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SIMULATION IN PRIMITIVE VARIABLES OF INCOMPRESSIBLE FLOW WITH PRESSURE NEUMANN CONDITION

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Abstract

We develop a velocity–pressure algorithm, in primitive variables and finite differences, for incompressible viscous flow with a Neumann pressure boundary condition. The pressure field is initialized by least–squares and up-dated from the Poisson equation in a direct weighted manner. Simulations with the cavity problem were made for several Reynolds numbers. It was obtained the expected displacement of the central vortex as well as the development of secondary and terciary eddies.

1 Introduction

We develop a velocity–pressure algorithm for incompressible viscous flow in primitive variables by using finite differences, a Neumann boundary condition for the pressure and without any iteration method for updating the pressure.

The incorporation of the Neumann condition for incompressible flow has been deeply discussed on a remarkable work of Gresho and Sani[15] (1987). From a mathematical point of view, the system of equations governing an incompressible flow is singular with respect to the pressure. There is no an evolutive equation for the pressure. In practice, the system is usually considered as the momentum equation subject to a solenoidal restriction for the velocity field. The initial and boundary conditions being prescribed only for the velocity field.

The discretization by difference methods of the Navier–Stokes equations on a staggered grid, when formulated in matrix terms, allows to identify a singular evolutive matrix system. When we derive the Poisson equation for the pressure and perform its integration, we can observe that a clear influence of the Neumann condition arises. From this we can extract a non–singular system for determining the pressure values at the interior points. The initialization process of the pressure, by a least–squares procedure, somehow incorporates an optimal pressure as a starting point, instead of employing an arbitrary constant as it usually made with iterative methods. The values of the velocity and pressure at interior points can be well determined by the forward Euler method for the velocity and by solving a non-singular Poisson equation without iteration. This later means that we incorporate the values of the pressure and velocity as soon as they are computed.

The formulation of our algorithm follows the unified operator approach introduced by Casulli[9] (1988) which allows to consider, with minor modifications, the up-wind and semi-lagrangean methods.

This velocity–pressure algorithm with central differences has been tested with the cavity flow problem for a wide range of Reynolds numbers. The displacement of the central vortex to the geometrical center of the cavity was obtained when increasing the Reynolds number, as earlier established by Burgggraf[6](1966), Ghia et al[14](1982) and Schreiber and Keller[22](1983), among others. Also, the development of secondary and terciary vortices.

2 The Continuum Equations for Incompressible Flow

In this section we give a brief account about the prescription of the Neumann condition as done by Gresho and Sani (1987). The Navier–Stokes equations for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure

$p(\mathbf{x}, t)$ with initial and boundary conditions for the velocity constitute the system

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u} \ , \quad t > 0 \tag{1}$$

$$\nabla.\mathbf{u} = 0 \tag{2}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \ , \qquad \mathbf{x} \ \text{ in } \overline{\Omega} = \Omega \oplus \Gamma \tag{3}$$

$$\mathbf{u} = \mathbf{w}(\mathbf{x}, t) \quad \text{in} \quad \Gamma = \partial\Omega \ . \tag{4}$$

Here Ω denotes a limited 2D region, Γ its boundary and

$$\nabla.\mathbf{u}_0 = 0 \quad \text{in} \quad \Omega \ , \tag{5}$$

an initial solenoidal velocity field. From the above system follows the initial normal velocity

$$\mathbf{u}_0.\mathbf{n} = \mathbf{w}(\mathbf{x}, 0).\mathbf{n} \quad \text{on} \quad \Gamma. \tag{6}$$

and the global mass conservation

$$\int_\Gamma \mathbf{u}.\mathbf{n} dx = 0 \ . \tag{7}$$

We observe that no initial nor boundary conditions are prescribed for the pressure. Thus p is determined up to an additive constant corresponding to the level of hydrostatic pressure.

The conditions of an initial velocity field solenoidal and normal velocity compatible with the above conditions are required for the problem to have a well defined, unique and solenoidal solution for all $t \geq 0$.[23] The initial tangential velocity field is not required to be compatible with the conditions, if so, then the solution may be smoother.

By assuming adequate differentiability hypotheses, we can derive the Poisson equation by taking divergence of the momentum equation and using the vector identity

$$\nabla^2 \mathbf{u} = \nabla(\nabla.\mathbf{u}) - \nabla \times \nabla \times \mathbf{u}. \tag{8}$$

We have that

$$\nabla.(\mathbf{u}.\nabla \mathbf{u}) + \nabla^2 p = (\nu \ \nabla^2 \Theta - \frac{\partial \Theta}{\partial t}) \quad \text{in} \ \ \Omega \ , \tag{9}$$

where $\Theta = \nabla.\mathbf{u}$.

Since the prescribed conditions for $\mathbf{u}$ in Γ are valid everywhere on time, that is,

$$\nabla.\mathbf{u} = 0 \quad \text{in} \ \ \overline{\Omega} \ \ \text{for} \ \ t \geq 0 \ . \tag{10}$$

we can substitute it in the equation for Θ and obtain Poisson's equation for the pressure

$$\nabla^2 p = -\nabla.(\mathbf{u}.\nabla \mathbf{u}) \quad \text{in} \ \ \Omega \ \ \text{for} \ \ t \geq 0 \ . \tag{11}$$

Let us consider the equivalent equation

$$\nabla^2 p = \nabla.(\nu \nabla^2 \mathbf{u} - \mathbf{u}.\nabla \mathbf{u}) \ . \tag{12}$$

In order to complete the specification of the problem for the pressure, we should impose boundary conditions for the pressure on Γ. Since the last two equations have derived, the boundary conditions should be also derived. One obvious manner is to set the momentum equation valid on the boundary. However, this is a vector equation and only one scalar boundary condition is required. We can choose either the normal or the tangential projection of the momentum equation upon Γ. The first option gives us

$$\mathbf{n}.\nabla p = \frac{\partial p}{\partial n} = \nu \nabla^2 u_n - (\frac{\partial u_n}{\partial t} + \mathbf{u}.\nabla u_n) \quad \text{in } \Gamma \quad \text{for } t \geq 0 \tag{13}$$

Thus equations (11)–(12) and (13) constitute a Neumann problem for the pressure.

On the other hand , the tangential component of the momentum equation upon Γ gives a Dirichlet condition type

$$\tau.\nabla p = \frac{\partial p}{\partial \tau} = \nu \nabla^2 u_\tau - (\frac{\partial u_\tau}{\partial t} + u.\nabla u_\tau) \ , \tag{14}$$

where the value of p on Γ, that is, Dirichlet data, is provided by integration of (14) through τ.

The determination of the solution of the Poisson equation (11) with Neumann boundary conditions (13) requires to hold the compatibility relationship

$$\int\int_\Omega \nabla^2 p \ d\Omega = \int\int_\Omega -\nabla.(\mathbf{u}.\nabla \mathbf{u}) \ d\Omega = \oint_\Gamma p_n \ d\Gamma \tag{15}$$

where $p_n = \mathbf{n}.\nabla p$, and $\mathbf{n}$ an exterior normal unit vector to Γ.

