\Lambda}}_1 = -\mathbf{\Lambda}_1^T\mathbf{Y}_1^T\mathbf{B}\mathbf{\Gamma}_d \tag{14}$$

If $\mathbf{Z}_d$ is invertible, and

$$\mathbf{\Phi}_d = \mathbf{\Gamma}_d\left(\mathbf{Z}_d^T\bar{\mathbf{\Lambda}}_1\right)^{-1} \tag{15}$$

then it has been shown in Zhang et al.[24] that

$$\mathbf{F}_d = \mathbf{\Phi}_d \mathbf{\Lambda}_1^T \mathbf{Y}_1^T \mathbf{M}, \ \ \mathbf{G}_d = -\mathbf{\Phi}_d \mathbf{Y}_1^T \mathbf{K} \tag{16}$$

Spectrum sensitivity

We define now eight sensitivities related to the perturbations of the sum and the product of the eigenvalues with respect to changes in the system matrices **M**, **C**, and **K** as follows

$$\mathbf{S}_{\Pi\mathbf{K}s} = \frac{\partial \prod \lambda_s^c}{\partial \mathbf{K}} = \frac{det\ (\mathbf{K} - \mathbf{B}\mathbf{G}_s)}{det\ \mathbf{M}} (\mathbf{K} - \mathbf{B}\mathbf{G}_s)^{-T} \tag{17}$$

$$\mathbf{S}_{\Pi\mathbf{M}s} = \frac{\partial \prod \lambda_s^c}{\partial \mathbf{M}} = -\frac{det\ (\mathbf{K} - \mathbf{B}\mathbf{G}_s)}{det\ \mathbf{M}} \mathbf{M}^{-T} \tag{18}$$

$$\mathbf{S}_{\Sigma\mathbf{C}s} = \frac{\partial \sum \lambda_s^c}{\partial \mathbf{C}} = -\mathbf{M}^{-T} \tag{19}$$

$$\mathbf{S}_{\Sigma\mathbf{M}s} = \frac{\partial \sum \lambda_s^c}{\partial \mathbf{M}} = -\mathbf{M}^{-T}(\mathbf{C} - \mathbf{B}\mathbf{F}_s)^T \mathbf{M}^{-T} \tag{20}$$

$$\mathbf{S}_{\Pi\mathbf{K}d} = \frac{\partial \prod \lambda_d^c}{\partial \mathbf{K}} = \frac{det\ \mathbf{K}}{det\ (\mathbf{M} - \mathbf{B}\mathbf{G}_d)} \mathbf{K}^{-T} \tag{21}$$

$$\mathbf{S}_{\Pi\mathbf{M}d} = \frac{\partial \prod \lambda_d^c}{\partial \mathbf{M}} = -\frac{det\ \mathbf{K}}{det\ (\mathbf{M} - \mathbf{B}\mathbf{G}_d)} (\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T} \tag{22}$$

$$\mathbf{S}_{\Sigma\mathbf{C}d} = \frac{\partial \sum \lambda_d^c}{\partial \mathbf{C}} = -(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T} \tag{23}$$

$$\mathbf{S}_{\Sigma\mathbf{M}d} = \frac{\partial \sum \lambda_d^c}{\partial \mathbf{M}} = -(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{C} - \mathbf{B}\mathbf{G}_d)^T(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T} \tag{24}$$

In the above formulas on the sensitivities, the subscripts s and d stand, respectively, for the state feedback and the derivative feedback, and λ_s^c and λ_d^c stand for the respective closed-loop eigenvalues.

Robustness with spectrum sensitivity. Based on the concepts of eigenvalue sensitivities stated above, we now propose to study the robustness issue in PQEVAP by minimizing the following parametric objective functions which are composed of only the closed-loop system and feedback matrices. Each term of the objective functions is intimately related to spectrum sensitivities as justified below.

Minimize

$$f_s(\mathbf{\Gamma}_s) = \frac{1}{2} w_{1s} ||(\mathbf{K} - \mathbf{B}\mathbf{G}_s)^{-T}||_F^2 + \frac{1}{2} w_{2s} ||\mathbf{M}^{-T}(\mathbf{C} - \mathbf{B}\mathbf{F}_s)^T \mathbf{M}^{-T}||_F^2 \tag{25}$$

The case of derivative feedback is similar.

Minimize

$$f_d(\mathbf{\Gamma}_d) = \frac{1}{2} w_{1d} ||(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}||_F^2 + \frac{1}{2} w_{2d} ||(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{C} - \mathbf{B}\mathbf{F}_d)^T(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}||_F^2 \tag{26}$$

Note that the first term of equation (25) is related to equations (17) and (18), which concern the sensitivities of the product of the closed-loop eigenvalues with respect to changes in the stiffness and mass matrices. Similarly, the second term of equation (25) relates to the sensitivities of the sum of the closed-loop eigenvalues with respect to damping and mass matrices through equations (19) and (48). Thus, minimization of equation (25) is related to the minimization of the sensitivities of closed-loop eigenvalues. Similar remarks apply to expression (26). The weights w_{1s}, w_{2s}, w_{1d}, and w_{2d} can be chosen by the designer to avoid dominance in the minimization problems (25) or (26).

In order to minimize equations (25) and (26), the corresponding gradient functions must be computed. In the following, we show how to do so by stating two propositions. The interested readers are referred to the proof in Appendix 1.

