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SÉRIE A: TRABALHO DE PESQUISA

SIMULATION IN PRIMITIVE VARIABLES FOR INCOMPRESSIBLE
FLOW WITH PRESSURE NEUMANN CONDITION

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Abstract

We develop a velocity-pressure algorithm, in primitive variables and finite differences, for incompressible viscous flow with a Neumann pressure boundary condition. The pressure field is initialized by least-squares and up-dated from the Poisson equation in one step without iteration. Simulations with the square cavity problem were made for several Reynolds numbers. It was obtained the expected displacement of the central vortex and the appearance of secondary and tertiary eddies. Different geometry ratios for the cavity were also considered. Simulations for a 3D cavity were carried out with an Adams-Bashforth method.

1 Introduction

We develop a velocity-pressure algorithm for incompressible viscous flow in primitive variables by using finite differences and a Neumann pressure boundary condition, as discussed by Gresho and Sami (1987).

The discretization by difference methods of the Navier-Stokes equations on a staggered grid, as made by Casulli (1988), when formulated in matrix terms, allows to identify a singular evolutive matrix system. When we derive the Poisson equation for the pressure and perform its integration, we can observe that a clear influence of the Neuman condition arises. From this we can extract a non-singular system for determining the pressure values at the interior points. The initialization process of the pressure, by a least-squares procedure, somehow incorporates an optimal pressure as a starting point, instead of employing an arbitrary constant as it usually made with iterative methods. The values of the velocity at interior points can then be well determined by a forward Euler method or Adams-Basforth. For the pressure we solve a non-singular Poisson equation without iteration. This later means that we incorporate the values of the pressure and velocity as soon as they are computed.

This velocity-pressure algorithm with central differences has been tested with the cavity problem for a wide range of Reynolds numbers and geometric ratios that include square, deep and shallow cavities. For a square cavity the displacement of the central vortex to the geometrical center of the cavity was obtained by increasing the Reynolds number, as earlier established by Burggraf(1966), Guia et al(1982) and Schreiber and Keller(1983), among others. Also, the apparition of secondary and tertiary vortices.

The proposed algorithm, described in detail for 2D regions, can be appropriately modified for 3D regions. Simulations were made for a 3D cavity.

2 The Continuum Equations for Incompressible Flow

The Navier-Stokes equations for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$ with initial and boundary conditions for the velocity constitute the system

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u}, \quad t > 0 \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \text{ in } \bar{\Omega} = \Omega \cup \Gamma \quad (3)$$

$$\mathbf{u} = \mathbf{w}(\mathbf{x}, t) \quad \text{in } \Gamma = \partial\Omega. \quad (4)$$

Here Ω denotes a limited region, Γ its boundary and

$$\nabla \cdot \mathbf{u}_0 = 0 \quad \text{in } \Omega, \quad (5)$$

an initial solenoidal velocity field. From the above system follows the initial normal velocity

$$\mathbf{u}_0 \cdot \mathbf{n} = \mathbf{w}(\mathbf{x}, 0) \cdot \mathbf{n} \quad \text{em } \Gamma. \quad (6)$$

and the global mass conservation

$$\int_{\Gamma} \mathbf{u} \cdot \mathbf{n} dl = 0 . \quad (7)$$

We observe that no initial nor boundary conditions are prescribed for the pressure. Thus p is determined up to an additive constant corresponding to the level of hydrostatic pressure. By taking divergence on the momentum equations, with adequate differentiability hypotheses, we obtain the Poisson equation

$$\nabla^2 p = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \quad \text{in } \Omega \text{ for } t \geq 0 . \quad (8)$$

or the equivalent equation

$$\nabla^2 p = \nabla \cdot (\nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}) . \quad (9)$$

Here we shall prescribe the Neuman condition

$$\mathbf{n} \cdot \nabla p = \frac{\partial p}{\partial n} = \nu \nabla^2 u_n - \left(\frac{\partial u_n}{\partial t} + \mathbf{u} \cdot \nabla u_n \right) \quad \text{in } \Gamma \text{ for } t \geq 0 \quad (10)$$

whose discussion has been made by Gresho and Sani.