3 Discretization of the Navier–Stokes Equations

We have for a nondimensional 2D incompressible viscous flow, the primitive equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{16}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{17}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{18}$$

where $u(x,y,t)$ and $v(x,y,t)$ denote the velocity components in x and y directions, $p(x,y,t)$ the pressure and $Re \geq 0$ the Reynolds number. This system can be written in the operator compact form

$$M\frac{\partial U}{\partial t} + NU = -PU + LU \tag{19}$$

where

$$U = \begin{bmatrix} u \\ v \\ p \end{bmatrix} , \qquad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

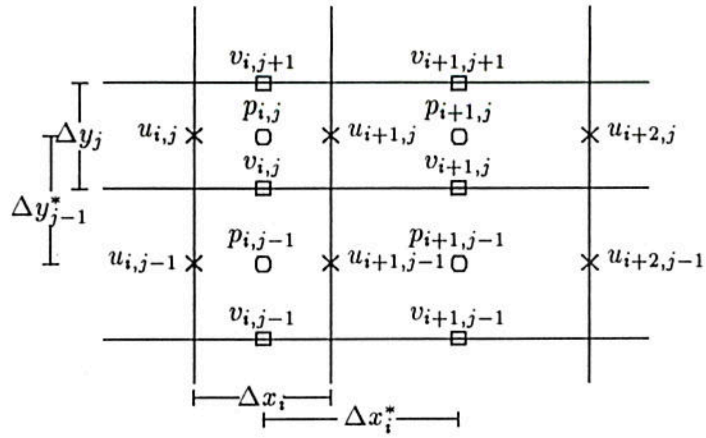

Figure 1: Staggered Grid

$$N = \begin{bmatrix} (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}) & 0 & 0 \\ 0 & (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} \frac{1}{Re}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) & 0 & 0 \\ 0 & \frac{1}{Re}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{bmatrix}.$$

We now use central differences for approximating the spatial derivatives. For simplicity, we shall restrict our formulation to a rectangular staggered grid (Figure 1).

Setting $\Delta x_i = \Delta x$, $i = 0...n$ and $\Delta y_j = \Delta y$, $j = 0...m$, let

$$u_{i,j} = u(i\Delta x, (j+1/2)\Delta y),$$

$$v_{i,j} = v((i+1/2)\Delta x, j\Delta y),$$

$$p_{i,j} = p((i+1/2)\Delta x, (j+1/2)\Delta y).$$

We write (16), (17) as

$$\frac{\partial u}{\partial t} = F_1(u,v) - G_1(p) \tag{20}$$

$$\frac{\partial v}{\partial t} = F_2(u,v) - G_2(p) \tag{21}$$

respectively, then we apply spatial central differences. It turns out that

$$F_1(u,v) = -u_{i,j}\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} - v|_{u_{i,j}}\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y} +$$
$$\frac{1}{Re}\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\right) \tag{22}$$
$$F_2(u,v) = -u|_{v_{i,j}}\frac{v_{i+1,j}-v_{i-1,j}}{2\Delta x} - v_{i,j}\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y} +$$
$$\frac{1}{Re}\left(\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{\Delta x^2} + \frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{\Delta y^2}\right) \tag{23}$$

Similarly,

$$G_1(p) = \frac{p_{i,j}-p_{i-1,j}}{\Delta x} \tag{24}$$
$$G_2(p) = \frac{p_{i,j}-p_{i,j-1}}{\Delta y} \tag{25}$$

Here $v|_{u_{i,j}}$ and $u|_{v_{i,j}}$ denote the average values

$$v|_{u_{i,j}} = \frac{v_{i,j+1}+v_{i,j}+v_{i+1,j+1}+v_{i+1,j-1}}{4} \tag{26}$$

and

$$u|_{v_{i,j}} = \frac{u_{i+1,j}+u_{i,j}+u_{i+1,j+1}+u_{i,j+1}}{4} \tag{27}$$

3.1 Matrix Formulation

The above spatial discretization procedure, together with the pressure Neumann condition, for the Navier–Stokes equations on a rectangular staggered grid amounts, in matrix terms, to replace (19) by the spatial approximation

$$\mathbb{M}\frac{d\mathbb{U}}{dt} + \mathbb{N}(\mathbb{U})\mathbb{U} + \mathbb{N}^F(\mathbb{U},\mathbb{U}^F)\mathbb{U}^F = -\mathbb{P}\mathbb{U} - \mathbb{P}^F(\mathbb{U},\mathbb{U}^F) + \mathbb{L}\mathbb{U} + \mathbb{L}^F\mathbb{U}^F\,. \tag{28}$$

Here $\mathbb{U} = [U_{i,j}]$, where $U_{i,j}$ includes the values $u_{i+1/2,j}$, $v_{i,j+1/2}$, $p_{i,j}$ at a cell (i,j). The vector $\mathbb{U}^F$ corresponds to the boundary values of u, v e p. The matrices $\mathbb{M}$, $\mathbb{N}$, $\mathbb{L}$, $\mathbb{P}$ are the corresponding spatial approximation of the continuous terms and $\mathbb{N}^F$, $\mathbb{L}^F$, $\mathbb{P}^F$ are matrices that contain boundary values. The above systems are singular once $\mathbb{M}$ is a singular matrix.

If we use an up-wind approximation for the velocity field and keep central differences for the pressure gradient, it will turn out that the matrix formulation reads

$$\mathbb{M}\frac{d\mathbb{U}}{dt} + \mathbb{N}(\mathbb{U},\mathbb{U}^F)\mathbb{U} + \mathbb{N}^F(\mathbb{U},\mathbb{U}^F)\mathbb{U}^F = -\mathbb{P}\mathbb{U} - \mathbb{P}^F(\mathbb{U},\mathbb{U}^F) + \mathbb{L}\mathbb{U} + \mathbb{L}^F\mathbb{U}^F \tag{29}$$

From such equations, we observe that, although the $\mathbb{N}$ matrix is different for each method, there is a direct influence of boundary values on the non-linear convection matrix when discretized by the up-wind method. This is not the case for central differences and, in numerical terms, it means, that boundary values do not interfere with the numerical convection at interior points.

The above matrix formulation will be of a similar nature for non-rectangular domains.

3.2 Time Discretization

The time discretization of the momentum equations, besides modifying the order of approximation, raises numerical stability problems. Although implicit methods have better stability properties, they are expensive to implement. In this paper, we shall employ the explicit Euler or Adams-Bashforth methods.

The Adams-Bashforth method, applied to the equations (20) and (21) can be written as

$$u^{k+1} = u^k + \Delta t \sum_{l=0}^{n_p-1} \alpha_l [F_1(u,v) - G_1(p)] \tag{30}$$

$$v^{k+1} = v^k + \Delta t \sum_{l=0}^{n_p-1} \alpha_l [F_2(u,v) - G_2(p)] \tag{31}$$

where $k = t/\Delta t$ represents the time steps and the coefficients n_p and α_l define a specific method: $n_p = 1, \alpha_o = 1$ (first-order forward Euler), $n_p = 2, \alpha_o = 3/2, \alpha_1 = -1/2$ (second-order Adams-Bashforth) and $n_p = 3, \alpha_o = 23/12, \alpha_1 = -4/3, \alpha_2 = 5/12$ (third-order Adams-Bashforth).

4 The Corrected Pressure Equation

Gresho and Sani derived an equation for the pressure in such a way that for $\nabla.\mathbf{u}_0 = 0$, the system given by (1),(2) can be replaced by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = \frac{1}{Re}\nabla^2\mathbf{u} \tag{32}$$

$$\nabla^2 p = \nabla.(\frac{1}{Re}\nabla^2\mathbf{u} - \mathbf{u}.\nabla\mathbf{u}) \tag{33}$$

where the boundary condition for p is given by

$$\mathbf{n}.\nabla p = \frac{\partial p}{\partial n} = \frac{1}{Re}\nabla^2 u_n - (\frac{\partial u_n}{\partial t} + \mathbf{u}.\nabla u_n) \quad \text{for } t \geq 0 \tag{34}$$

The discretization of the Pressure equation, being a derived one, requires a special care, in order to control the accumulation of numerical errors that might invalidate the continuity equation. This means the need of introducing corrective terms to the pressure equation.