Computations of gradient formulae and the associated robust feedback algorithms

Gradient formula for state-feedback objective function

Proposition 1: *Suppose that* $\mathbf{U}$, $\mathbf{V}$ *are the solutions of the following Sylvester equations*

$$\bar{\mathbf{\Lambda}}_1\mathbf{U} - \mathbf{U}\mathbf{\Lambda}_1^T = -\mathbf{Z}_s^{-T}\mathbf{P}\boldsymbol{\gamma}\mathbf{B}\mathbf{\Gamma}\mathbf{Z}_s^{-T} \tag{27}$$

$$\bar{\mathbf{\Lambda}}_1\mathbf{V} - \mathbf{V}\mathbf{\Lambda}_1^T = -\mathbf{Z}_s^{-T}\mathbf{Q}\mathbf{\Theta}\mathbf{B}\mathbf{\Gamma}\mathbf{Z}_s^{-T} \tag{28}$$

and $\mathbf{Z}_s$ *is the same as in equation (11).*

Then, the gradient $\nabla_{\mathbf{\Gamma}_s} f_s$ *is given by*

$$\nabla_{\mathbf{\Gamma}_s} f_s = \left\{ \left[\frac{1}{2}\mathbf{Z}_s^{-T}\left(\mathbf{Q}\mathbf{\Theta} - \mathbf{P}\boldsymbol{\gamma}\right) + \frac{1}{2}(-\mathbf{V} + \mathbf{U})\mathbf{Y}_1^T \right] \mathbf{B} \right\}^T \tag{29}$$

where

$$\mathbf{\Theta} = w_{1s}(\mathbf{K} - \mathbf{B}\mathbf{G}_s)^{-1}(\mathbf{K} - \mathbf{B}\mathbf{G}_s)^{-T}(\mathbf{K} - \mathbf{B}\mathbf{G}_s)^{-1} \tag{30}$$

$$\boldsymbol{\gamma} = \mathrm{w}_{2\mathrm{s}}\mathbf{M}^{-2}(\mathbf{C} - \mathbf{B}\mathbf{F}_{\mathrm{s}})^{\mathrm{T}}\mathbf{M}^{-2} \tag{31}$$

$$\mathbf{P} = \mathbf{Y}_1^T\mathbf{M} \tag{32}$$

$$\mathbf{Q} = \mathbf{\Lambda}_1\mathbf{Y}_1^T\mathbf{M} + \mathbf{Y}_1^T\mathbf{C} \tag{33}$$

Robust state-feedback algorithm. Based on the gradient formula obtained above, we now state the following algorithm for robust feedback computation in the state-feedback case as SFRPQEVAP.

Algorithm 1. Robust state**-feedback algorithm**
Input: The matrices $\mathbf{K}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{\Lambda}_1, \bar{\mathbf{\Lambda}}_1, \mathbf{Y}_1$; the maximum number of iterations *maxiter*; the tolerance ϵ
Output: The feedback matrices $\mathbf{F}_s$, $\mathbf{G}_s$
1 Step 1: Set $k = 1$ and choose $\mathbf{\Gamma}_s^{(1)} = [\gamma_1 \ldots \gamma_p] \in \mathrm{R}^{m\times p}$;
2 Step 2: Compute $\mathbf{Z}_s$, $\mathbf{F}_s$, $\mathbf{G}_s$, using equations (9) to (11);
3 Step 3: Compute $\nabla_{\mathbf{\Gamma}_s} f^{(k)}$ using equations (27) to (33);
4 Step 4: If $||\nabla_{\mathbf{\Gamma}_s} f^{(k)}||_F \leq \epsilon$ or $k = maxiter$, stop. Otherwise, set $k \Leftarrow k+1$ and compute a new $\mathbf{\Gamma}_s^{(k+1)}$ using a gradient-based technique—BFGS,
Levenberg–Marquardt or other; other, return to Step 2.

Gradient formula for derivative feedback objective function

Proposition 2: *Suppose that* $\mathbf{U}$, $\mathbf{V}$ *are the solutions of the following Sylvester equations*

$$\bar{\mathbf{\Lambda}}_1\mathbf{U} - \mathbf{U}\mathbf{\Lambda}_1^T = -\mathbf{Z}_d^{-T}\mathbf{P}\boldsymbol{\gamma}\mathrm{B}\mathbf{\Gamma}\mathbf{Z}_{\mathrm{d}}^{-\mathrm{T}} \tag{34}$$

$$\bar{\mathbf{\Lambda}}_1\mathbf{V} - \mathbf{V}\mathbf{\Lambda}_1^T = -\mathbf{Z}_d^{-T}\mathbf{Q}\mathbf{\Theta}\mathbf{B}\mathbf{\Gamma}\mathbf{Z}_d^{-T} \tag{35}$$

and $\mathbf{Z}_d$ *be the same as in equation (14).*

Then, the gradient $\nabla_{\mathbf{\Gamma}_d} f_d$*, equation (26) is given by*

$$\nabla_{\mathbf{\Gamma}_d} f_d = \left\{ \left[\frac{1}{2}\mathbf{Z}_d^{-T}\Big(\mathbf{Q}\mathbf{\Theta} - \mathbf{P}\boldsymbol{\gamma}) + \frac{1}{2}(-\mathbf{V} + \mathbf{U})\mathbf{Y}_1^T \right] \mathbf{B} \right\}^T \tag{36}$$

where

$$\begin{aligned} \mathbf{\Theta} = {} & w_{1d}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1} + w_{2d} \\ & \times \Big[(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1}(\mathbf{C} - \mathbf{B}\mathbf{F}_d)(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{C} - \mathbf{B}\mathbf{F}_d)^T \\ & \times (\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1} + (\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{C} - \mathbf{B}\mathbf{F}_d)^T \\ & \times (\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-T}(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1}(\mathbf{C} - \mathbf{B}\mathbf{F}_d)(\mathbf{M} - \mathbf{B}\mathbf{G}_d)^{-1}\Big] \end{aligned} \tag{37}$$

$$\boldsymbol{\gamma} = \mathrm{w}_{2\mathrm{d}}\Big[(\mathbf{M} - \mathbf{B}\mathbf{G}_\mathrm{d})^{-1}(\mathbf{M} - \mathbf{B}\mathbf{G}_\mathrm{d})^{-\mathrm{T}}(\mathbf{C} - \mathbf{B}\mathbf{F}_\mathrm{d})^\mathrm{T}(\mathbf{M} - \mathbf{B}\mathbf{G}_\mathrm{d})^{-\mathrm{T}}(\mathbf{M} - \mathbf{B}\mathbf{G}_\mathrm{d})^{-1}\Big] \tag{38}$$

$$\mathbf{P} = -\mathbf{Y}_1^T\mathbf{K} \tag{39}$$

$$\mathbf{Q} = \mathbf{\Lambda}_1^T\mathbf{Y}_1^T\mathbf{M} \tag{40}$$

Robust derivative-feedback algorithm

An algorithm for robust feedback computation is now stated below for a solution of RPQEVAP in the derivative feedback case.