The determination of the solution of the Poisson equation with Neumann boundary conditions requires that the following compatibility relation holds

$$\int_{\Omega} -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \, d\Omega = \oint_{\Gamma} p_n \, d\Gamma \quad (11)$$

where $p_n = \mathbf{n} \cdot \nabla p$, and \mathbf{n} an exterior normal unit vector to Γ .

3 Discretization of the Navier–Stokes Equations

The primitive equations for a 2D incompressible viscous flow are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (12)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (13)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (14)$$

where $u(x, y, t)$ and $v(x, y, t)$ denote the velocity components in x and y directions, $p(x, y, t)$ the pressure and $\nu \leq 0$ kinematic viscosity coefficient. This system can be written in the operator compact form

$$M \frac{\partial U}{\partial t} + NU = -PU + LU \quad (15)$$

where

$$U = \begin{bmatrix} u \\ v \\ p \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) & 0 & 0 \\ 0 & (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} \nu(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) & 0 & 0 \\ 0 & \nu(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{bmatrix}.$$

We now use central differences for approximating the spatial derivatives and the explicit Euler method for approximating the time derivative. Thus, with reference to the staggered grid, we have

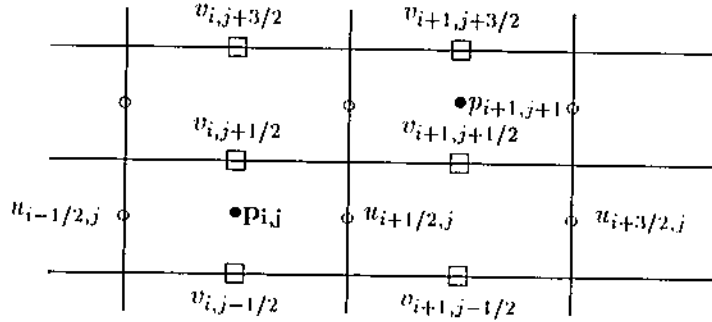


Figure 1: Staggered Grid

$$\begin{aligned} & \frac{u_{i+1/2,j}^{k+1} - u_{i+1/2,j}^k}{\Delta t} + u_{i+1/2,j}^k \frac{u_{i+3/2,j}^k - u_{i-1/2,j}^k}{2\Delta x} \\ & + \bar{v}_{i+1/2,j}^k \frac{u_{i+1/2,j+1}^k - u_{i+1/2,j-1}^k}{2\Delta y} = -\frac{p_{i+1,j}^k - p_{i,j}^k}{\Delta x} \\ & + \nu \left(\frac{u_{i+3/2,j}^k - 2u_{i+1/2,j}^k + u_{i-1/2,j}^k}{(\Delta x)^2} + \frac{u_{i+1/2,j+1}^k - 2u_{i+1/2,j}^k + u_{i+1/2,j-1}^k}{(\Delta y)^2} \right) \end{aligned}$$

or equivalently

$$u_{i+1/2,j}^{k+1} = F_1 u_{i+1/2,j}^k - \Delta t \frac{p_{i+1,j}^k - p_{i,j}^k}{\Delta x} \quad (16)$$

$$i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m$$

where the finite difference operator F_1 is given by

$$\begin{aligned}
F_1 u_{i+1/2,j}^k &= u_{i+1/2,j}^k \\
&- \Delta t \left[u_{i+1/2,j}^k \frac{u_{i+3/2,j}^k - u_{i-1/2,j}^k}{2\Delta x} + v_{i+1/2,j}^k \frac{u_{i+1/2,j+1}^k - u_{i+1/2,j-1}^k}{2\Delta y} \right] \\
&+ \nu \Delta t \left(\frac{u_{i+3/2,j}^k - 2u_{i+1/2,j}^k + u_{i-1/2,j}^k}{(\Delta x)^2} + \frac{u_{i+1/2,j+1}^k - 2u_{i+1/2,j}^k + u_{i+1/2,j-1}^k}{(\Delta y)^2} \right)
\end{aligned} \tag{17}$$