Let us consider the momentum equations discretized as

$$\mathbf{u}_{i,j}^{k+1} = \mathbf{u}_{i,j}^{k} + \Delta t \sum_{l=0}^{n_p-1} \alpha_l [\mathbf{F}(\mathbf{u}_{i,j}^{k-l}) - \nabla p_{i,j}^{k-l}] \tag{35}$$

where, n_p and α_l depend upon the employed Adams-Bashforth method, $\mathbf{F}(\mathbf{u}^k)$ being the discretization operator of the convective add diffusive terms.

Applying the divergence operator in both sides of (35), we obtain

$$\nabla.\mathbf{u}_{i,j}^{k+1} = \nabla.\mathbf{u}_{i,j}^{k} + \Delta t \sum_{l=0}^{n_p-1} \alpha_l [\nabla.\mathbf{F}(\mathbf{u}_{i,j}^{k-l}) - \nabla^2 p_{i,j}^{k-l}] \tag{36}$$

The incompressibility condition at the (k+1)th-time step, $\nabla.\mathbf{u}^{k+1} = 0$, is then characterized by

$$\nabla^2 p^k_{i,j} = \nabla.\mathbf{F}(\mathbf{u}^k_{i,j}) + \frac{\nabla.\mathbf{u}^k_{i,j}}{\alpha_0 \Delta t} + \frac{1}{\alpha_0}\sum_{l=1}^{n_p-1} \alpha_l[\nabla.\mathbf{F}(\mathbf{u}^{k-l}_{i,j}) - \nabla^2 p^{k-l}_{i,j}]. \tag{37}$$

We observe that for the Euler method, the above correction coincides with the dilatation term $D_t = \frac{\nabla.\mathbf{u}^n}{\Delta t}$.[3,13,21]

The equation (37) can be written in the compact form

$$\nabla^2 p^k = \nabla.\mathbf{H}(\mathbf{u}^k) \tag{38}$$

where

$$\mathbf{H}(\mathbf{u}^k) = \mathbf{F}(\mathbf{u}^k) + \frac{\mathbf{u}^k}{\alpha_0 \Delta t} + \frac{1}{\alpha_0}\sum_{l=1}^{n_p-1} \alpha_l[\mathbf{F}(\mathbf{u}^{k-l}) - \nabla p^{k-l}] \tag{39}$$

We now discretize the laplacian of the pressure with second-order central differences on a staggered grid

$$\nabla^2 p_{i,j} \approx \frac{p_{i-1,j} + p_{i,j-1} - 4p_{i,j} + p_{i+1,j} + p_{i,j+1}}{h^2} \tag{40}$$

where $h = \Delta x = \Delta y$. Thus (38) becomes

$$p_{i-1,j} + p_{i,j-1} - 4p_{i,j} + p_{i+1,j} + p_{i,j+1} = h(H_{1_{i+1,j}} - H_{1_{i,j}} + H_{2_{i,j+1}} - H_{2_{i,j}}) \tag{41}$$

where $H_{1_{i,j}}$ and $H_{2_{i,j}}$ are the x and y components of $\mathbf{H}(\mathbf{u})$, respectively, applied at the points $(i\Delta x, (j+1/2)\Delta y)$ for the first component and $((i+1/2)\Delta x, j\Delta y)$ for the second one.

For a good convergence of the discretized Poisson equation with a Neumann condition, the compatibility relationship (15) must hold exactly on the discretized domain, that is[1]

$$\sum_{i,j\in\Omega} \nabla^2 p_{i,j} = \sum_{i,j\in\Gamma} \frac{\partial p_{i,j}}{\partial \eta} \tag{42}$$

By adding (41) for all points of the square domain, we have

$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} p_{i-1,j} + p_{i,j-1} - 4p_{i,j} + p_{i+1,j} + p_{i,j+1} = h\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} H_{1_{i+1,j}} - H_{1_{i,j}} + H_{2_{i,j+1}} - H_{2_{i,j}} \tag{43}$$

and making simplifications, we obtain

$$\sum_{j=1}^{m-1}(p_{0,j} - p_{1,j} - p_{n-1,j} + p_{n,j}) + \sum_{i=1}^{n-1}(p_{i,0} - p_{i,1} - p_{i,m-1} + p_{i,m}) =$$

$$h\sum_{j=1}^{m-1}(H_{1_{n,j}} - H_{1_{1,j}}) + h\sum_{i=1}^{n-1}(H_{2_{i,m}} - H_{2_{i,1}}) \tag{44}$$

Then the discrete Neumann condition holds for

$$p_{0,j} = p_{1,j} - hH_{1_{1,j}} \tag{45}$$

$$p_{n,j} = p_{n-1,j} + hH_{1_{n,j}} \tag{46}$$

$$p_{i,0} = p_{i,1} - hH_{2_{i,1}} \tag{47}$$

$$p_{i,m} = p_{i,m-1} + hH_{2_{i,m}} \tag{48}$$

We should observe that these boundary conditions are discretizations of (34).

5 The Velocity–Pressure Algorithm

We now give an algorithm for integrating the Navier–Stokes equations. First, the pressure is initialized by solving a singular system that arises from the discretization of the pressure equation with the Neumann conditions. Second, the momentum equations are solved for the velocity field at each time step. Third, the pressure is up-dated by solving a Poisson equation, giving a special treatment for the interior points that correspond to interior cells and to the adjacent cells in such a way that the compatibility condition is verified. This updating contains corrective terms for the direct calculation of the pressure at interior points of interior cells. This is done by incorporating the already known pressure values at neighboring points.

5.1 Pressure Initialization

To initialize the pressure, we consider the equation (38) with $k = 0$.

$$\nabla^2 p^0 = \nabla.\mathbf{H}(\mathbf{u}^0) \tag{49}$$

Here we use $n_p = 1$ and $\alpha_0 = 1$, as $\nabla.\mathbf{u}^0 = 0$, $\mathbf{H}(\mathbf{u}^0) = \mathbf{F}(\mathbf{u}^0)$. No corrective term appear on the initialization and the discretize boundary conditions are (45)-(48).

The equation (49) discretized as done in (41) associated to the boundary conditions, when written in matrix terms, turns out

$$A\ \mathbf{p}^0 = \mathbf{b} \tag{50}$$

where A is the singular matrix

$$A = \begin{bmatrix} S_1 & I & & & & \\ I & S_2 & I & & & \\ & I & S_2 & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & S_2 & I \\ & & & & I & S_1 \end{bmatrix}_{(m\times n)\times(m\times n)}$$

with

$$S_1 = \begin{bmatrix} -2 & 1 & & & \\ 1 & -3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -3 & 1 \\ & & & 1 & -2 \end{bmatrix}_{n\times n}, \quad S_2 = \begin{bmatrix} -3 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -3 \end{bmatrix}_{n\times n}$$

and I is the identity matrix of order n.