Algorithm 2. Robust **derivative-feedback algorithm**
Input: The matrices $\mathbf{K}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{\Lambda}_1, \bar{\mathbf{\Lambda}}_1, \mathbf{Y}_1$; the maximum number of iterations *maxiter*; the tolerance ϵ
Output: The feedback matrices $\mathbf{F}_d$, $\mathbf{G}_d$
1 Step 1: Set $k = 1$ and choose $\mathbf{\Gamma}_d^{(1)} = [\gamma_1 \ldots \gamma_p] \in \mathrm{R}^{m \times p}$;
2 Step 2: Compute $\boldsymbol{Z_d}$, $\boldsymbol{F_d}$, $\boldsymbol{G_d}$, using equations (12) to (14);
3 Step 3: Compute $\nabla_{\mathbf{\Gamma}_d} f^{(k)}$ using equations (34) to (40);
4 Step 4: If $||\nabla_{\mathbf{\Gamma}_d} f^{(k)}||_F \leq \epsilon$ or $k = maxiter$, stop. Otherwise, set $k \Leftarrow k + 1$ and compute a new $\mathbf{\Gamma}_d^{(k+1)}$ using a gradient-based technique as BFGS,
Levenberg–Marquardt, or other; return to Step 2;

A Remark on Computational Complexity Algorithms 1 and 2: The computational complexity of the proposed algorithms is dominated mainly by the matrix inversions and products necessary to compute the matrices $\mathbf{\Theta}$ and $\boldsymbol{\gamma}$ which are used in gradient calculation for the feedbacks. Thus, the algorithms are $O(n^3)$, and therefore efficient. Also, note that both the algorithms can be implemented with the help of only a small number of eigenvalues and eigenvectors that need to be replaced.

Numerical experiments and comparisons

In this section, we present the results of the proposed method, comparing them with those of other existing methods. Specifically, the following methods are considered for our comparisons:

(I) Proposed Robust State Feedback method (Method I—Algorithm 1);
(II) Proposed Robust Derivative Feedback (Method II—Algorithm 2);
(III) The method of Cai et al.[22] (Method III);

(IV) The method of Wang[25] (Method IV);
(V) The method of Bai et al.[21] (Method V).

In the first two experiments, the matrices **M**, **C** and **K** are perturbed by the following quantities

$$||\Delta\mathbf{M}||_F \leq 0.0001||\mathbf{M}||_F, \; ||\Delta\mathbf{C}||_F \leq 0.0001||\mathbf{C}||_F, \; ||\Delta\mathbf{K}||_F \leq 0.0001||\mathbf{K}||_F$$

Let λ_j^c and $\tilde{\lambda}_j^c$ stand for the jth closed-loop eigenvalue of the unperturbed and the perturbed system, respectively. Then, we define

$$D_{en} = \left[\sum_{j=1}^{2n}\left(\lambda_j^c - \tilde{\lambda}_j^c\right)^2\right]^{\frac{1}{2}} \quad (41)$$

This quantity is the deviation of the perturbed closed-loop eigenvalues from the unperturbed ones

$$R_{ss} = \frac{||\mathbf{F}||_{\mathbf{F}}^2 + ||\mathbf{G}||_{\mathbf{F}}^2}{||\mathbf{F}_{mn}||_{\mathbf{F}}^2 + ||\mathbf{G}_{mn}||_{\mathbf{F}}^2} \quad (42)$$

is the relative change to minimum norm feedbacks. $||\mathbf{F}_{mn}||_{\mathbf{F}}$ and $||\mathbf{G}_{mn}||_{\mathbf{F}}$ stand for the minimum feedback norms as computed in the paper of Brahma and Datta.[19]

Experiment I: Random example

In this experiment, we consider a random example of order 5 from MATLAB gallery ("randcorr", n)

$$\mathbf{M} = \begin{bmatrix} 1 & 0.020074 & 0.16178 & -0.00084629 & -0.039004 \\ 0.020074 & 1 & 0.25089 & 0.090954 & 0.14549 \\ 0.16178 & 0.25089 & 1 & -0.13847 & 0.0026833 \\ -0.00084629 & 0.090954 & -0.13847 & 1 & -0.13832 \\ -0.039004 & 0.14549 & 0.0026833 & -0.13832 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -0.044725 & -0.093248 & -0.16885 & 0.18645 \\ -0.044725 & 1 & 0.05047 & 0.38706 & -0.29389 \\ -0.093248 & 0.05047 & 1 & 0.0028751 & -0.086355 \\ -0.16885 & 0.38706 & 0.0028751 & 1 & 0.034282 \\ 0.18645 & -0.29389 & -0.086355 & 0.034282 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 1 & -0.63971 & -0.16469 & 0.042341 & -0.50555 \\ -0.63971 & 1 & 0.19923 & 0.072314 & 0.49672 \\ -0.16469 & 0.19923 & 1 & 0.64109 & -0.24001 \\ 0.042341 & 0.072314 & 0.64109 & 1 & -0.403 \\ -0.50555 & 0.49672 & -0.24001 & -0.403 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.3971 & 0.9226 \\ 0.1576 & 0.4583 \\ 0.7275 & 0.7742 \\ 0.9719 & 0.3286 \\ 0.1564 & 0.3638 \end{bmatrix}$$

The eigenvalues $-0.2551 \pm 1.3772i$ are reassigned to $-1, \ -2$, respectively. The weights w_{1s} and w_{2s} are set to be 1 for Method I.