Here $\bar{v}_{i+1/2,j}^k$ denotes the average value

$$\bar{v}_{i+1/2,j}^k = \frac{v_{i,j+1/2}^k + v_{i,j-1/2}^k + v_{i+1,j+1/2}^k + v_{i+1,j-1/2}^k}{4} \tag{18}$$

For the vertical velocity component v we make a similar discretization as the one done for u . Thus

$$\begin{aligned}
v_{i,j+1/2}^{k+1} &= F_2 v_{i,j+1/2}^k - \Delta t \frac{p_{i,j+1}^k - p_{i,j}^k}{\Delta y} \\
i &= 1, 2, \dots, n, \quad j = 1, 2, \dots, m-1
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
F_2 v_{i,j+1/2}^k &= v_{i,j+1/2}^k \\
&- \Delta t \left[\bar{u}_{i,j+1/2}^k \frac{v_{i+1,j+1/2}^k - v_{i-1,j+1/2}^k}{2\Delta x} + v_{i,j+1/2}^k \frac{v_{i,j+3/2}^k - v_{i,j-1/2}^k}{2\Delta y} \right] \\
&+ \nu \Delta t \left(\frac{v_{i+1,j+1/2}^k - 2v_{i,j+1/2}^k + v_{i-1,j+1/2}^k}{(\Delta x)^2} + \frac{v_{i,j+3/2}^k - 2v_{i,j+1/2}^k + v_{i,j-1/2}^k}{(\Delta y)^2} \right)
\end{aligned} \tag{20}$$

As before, we consider $\bar{u}_{i,j+1/2}^k$ as a mean value of the known neighbouring points, that is

$$\bar{u}_{i,j+1/2}^k = \frac{u_{i+1/2,j}^k + u_{i-1/2,j}^k + u_{i+1/2,j+1}^k + u_{i-1/2,j+1}^k}{4} \tag{21}$$

The spatial discretization procedure for the Navier-Stokes equations amounts, in matrix terms, amounts to replace (15) by the semi-discrete approximation matrix equation

$$\mathbf{M} \frac{d\mathbf{U}}{dt} + \mathbf{N}(\mathbf{U})\mathbf{U} + \mathbf{N}^F(\mathbf{U}, \mathbf{U}^F)\mathbf{U}^F = -\mathbf{P}\mathbf{U} - \mathbf{P}^F(\mathbf{U}, \mathbf{U}^F) + \mathbf{L}\mathbf{U} + \mathbf{L}^F\mathbf{U}^F \tag{22}$$

Here $\mathbf{U} = [U_{i,j}]$ where $U_{i,j}$ includes the values $u_{i+1/2,j}$, $v_{i,j+1/2}$, $p_{i,j}$ at a cell (i, j) . The vector \mathbf{U}^F corresponds to the boundary values of u , v e p . The matrices \mathbf{M} , \mathbf{N} , \mathbf{L} , \mathbf{P} are the corresponding spatial approximations of the continuous terms and \mathbf{N}^F , \mathbf{L}^F , \mathbf{P}^F are matrices that contain boundary values. The above systems are singular once \mathbf{M} is a singular matrix. We have that \mathbf{F} is difference operator corresponding to the discretization of the convective and viscous terms, including the continuity equation, and \mathbf{P} corresponds to the spatial discretization of the pressure gradient.

4 The Pressure Equation Discretization

The Poisson equation for the pressure is given by

$$\Delta p = -\nabla \cdot (u, \nabla u) - D_t \quad (23)$$

where the dilation term

$$D = u_x + v_y .$$

is included for numerical stability purposes. We now restrict our discussion to a rectangular domain.