At time $k = 0$, the vector $\mathbf{p}^0$ contains the values of the pressure at interior points, that is,

$$\mathbf{p}^0 = [\ p^0_{1,1}\ \ p^0_{2,1}\ \cdots\ p^0_{n,1}\ \ p^0_{1,2}\ \ p^0_{2,2}\ \cdots\ p^0_{n,2}\ \cdots\ p^0_{1,m}\ \ p^0_{2,m}\ \cdots\ p^0_{n,m}\ ]^T .$$

The vector $\mathbf{b}$ contains all values $u^0_{i,j}$, $v^0_{i,j}$ from the right hand side of (45)–(48), which are given initial values, and this has the particular form,

$$\mathbf{b} = [\ 0\ \ldots\ 0\ \ b_{mn-n+1}\ \ 0 \ldots\ 0\ \ b_{mn}\ ]^T_{m\times n}$$

where

$$b_{mn-n+1} = \frac{2\nu}{h} \qquad \text{and} \qquad b_{mn} = -\frac{2\nu}{h}$$

Hence, **b** is a non-zero vector.

The above singular system can be solved by several methods: least–squares, iterative or LU.

5.2 The One Step Pressure Updating

Once the pressure is initialized, the interior pressure values $p_{i,j}$ at time $t+\Delta t$ are computed with the following one step and explicit scheme (Fig. 2):

$$p_{i,j}^{k+1} = \frac{1}{4}[p_{i-1,j}^{k+1} + p_{i,j-1}^{k+1} + p_{i+1,j}^{k} + p_{i,j+1}^{k}] - \frac{h^2}{4}\nabla.\mathbf{H}(\mathbf{u}_{i,j}^{k+1}) \tag{51}$$

which incorporates by a simple averaging old and new values for the pressure.

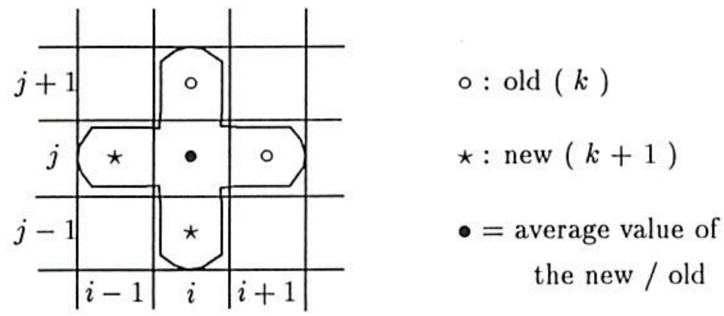

Figure 2: Pressure molecule

This updating of the pressure field can be written in matrix terms as

$$B\ \mathbf{p}^{\mathbf{k+1}} + C\ \mathbf{p}^{\mathbf{k}} = -\mathbb{Q}(\mathbf{u}^{k+1}) \tag{52}$$

where

$$B = \begin{bmatrix} B_1 & & & & \\ I & B_1 & & & \\ & I & B_1 & & \\ & & \ddots & \ddots & \\ & & & I & B_1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & I & & & \\ & C_1 & I & & \\ & & C_1 & \ddots & \\ & & & \ddots & I \\ & & & & C_1 \end{bmatrix}$$

with

$$B_1 = \begin{bmatrix} -4 & & & & \\ 1 & -4 & & & \\ & 1 & -4 & & \\ & & \ddots & \ddots & \\ & & & 1 & -4 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

where B and C are matrix of order $((m-2)\times(n-2))\times((m-2)\times(n-2))$, B_1 and C_1 of order $(n-2)\times(n-2)$ and I is the identity matrix of order $(n-2)$.

The term $\mathbb{Q}(\mathbf{u}^{k+1})$ contains the velocity and corrective terms of (51). The above matrix equation for updating the pressure must not considered as being iterative. It is just a compact form of writing (51).

5.3 Velocity–Pressure Algorithm

The algorithm for solving an incompressible viscous flow with prescribed Neumann condition for the pressure is given as follows.

1. Introduction of the initial velocity components at time $t_0 = 0$, corresponding to level $k = 0$, and the boundary conditions for the velocity field.

2. Initialization of the pressure by solving a singular linear system of the type

$$A\ \mathbf{p}^0 = \mathbf{b}$$

   through least-squares or iterative or LU decomposition.

3. Computation of the velocity field $u_{i,j}^{k+1}$ and $v_{i,j}^{k+1}$ by using (30), (31),respectively.

4. Computation of the pressure p at level time $k+1$ through (51).

5. Up-dating of the pressure and velocity field by setting p^{k+1} instead of p^k and $\mathbf{u}^{k+1}$ for $\mathbf{u}^k$.

6. To perform steps (3)–(5) for $k = 1, 2, \ldots$.

7. End the calculations.

Remarks

1.This algorithm computes corrected pressure values at interior points without any iteration.

2.The algorithm can handle non-rectangular geometries. The only modifications are related to boundary rows and columns of matrices A, B and C, the nonlinear term $\mathbb{Q}$ and the initial vector $\mathbf{b}$. In this work, for simplicity and numerics with a broad range of Re, the algorithm was set up from a 2D square cavity discussion. However, simulations were made for cavities with a non-rectangular bottom.

3.The above algorithm have been successfully employed with 3D rotating convective flow and with the inclusion of viscoelastic terms. It is a matter of a forthcoming work.

5.4 Extension to 3D Domains

The case of a 3D cubic cavity can be easily handled. For a velocity field $\mathbf{u}$=(u,v,w) on a staggered grid, we consider

$$u_{i_x,i_y,i_z} = u(i_x\Delta x, (i_y+1/2)\Delta y, (i_z+1/2)\Delta z)$$
$$v_{i_x,i_y,i_z} = v((i_x+1/2)\Delta x, i_y\Delta y, (i_z+1/2)\Delta z)$$

$$w_{i_x,i_y,i_z} = w((i_x+1/2)\Delta x, (i_y+1/2)\Delta y, i_z\Delta z)$$

$$p_{i_x,i_y,i_z} = p((i_x+1/2)\Delta x, (i_y+1/2)\Delta y, (i_z+1/2)\Delta z)$$

The discretized momentum equations read

$$\mathbf{u}^{k+1}_{i_x,i_y,i_z} = \mathbf{u}^k_{i_x,i_y,i_z} + \Delta t \sum_{l=0}^{n_p-1} \alpha_l [\mathbf{F}(\mathbf{u}^{k-l}_{i_x,i_y,i_z}) - \nabla p^{k-l}_{i_x,i_y,i_z}] \tag{53}$$

where the operator F have a similar meaning to the two dimensional case.

The equation for the pressure, including correcting terms, becomes

$$\nabla^2 p^k_{i_x,i_y,i_z} = \nabla.\mathbf{F}(\mathbf{u}^k_{i_x,i_y,i_z}) + \frac{\nabla.\mathbf{u}^k_{i_x,i_y,i_z}}{\alpha_0 \Delta t} + \frac{1}{\alpha_0}\sum_{l=1}^{n_p-1} \alpha_l [\nabla.\mathbf{F}(\mathbf{u}^{k-l}_{i_x,i_y,i_z}) - \nabla^2 p^{k-l}_{i_x,i_y,i_z}]. \tag{54}$$

The discretization of the laplacian operator on a cubic cavity by second-order central differences and the fulfillment of the discrete compatibility relationship lead to the one step and explicit scheme

$$\begin{aligned} p^{k+1}_{i_x,i_y,i_z} &= \frac{1}{6}[p^{k+1}_{i_x-1,i_y,i_z} + p^{k+1}_{i_x,i_y-1,i_z} + p^{k+1}_{i_x,i_y,i_z-1} + p^{k}_{i_x+1,i_y,i_z} + p^{k}_{i_x,i_y+1,i_z} + p^{k}_{i_x,i_y,i_z+1}] \\ &\quad -\frac{h^2}{6}\nabla.\mathbf{H}(\mathbf{u}^k_{i_x,i_y,i_z}) \end{aligned} \tag{55}$$

$$\text{where} \quad \mathbf{H}(\mathbf{u}^k) = \mathbf{F}(\mathbf{u}^k) + \frac{\mathbf{u}^k}{\alpha_0\Delta t} + \frac{1}{\alpha_0}\sum_{l=1}^{n_p-1}\alpha_l[\mathbf{F}(\mathbf{u}^{k-l}) - \nabla p^{k-l}] \tag{56}$$

Here $h = \Delta x = \Delta y = \Delta z$.