The results in Table 1 show that $\kappa_2(\mathbf{Y}_c)$ is comparable for all the four methods, while the other significant measures of robustness, namely $f(\Gamma_s), \ D_{en}$ and R_{ss} are much better with the Method I than with the others.

Experiment II: An example of oil rig

For this experiment, the matrices $\mathbf{M}, \ \mathbf{K} \in \mathrm{R}^{66\times 66}$ are obtained from Harwell–Boeing Collection BCSSTRUC1[26] which proposes a *statically condensed oil rig model*. Moreover, we set $\mathbf{C} = \mathbf{I}_{66\times 66}$ and $\mathbf{B}^T = [\mathbf{I}_{2\times 2} \quad 0_{62\times 2} \quad -\mathbf{I}_{2\times 2}]^T$. The eigenvalues $-4.7067 \pm 5.2347i, \ -5.1680 \pm 4.2682i, \ -5.2067 + 4.1522i$ of the model are reassigned to the positions: $-6 \pm i, \ -6 \pm 2i, \ -6 \pm 3i$. The weights are set as $w_{1s} = 1$ and $w_{2s} = 1e - 008$. As seen in Table 2, for the Method I, the condition number of the matrix of closed-loop eigenvectors is smaller than that obtained by Method V, whereas it is comparable with that of Method III. The other measures are substantially better for the Method I than the others.

Experiment III

The matrices for this experiment were taken from Qian and Xu[27]

$$\mathbf{M} = \mathbf{I}_{4\times 4}, \quad \mathbf{C} = diag([0.5 \quad 0 \quad 0 \quad 0.5]),$$

$$\mathbf{K} = \begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 10 & -5 \\ 0 & 0 & -5 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Table 1. Comparisons of Methods in a random example.

Method	$f_s(\Gamma_s)$	$\kappa_2(\mathbf{Y}_c)$	D_{en}	R_{ss}
I	43.9483	170.3181	0.0014	1.7502
III	100.2846	116.4019	0.0012	12.9499
IV	44.7336	159.3025	0.0013	2.6302
V	389.0345	114.7821	0.0012	77.3673

Table 2. Comparison of Methods in an example of oil rig.

Method	$f_s(\Gamma_s)$	$\kappa_2(\mathbf{Y}_c)$	D_{en}	R_{ss}
I	0.2979	4.4416e+004	0.2813	1.1355
III	0.3143	3.5915e+004	0.2906	1.8560
IV	3.6190	6.9402e+006	0.9580	101.8630
V	0.3524	9.0145e+004	0.2837	3.6009

For this experiment, the eigenvalues $-0.0385\pm4.1362i$ are reassigned to $-1\pm i$, and the perturbations in the system matrices are set as

$$||\Delta\mathbf{M}||_F \leq 0.01||\mathbf{M}||_F, \ ||\Delta\mathbf{C}||_F \leq 0.01||\mathbf{C}||_F, \ ||\Delta\mathbf{K}||_F \leq 0.01||\mathbf{K}||_F$$

The weights considered in this case are $w_{1s} = w_{2s} = 1$ for Method I and for Method II $w_{1d} = w_{2d} = 1$.

As seen from the results of Table 3, the condition numbers for the eigenvector closed-loop matrix for each of the Methods I and II are smaller than that of Method III. The quantity D_{en} for the Method I is comparable with that of Method III, while for Method II it is much better. Figure 1 shows the distribution of reassigned eigenvalues in both cases of the feedbacks.

Experiment IV on system response—vibration absorber of a machine

In this section, we present the results on system responses of a second-order model representing absorber of a machine, taken from Beards.[28] The matrices **M**, **C**, **K**, and **B** are given by

$$\mathbf{M} = \mathbf{I}_{3\times3}, \quad \mathbf{C} = 0$$
$$\mathbf{K} = \begin{bmatrix} 2 & 0 & -0.6 \\ 0 & 2 & -2 \\ -0.6 & -2 & 2.68 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Table 3. Condition number and eigenvalue perturbation for the control design in Experiment III.

Method	$\kappa_2(\mathbf{Y}_c)$	D_{en}
I	21.1073	0.2332
II	46.3772	0.0560
III	72.8761	0.2248

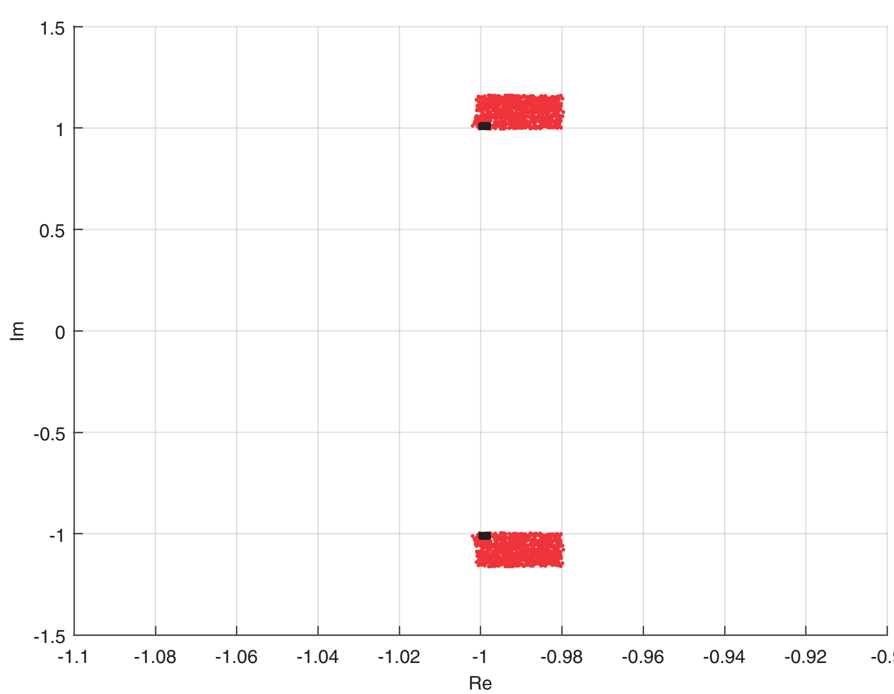

Figure 1. Distribution of the reassigned eigenvalues in Experiment III with state feedback (red) and derivative feedback (black), under linear perturbation in systems matrices of 1%.

The natural frequencies of the system are $\pm 2.1108i$, $\pm 1.4142i$, $\pm 0.4737i$. Now, an external excitation of the form $f(t) = 0.1 sin(2.1108t)$ is applied to the system. It is clear that the eigenpair $\pm 2.1108i$ will cause resonance. This eigenpair is then reassigned to $-1 \pm i$ to control the vibration due to the resonance and the feedback matrices are computed using Algorithms 1 and 2. The system responses then are determined for the open-loop system and for the closed-loop system using the Algorithm 1 (Method I), Algorithm 2 (Method II), and Method V. These system responses are displayed in Figure 2. It is seen that the oscillations due to resonance (Figure 2(a)) are well controlled by applying feedback control forces in both cases (Figure 2(b) and (c)). Finally, we study the closed-loop steady-state system response in a long term—after 450 s from initial time—for Algorithms 1 and 2 and Method V under a perturbation of +10% and –10%, respectively, in matrices **M** and **K**. The results are displayed in Figure 3. In Figure 3(a) the horizontal displacement (red lines) and the torsional tilt (green lines) under perturbations—dashed lines for the closed-loop system determined by Method I are displayed. The corresponding results for the closed-loop system obtained by Method II are displayed in Figure 3(b), and the results for the Method V are displayed in Figure 3(c). Despite the phase difference in the three responses, all the Methods were able to achieve the control of resonance under severe perturbations in the matrices **M** and **K**, and thus confirming that they deliver good robustness.

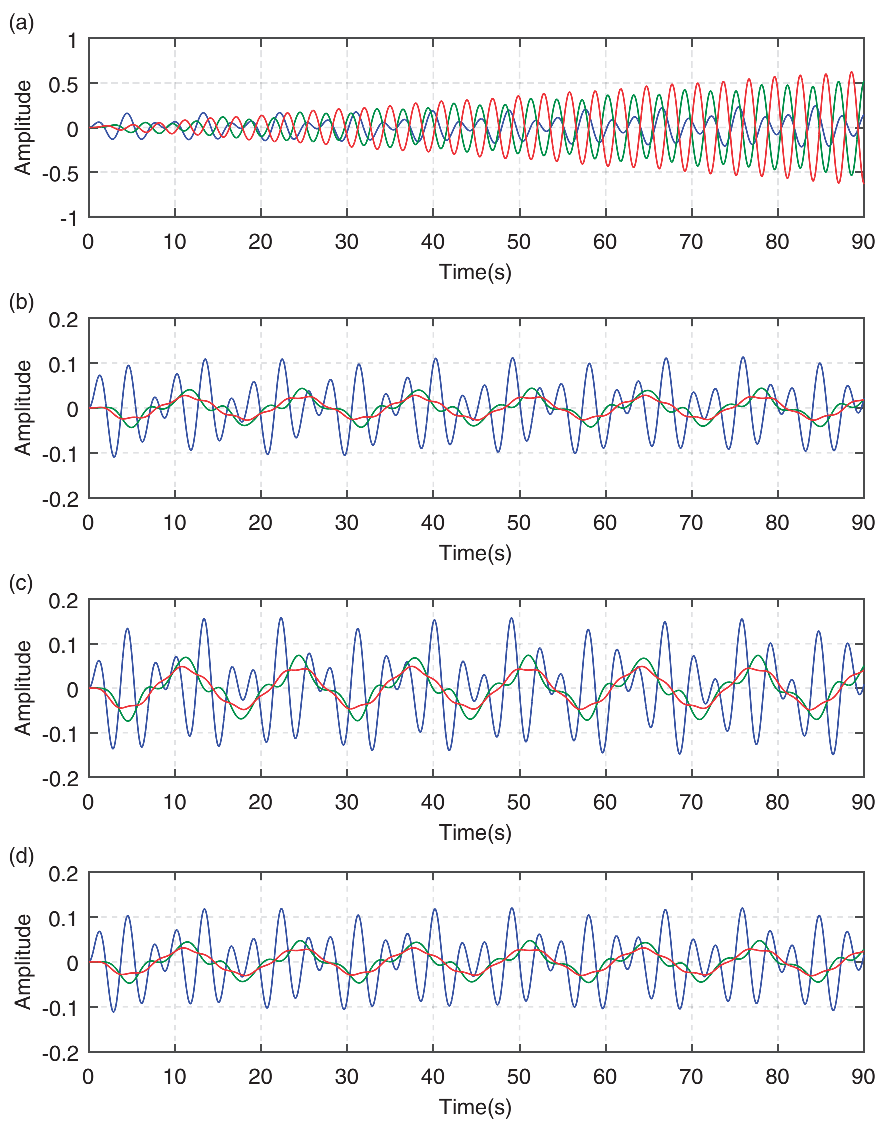

Figure 2. A study of controlling resonant vibrations on vertical (blue lines) and horizontal (red lines) displacements and torsional tilt (green lines) by Methods I, II, and V: (a) Open-loop, (b) closed-loop with Method I, (c) closed-loop with Method II, and (d) closed-loop with Method V.