6 Numerical Simulations

We consider the incompressible viscous flow within a cavity that it is induced by the shear movement of the the upper wall, with uniform horizontal velocity $u_T = 1$, and keeping fixed the other walls. The driven cavity flow is often employed for testing and comparison of numerical techniques for solving the Navier-Stokes equations.[9,12,16,18]

We have the horizontal velocity boundary conditions

$$u(x,0,t) = v(0,y,t) = v(X,y,t) = 0 \quad , \qquad u(x,Y,t) = u_T = 1$$

and the normal velocity conditions

$$u(0,y,t) = u(X,y,t) = v(x,0,t) = v(x,Y,t) = 0$$

where X and Y are the linear dimensions of the cavity. We shall assume that at time $t_0 = 0$ the velocity field is zero.

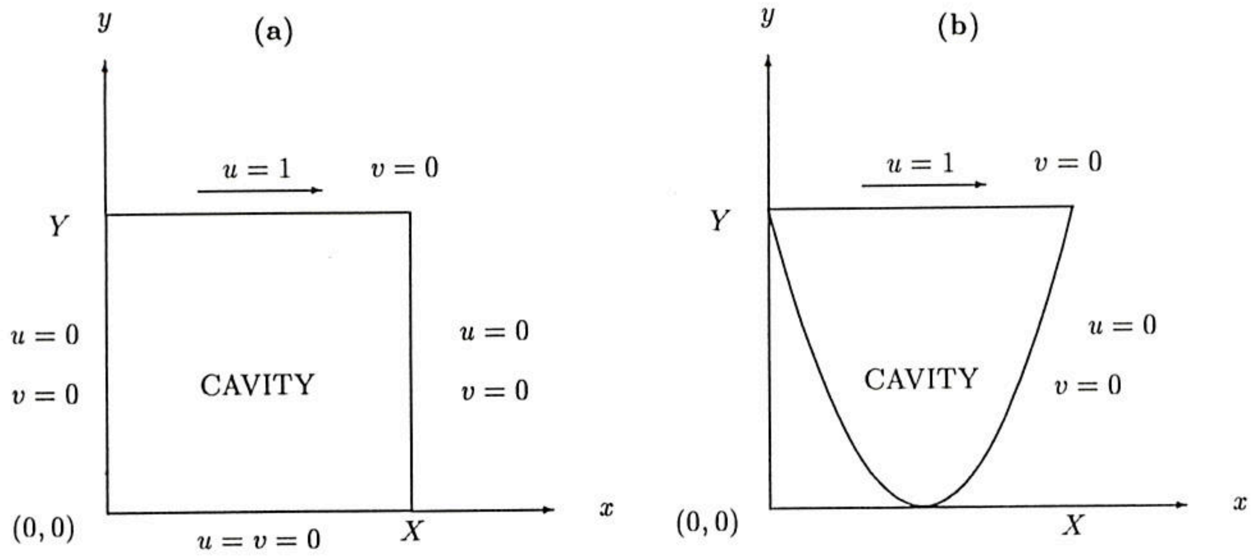

Figure 3: Cavity driven flow (a) rectangular ; (b) curved.

The governing equations were considered in nondimensional form for

$$\overline{x} = \frac{x}{X} \quad \overline{y} = \frac{y}{X} \quad \overline{u} = \frac{u}{u_T} \quad \overline{v} = \frac{v}{u_T}$$
$$\overline{t} = \frac{t u_T}{X} \quad \overline{p} = \frac{p}{\rho u_T^2} \quad Re = \frac{u_T X}{\nu}$$

The simulations were performed for $Re = 100$, $Re = 1000$ and $Re = 5000$ and compared with the results of Ghia et al.[14]. Figure 4 shows the streamlines and the appearance of the primary vortex, the secondary and terciary vortexs at the lower and upper left corners of the cavity. Figure 5 shows the velocity profiles at the centerlines of the cavity ($y = 0.5$ and $x = 0.5$), compared with those of Ghia et al. For $Re = 100$ and $Re = 1000$, it was enough to consider a 66×66 non-uniform grid, refined at the walls, while for $Re = 5000$, we considered a 130×130 refined grid. However, good results can be obtained with smaller refined grids. Figure 6 shows the flow for $Re = 7000$ with a refined grid 50×50.

The proposed algorithm was derived, for simplicity, with a rectangular domain. However, it works too with more complex geometries as shown by the simulations made for a curved cavity. Figure 7 exhibits the results for a parabolic bottom with $Re = 10$, $Re = 400$, $Re = 1000$ and $Re = 2000$.

In order to test the non-iterative one-step pressure updating for 3D domains, we considered a cubic cavity. Although the geometry is relatively simple, the flow is quite complex and appropriate for testing computational codes.[24]

Figure 8 shows the flow for $Re = 400$ with a grid $60 \times 60 \times 60$. We can observe from the upper and frontal views that the flow exhibits some kind symmetry which is to be expected from the boundary conditions. Some authors[25], make use of this observation for reducing the

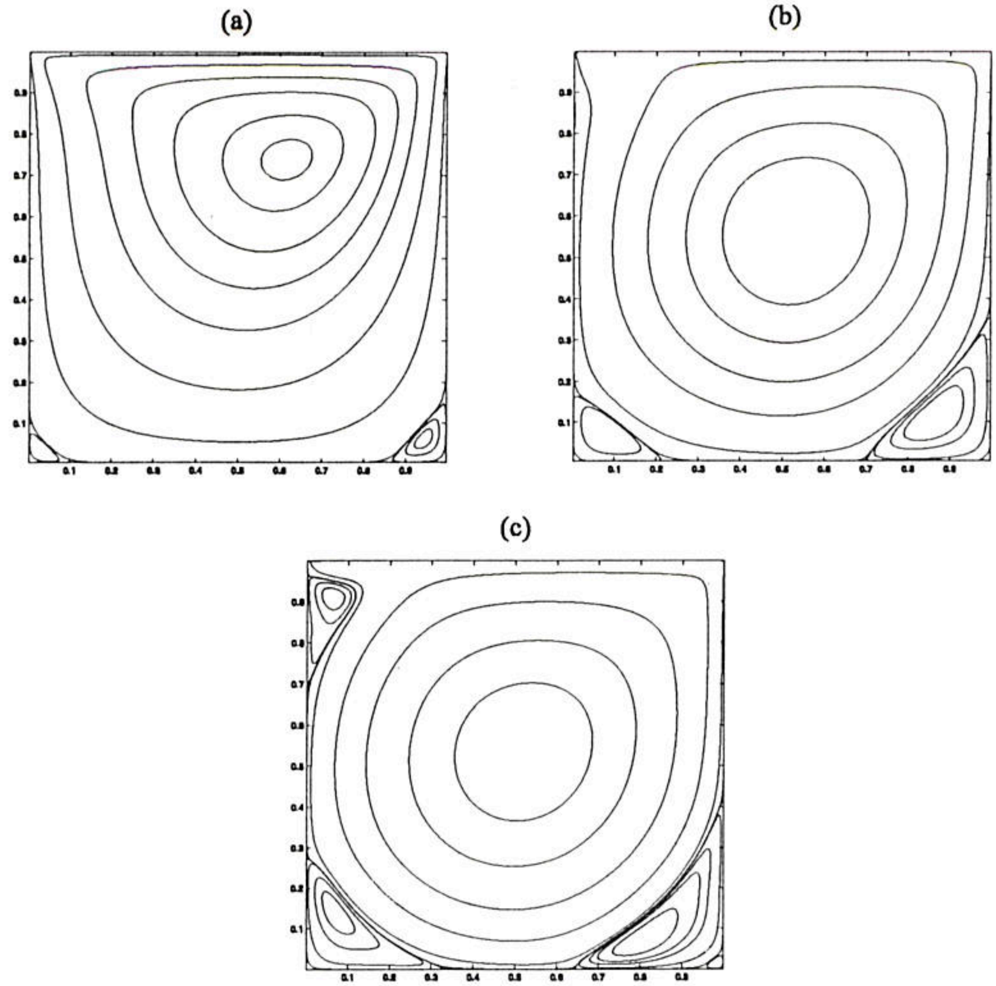

Figure 4: Streamlines (a) $Re = 100$; (b) $Re = 1000$; (c) $Re = 5000$.

computational time. However, we did not need to make use of such device. The reason being that we wanted to observe if such symmetry could be detected with the proposed non-iterative pressure algorithm which do not need to update the pressure in a symmetric way.

The pattern of the streamlines for a cubic cavity is certainly more complex than the 2D case. We observe in Figure 9 that the flow moves between the wall and the center of the cavity, besides circulating around the axis of the main vortex. Figure 6 illustrates isobaric surfaces. The pressure at the interior of the cavity is near zero and the extreme values are obtained at the upper corners. The negative values of the pressure being at the upper left corner and center of the vortex, while the positive ones ocurr at the bottom and upper right corner of the cavity.

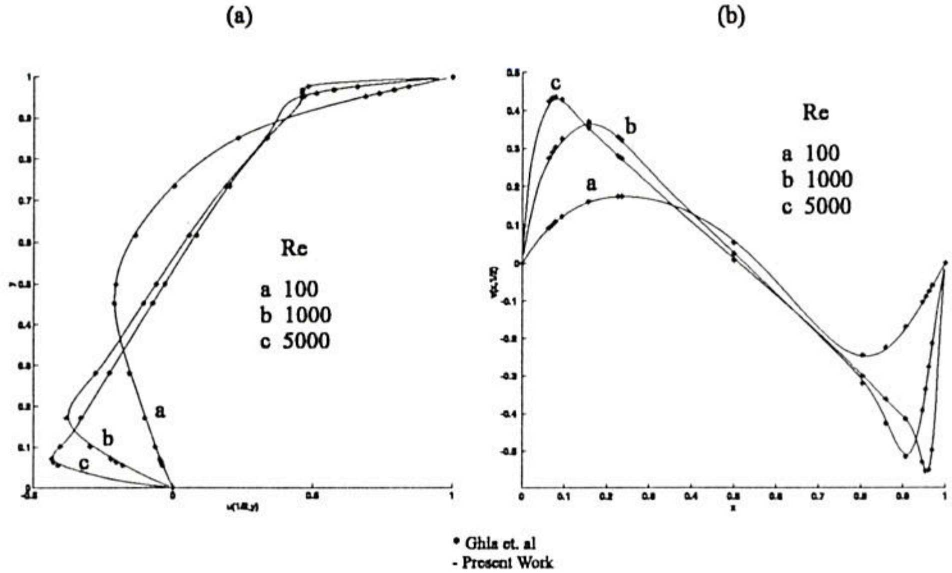

Figure 5: Velocity profiles (a) x=0.5; (b) y=0.5.

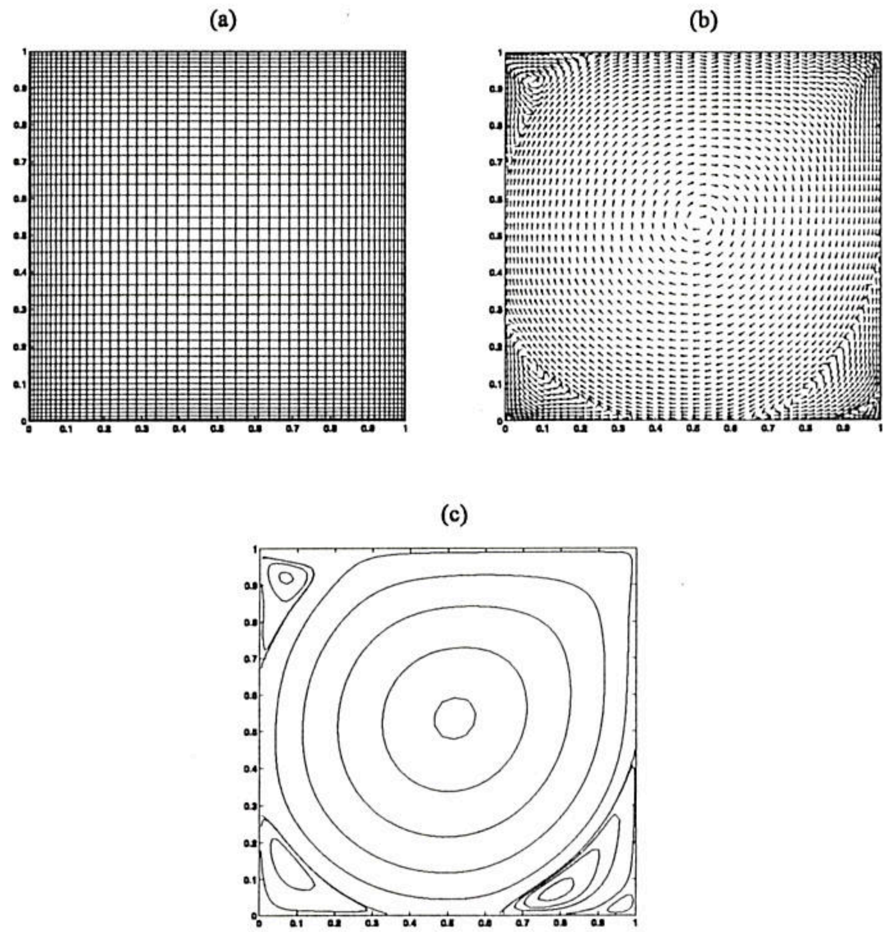

Figure 6: Nonuniform grid (a) grid 50×50; (b) normalized velocity field; (c) Streamlines.