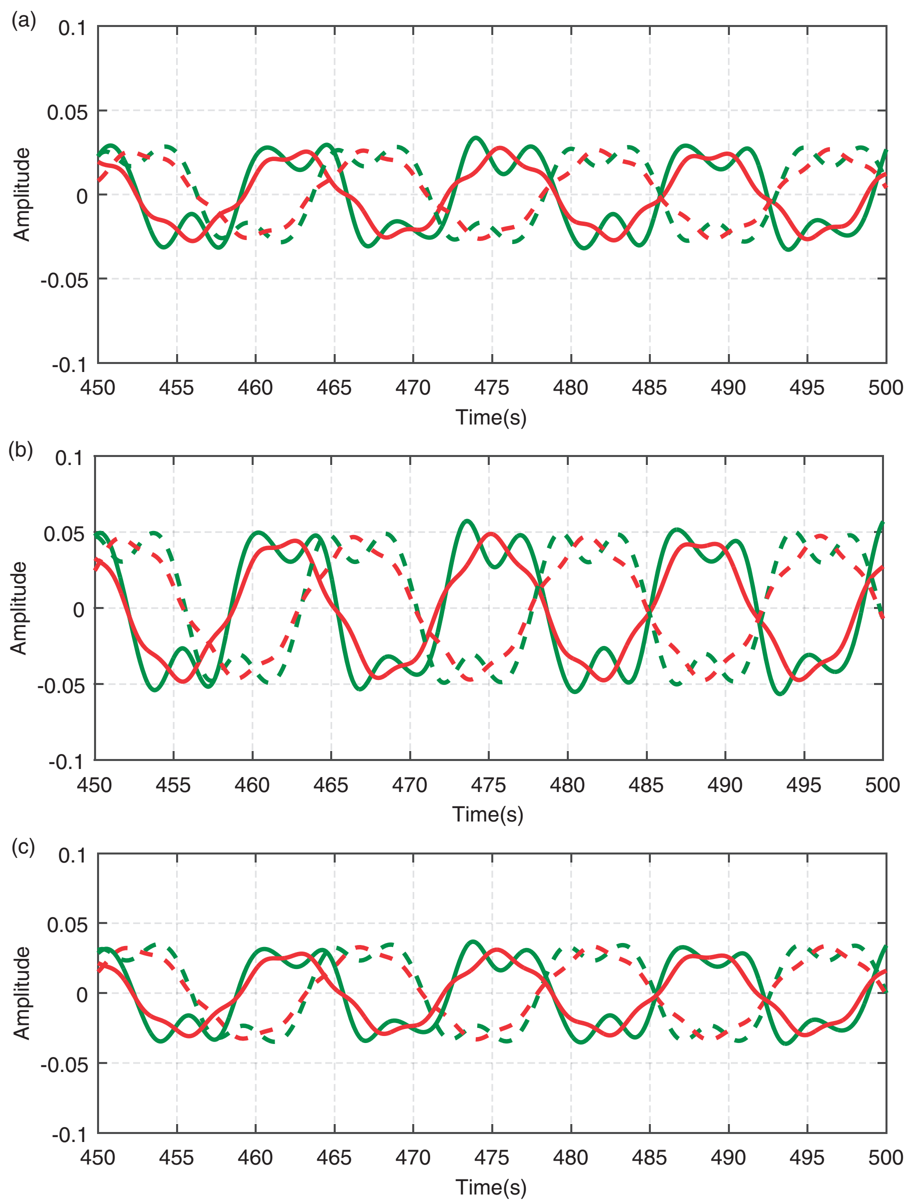

Figure 3. Deviations of the steady-state time domain responses in for horizontal displacement (red lines) and torsional tilt (green lines) under resonant excitation in unperturbed (continuous) and perturbed (dashed) closed-loop system: (a) Method I, (b) Method II, and (c) Method V.

Experiment V: Comparison of the proposed algorithms with a genetic algorithm (GA)

In this section, we compare the proposed algorithms with a GA which is believed to give a global solution to an optimization problem but which is heuristic in nature. The results are displayed in Table 4. Here the superscripts s, d, and GA stand for the respective quantities in cases of state feedback, derivative feedback, and GA. All these three algorithms are applied to the three examples in "Experiment I: Random example," "Experiment II: An example of oil rig," and "Experiment III" subsections. It is seen that the results of our algorithm are very close or same as those obtained by GA for Experiment I and Experiment III.

Experiment VI: A study of eigenvector condition number reduction

In this experiment, we evaluate the capability of Algorithm 1 in reducing the closed-loop eigenvector condition number and compare it with the results obtained by Method V. It is to be noted that Method V is designed to

Table 4. Comparison of the proposed gradient-based Algorithms 1 and 2 against the meta-heuristic GA optimization.

		$f_s(\boldsymbol{\Gamma}^*)$	$f_d(\boldsymbol{\Gamma}^*)$	$\|\|\mathbf{F}_I - \mathbf{F}_{GA}\|\|_2$	$\|\|\mathbf{G}_I - \mathbf{G}_{GA}\|\|_2$	$\|\|\mathbf{F}_{II} - \mathbf{F}_{GA}\|\|_2$	$\|\|\mathbf{G}_{II} - \mathbf{G}_{GA}\|\|_2$
5.1	I	43.9483	–	0.0056	0.0410	–	–
	GA	43.9487					
5.2	I	0.2979	–	0.6917	3.6533	–	–
	GA	0.2963					
5.3	I	16.6393	2.1972	0.1062	0.0077	0.0560	0.0097
	GA	16.6451	2.1972				

GA: genetic algorithm.