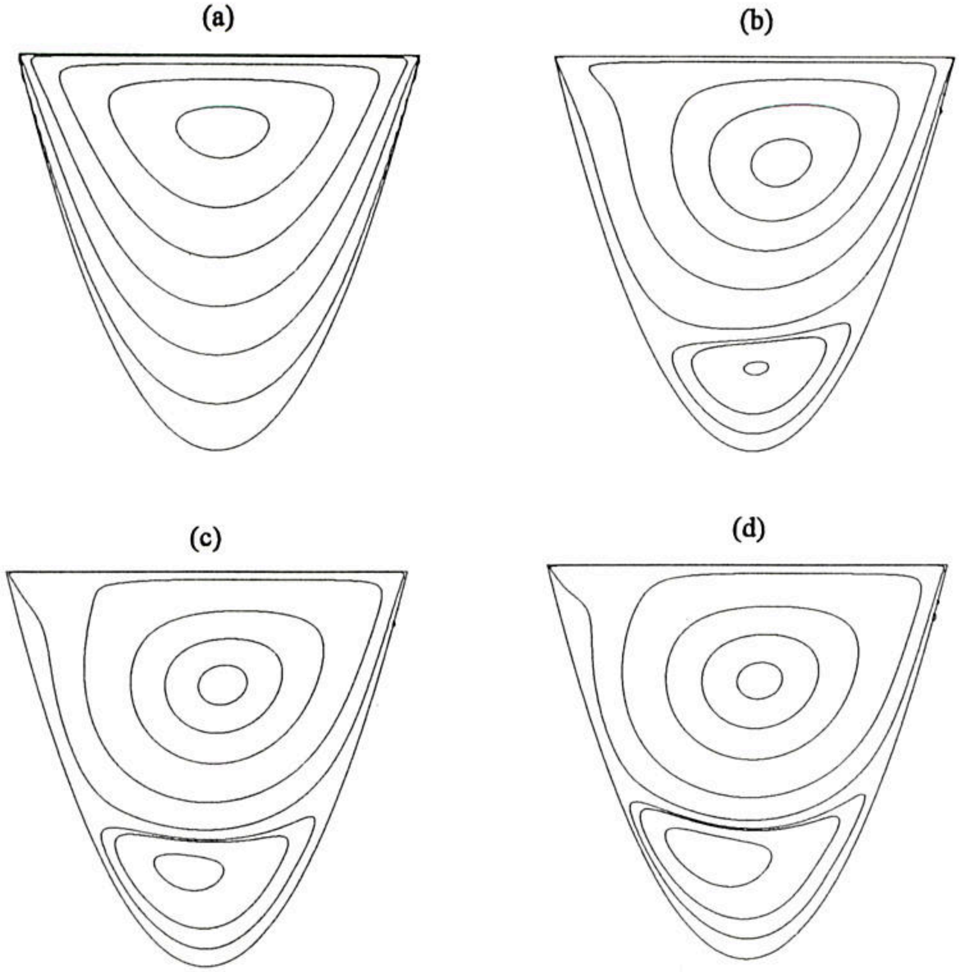

Figure 7: Curved cavity (a) $Re = 10$; (b) $Re = 400$; (c)$Re = 1000$; (d) $Re = 2000$.

7 Conclusions

An algorithm has been developed for the numerical solution of the incompressible Navier–Stokes equations with a central differences scheme in primitive variables and the Neumann boundary condition for the pressure on a staggered grid. The algorithm solve without any iteration a Poisson equation which is transient due to the Neumann condition for the pressure.

This algorithm was tested with the driven cavity flow problem for several Reynolds numbers, uniform and nonuniform grids, curved domains and a 3D cubic cavity. It has been observed the aparition of the central vortex and the recirculation with secondary and terciary eddies. As the Reynolds number increases, the central vortex moves toward the geometrical center of the cavity as shown before by Burggraf[6], 1966; Ghia et al.[14], 1982; Schreiber e Keller[22], 1983, etc..

The matrix formulation allows to follow the influence of the Neumann conditions for the pressure when integrating the velocity and pressure fields at interior points. A correction of the pressure equation was introduced and for increasing the time step and to diminish the number of iterations, we can use other time integration methods.

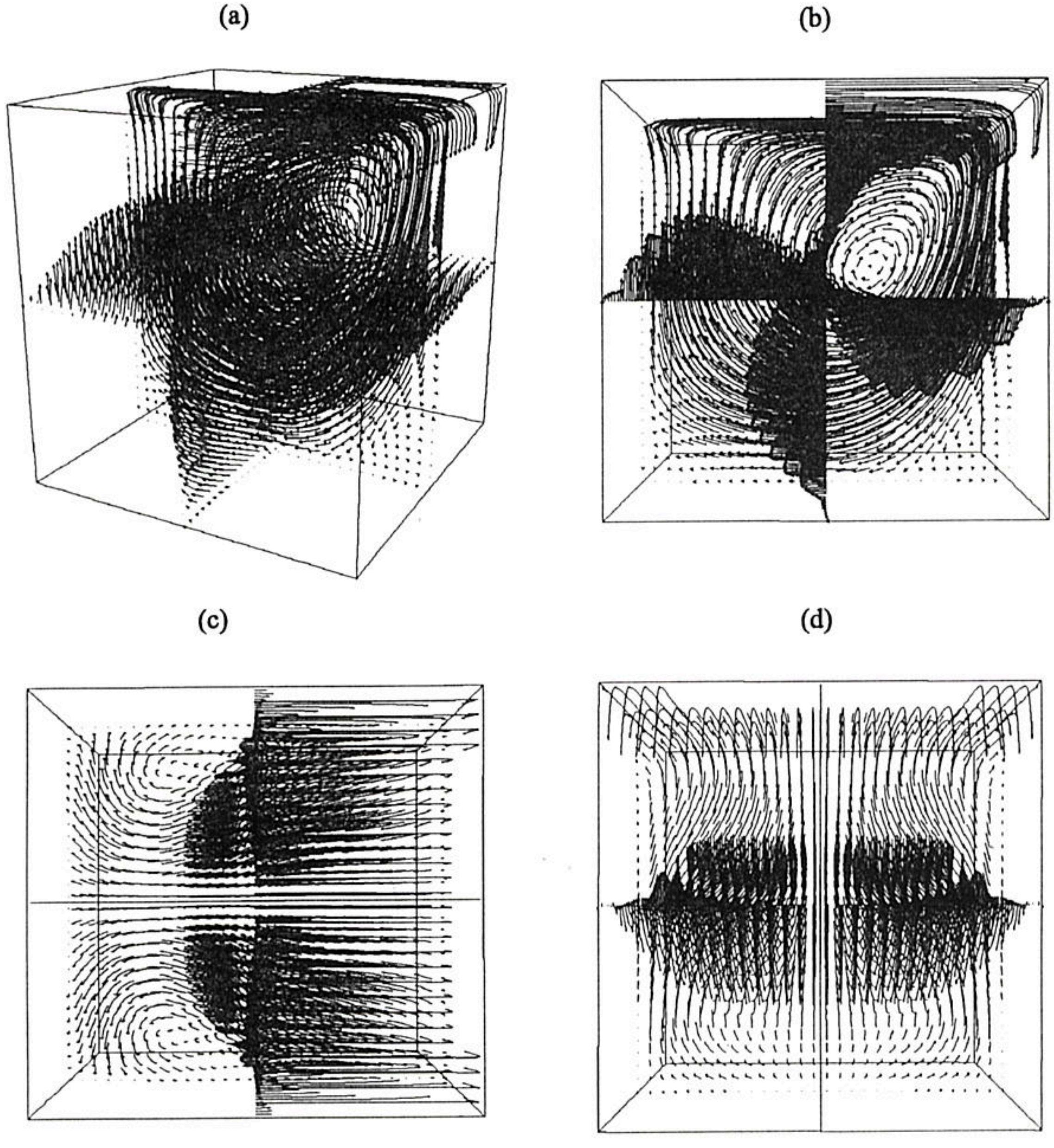

Figure 8: Cubic driven cavity flow at $Re = 400$ (a) Perspective; (b) Lateral view; (c) Upper view; (d) Frontal view.