Table 5. Reduction for the condition number and deviation of the eigenvalues for Experiment VI.

Method	$\Delta\kappa_2(\mathbf{Y}_C)\%$	D_{en}
I	49.05%	0.0412
V	63.57%	0.0451

minimize the condition number of the closed-loop eigenvector matrix. Here we consider the example from Ram and Elhay[18] and Bai et al.[21] with matrices

$$\mathbf{M} = \mathbf{I}_n, \ \ \mathbf{C} = \mathbf{0}, \ \ \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \ \ \mathbf{B} = \begin{bmatrix} \mathbf{I}_m \\ 0 \end{bmatrix}$$

where $n = 40$ and $m = 3$. The four eigenvalues with smallest absolute value are reassigned to $\lambda_{2k-1} = -k + \sqrt{-10k}$, $\lambda_{2k} = conj(\lambda_{2k-1})$, $k = 1, 2$. The weights are chosen as $w_{1s} = 0.1$ and $w_{2s} = 1$ for Method I. For the sake of comparison, the initial value of Γ_s for both methods is taken as

$$\boldsymbol{\Gamma}_0 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We then compute the reduction on the condition number for the methods

$$\Delta\kappa_2(\mathbf{Y}_C)\% = 100\frac{\kappa_2^0 - \kappa_2}{\kappa_2^0} \tag{43}$$

as well as the quantity D_{en}. The results are displayed in Table 5. We observe that, although Method I does not explicitly take into account the condition number in the formulation of the cost function, it gives a reasonable improvement on the condition number after its application, with a slightly favorable result for the Method V. However, the quantity D_{en} is better in Method I than in Method V for perturbations of 1% in both the matrices **K** and **M**.

Summary and conclusions

A practical aspect of the design of a controller is to ensure robustness in the closed-loop eigenvalues. Mathematically, this is equivalent to minimizing the condition number of the closed-loop eigenvector matrix. The task is computationally prohibitive for the active vibration controller design of a vibrating structure modeled by a system of second-order differential equations, because a major part of the closed-loop eigenvector matrix in this case is not known to the users. We propose an alternative approach to this problem. The approach involves minimizing a cost function that is composed of only the closed-loop system and feedback matrices. Therefore, its implementation does not require the explicit knowledge of eigenvalues and eigenvectors, making the approach practically implementable. Another distinguished feature of the approach is that it works for both state- and velocity-feedback designs. A comparative study of the algorithms based on this proposed approach with the existing algorithms which are especially designed to explicitly minimize the condition number of the closed-loop eigenvector matrix shows that the algorithms are competitive and in some cases give better results. Future research will be directed toward the development of such a strategy for partial eigenstructure assignment in a vibrating system where not only a few eigenvalues are assigned but the associated eigenvectors need to be assigned as well by feedback of different types.

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Appendix 1. Proof of Proposition 1

The cost function. $f(\mathbf{\Gamma}_s)$ given by equation (25) can be written using matrix traces. Then, from the definition of the gradient of a scalar function of matrices, the differential ∂f must contain some term of the type $tr(\nabla_\Gamma f^T \partial \mathbf{\Gamma}_s)$. By differentiating equation (25) with respect to. $\mathbf{F}_s$ and $\mathbf{G}_s$ and applying trace properties such as linearity and trace of cyclic permutations for the matrix product, one has

$$\partial f_s = \frac{1}{2} tr(\mathbf{\Theta B} \partial \mathbf{G}_s - \boldsymbol{\gamma} \mathbf{B} \partial \mathbf{F}_s) + \frac{1}{2} tr\left(\mathbf{B}^T \mathbf{\Theta}^T \partial \mathbf{G}_s^T - \mathbf{B}^T \boldsymbol{\gamma}^\mathrm{T} \partial \mathbf{F}_s^T\right) \tag{44}$$

The first term in equation (44) can be expressed as a function of $\partial \mathbf{\Gamma}_s$. Thus, by combining equations (12) and (13) in order to expand $\partial \mathbf{F}_s$ and $\partial \mathbf{G}_s$, this gives

$$\partial \mathbf{F}_s = \left(\partial \mathbf{\Gamma}_s - \mathbf{\Gamma}_s \mathbf{Z}_s^{-T} \partial \mathbf{Z}_s^T\right) \mathbf{Z}_s^{-T} \mathbf{P} \tag{45}$$

$$\partial \mathbf{G}_s = \left(\partial \mathbf{\Gamma}_s - \mathbf{\Gamma}_s \mathbf{Z}_s^{-T} \partial \mathbf{Z}_s^T\right) \mathbf{Z}_s^{-T} \mathbf{Q} \tag{46}$$

The differential $\partial \mathbf{Z}_s^T$ can be computed by applying a differentiation rule in equation (11)

$$\mathbf{\Lambda}_1^T \partial \mathbf{Z}_s^T - \partial \mathbf{Z}_s^T \bar{\mathbf{\Lambda}}_1 = -\mathbf{Y}_1^T \mathbf{B} \partial \mathbf{\Gamma}_s \tag{47}$$

Returning to equation (44) and developing the first argument of the trace term leads to

$$\mathbf{\Theta B} \partial \mathbf{G}_s - \boldsymbol{\gamma} \mathbf{B} \partial \mathbf{F}_s = \mathbf{\Theta B} \partial \mathbf{\Gamma}_s \mathbf{Z}_s^{-\mathrm{T}} \mathbf{Q} - \mathbf{\Theta B} \mathbf{\Gamma}_s \mathbf{Z}_s^{-\mathrm{T}} \partial \mathbf{Z}_s^\mathrm{T} \mathbf{Z}_s^{-\mathrm{T}} \mathbf{Q} - \boldsymbol{\gamma} \mathbf{B} \partial \mathbf{\Gamma}_s \mathbf{Z}_s^{-\mathrm{T}} \mathbf{P} + \boldsymbol{\gamma} \mathbf{B} \mathbf{\Gamma}_s \mathbf{Z}_s^{-\mathrm{T}} \partial \mathbf{Z}_s^\mathrm{T} \mathbf{Z}_s^{-\mathrm{T}} \mathbf{P} \tag{48}$$

Substituting equation (47) into equation (44) and using again the properties of the trace function yields

$$\frac{1}{2}tr(\mathbf{\Theta B}\partial \mathbf{G}_s - \boldsymbol{\gamma}\mathbf{B}\partial \mathbf{F}_s) = \frac{1}{2}tr\left[\mathbf{Z}_s^{-T}(\mathbf{Q\Theta} - \mathbf{P}\boldsymbol{\gamma})\mathbf{B}\partial\mathbf{\Gamma}_s\right] + \frac{1}{2}tr\left(-\mathbf{Z}_s^{-T}\mathbf{Q\Theta B\Gamma}_s\mathbf{Z}_s^{-T}\partial\mathbf{Z}_s + \mathbf{Z}_s^{-T}\mathbf{P}\boldsymbol{\gamma}\mathbf{B\Gamma}_s\mathbf{Z}_s^{-\mathrm{T}}\partial\mathbf{Z}_s^{\mathrm{T}}\right) \tag{49}$$

Next, consider the following solution for the Sylvester equation in equation (47) [23]

$$\partial\mathbf{Z}_s^T = \sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\gamma_{jk}\left(\mathbf{\Lambda}_1^T\right)^j\left(-\mathbf{Y}_1^T\mathbf{B}\partial\Gamma\right)\left(\bar{\mathbf{\Lambda}}_1\right)^k \tag{50}$$

Substituting equation (50) into equation (48), we obtain

$$\begin{aligned}\frac{1}{2}tr\left(-\mathbf{Z}_s^{-T}\mathbf{Q\Theta B\Gamma}_s\mathbf{Z}_s^{-T}\partial\mathbf{Z}_s^T + \mathbf{Z}_s^{-T}\mathbf{P}\boldsymbol{\gamma}\mathbf{B\Gamma}_s\mathbf{Z}_s^{-\mathrm{T}}\partial\mathbf{Z}_s^{\mathrm{T}}\right) &= \frac{1}{2}tr\left[-\mathbf{Z}_s^{-T}\mathbf{Q\Theta B\Gamma}_s\mathbf{Z}_s^{-T}\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\gamma_{jk}\left(\mathbf{\Lambda}_1^T\right)^j\left(-\mathbf{Y}_1^T\mathbf{B}\partial\mathbf{\Gamma}_s\right)\left(\bar{\mathbf{\Lambda}}_1\right)^k\right] \\ &+ \frac{1}{2}tr\left[\mathbf{Z}_s^{-T}\mathbf{P}\boldsymbol{\gamma}\mathbf{B\Gamma}_s\mathbf{Z}_s^{-\mathrm{T}}\sum_{\mathrm{j}=0}^{\mathrm{p}-1}\sum_{\mathrm{k}=0}^{\mathrm{p}-1}\gamma_{\mathrm{jk}}\left(\mathbf{\Lambda}_1^{\mathrm{T}}\right)^{\mathrm{j}}\left(-\mathbf{Y}_1^{\mathrm{T}}\mathbf{B}\partial\mathbf{\Gamma}_{\mathrm{s}}\right)\left(\mathbf{\Lambda}_1\right)^{\mathrm{k}}\right] \\ &= \frac{1}{2}tr\left[-\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\gamma_{jk}\left(\bar{\mathbf{\Lambda}}_1\right)^k\left(-\mathbf{Z}_s^{-T}\mathbf{P\Theta B\Gamma}_s\mathbf{Z}_s^{-T}\partial\mathbf{\Gamma}_s\right)\left(\mathbf{\Lambda}_1^T\right)^j\mathbf{Y}_1^T\mathbf{B}\right] \\ &+ \frac{1}{2}tr\left[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\gamma_{jk}\left(\bar{\mathbf{\Lambda}}_1\right)^k\left(-\mathbf{Z}_s^{-T}\mathbf{P}\boldsymbol{\gamma}\mathbf{B\Gamma}_s\mathbf{Z}_s^{-\mathrm{T}}\partial\mathbf{\Gamma}_{\mathrm{s}}\right)\left(\mathbf{\Lambda}_1^T\right)^j\mathbf{Y}_1^T\mathbf{B}\right] = \frac{1}{2}tr\left[(-\mathbf{V} + \mathbf{U})\mathbf{Y}_1^T\mathbf{B}\right]\end{aligned} \tag{51}$$

Thus, from equation (48), we obtain

$$\frac{1}{2}tr\left(\mathbf{\Theta B}\partial\mathbf{G}_s - \boldsymbol{\gamma}\mathbf{B}\partial\mathbf{F}_s\right) = \frac{1}{2}tr\left\{\left[\mathbf{Z}_s^{-T}(\mathbf{Q\Theta} - \mathbf{P}\boldsymbol{\gamma}) + (-\mathbf{V} + \mathbf{U})\mathbf{Y}_1^T\right]\mathbf{B}\partial\mathbf{\Gamma}_s\right\} \tag{52}$$

Since we have

$$\partial f = tr(\mathbb{J}_1\partial\mathbf{R} + \mathbb{J}_2\partial\mathbf{T}) \tag{53}$$

then $\nabla_{\mathbf{M}} f = \mathbb{J}_1^T$ .The proposition is then proved. ■

The proof of Proposition 2 is fully similar to the above development, and then is omitted here.