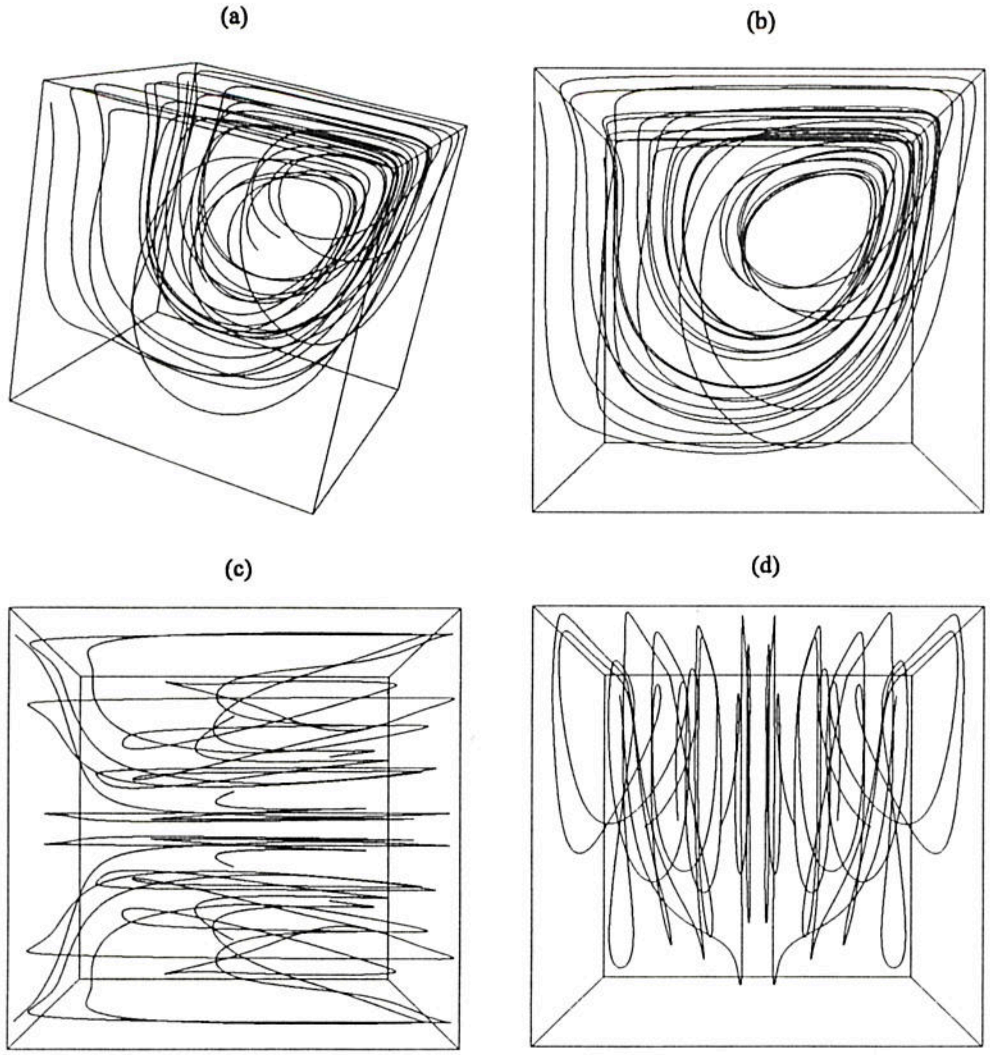

Figure 9: Driven Cavity Streamlines at $Re = 400$ (a) Perspective; (b) Lateral view; (c) Upper view; (d) Frontal view.

The formulation of our algorithm follows the unified operator approach introduced by Casulli[9] (1988) which allows to consider, with minor modifications, the up-wind and semi-lagrangean methods.

Numerical simulations were carried out for a broad range of Reynolds numbers. The results were compared with the existing solutions[14] showing a very good agreement. Besides this, simulations done for the cubic and parabolic cavities illustrate that the algorithm can handle 3D and non-rectangular domains.

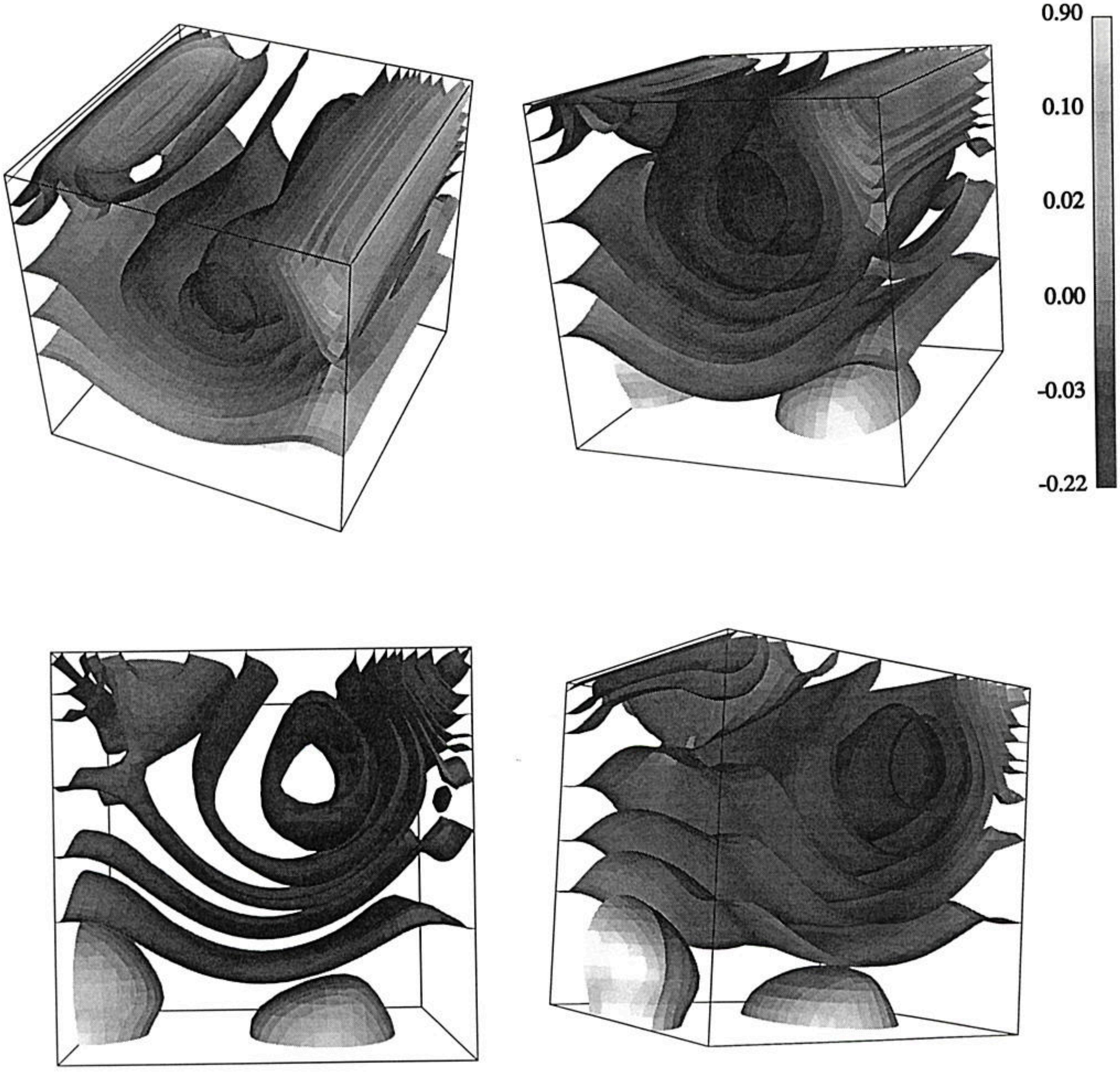

Figure 10: Driven cavity isobaric surfaces at $Re = 400$.

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