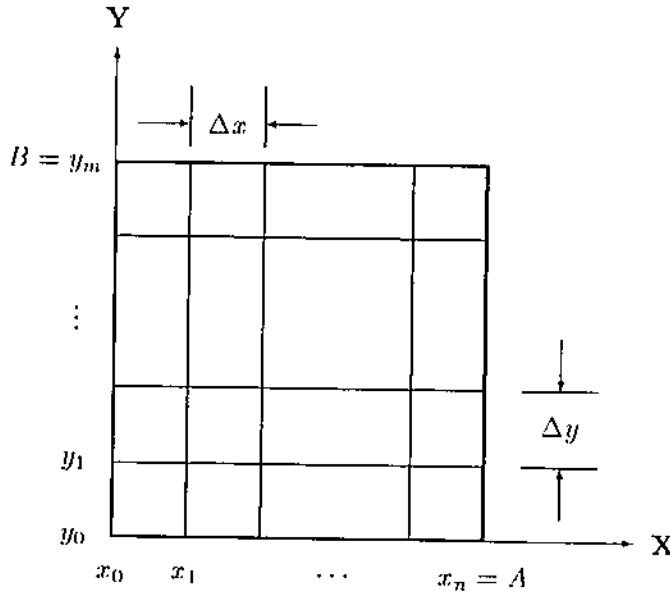


Figure 2: Rectangular Grid Ω

With respect to Fig. 2, the following Neumann boundary conditions for the pressure are obtained from the momentum equation on a solid boundary

$$-p_x = u_t + u u_x + v u_y + \nu (v_{xy} - u_{yy}) \quad \text{in} \quad x = 0, A \quad (24)$$

$$-p_y = v_t + u v_x + v v_y - \nu (v_{xx} - u_{xy}) \quad \text{in} \quad y = 0, B . \quad (25)$$

The Poisson equation (23) and the boundary conditions (23) (24) for the pressure are now approximated on a staggered grid with $\Delta x = \Delta y = h$. The spatial derivatives in (23) , (24) and (25) shall be now approximated by second-order central differences for interior cells and cells adjacent to the boundary.

4.1 Interior Cells

We consider the Poisson equation

$$p_{xx} + p_{yy} = -\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial y} \left(v \frac{\partial v}{\partial y} \right) - D_t . \quad (26)$$

As usual, the dilatation term D_t is approximated by

$$D_t \simeq \frac{D^{k+1} - D^k}{\Delta t} \quad (27)$$

where the superscript indexes k and $k+1$ refer to the time levels t and $t + \Delta t$. In order to satisfy the continuity equation (14), D^{k+1} is made equal to zero.

Let (i, j) refer to an interior cell, that is, without common sides with the boundary (Fig. 1). Then the Poisson equation (26) is approximated by

$$\begin{aligned} \left(p_{xx} + p_{yy} \right)_{p_{i,j}} &= - \left[\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) \right]_{p_{i,j}} - \left[\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) \right]_{p_{i,j}} - \left[\frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) \right]_{p_{i,j}} - \\ &\quad \left[\frac{\partial}{\partial y} \left(v \frac{\partial v}{\partial y} \right) \right]_{p_{i,j}} + \frac{1}{\Delta t} \left(\frac{\partial u}{\partial x} \Big|_{p_{i,j}} + \frac{\partial v}{\partial y} \Big|_{p_{i,j}} \right) \end{aligned} \quad (28)$$

$$i = 2, 3, \dots, n-1; \quad j = 2, 3, \dots, m-1,$$

where each term is given by

$$\begin{aligned} \left[\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) \right]_{p_{i,j}} &\approx \frac{1}{h} \left[\left(u \frac{\partial u}{\partial x} \right)_{u_{i+1/2,j}} - \left(u \frac{\partial u}{\partial x} \right)_{u_{i-1/2,j}} \right] \\ &\approx \frac{1}{h} \left[u_{i+1/2,j} \left(\frac{u_{i+3/2,j} - u_{i-1/2,j}}{2h} \right) - u_{i-1/2,j} \left(\frac{u_{i+1/2,j} - u_{i-3/2,j}}{2h} \right) \right] \end{aligned} \quad (29)$$

$$\begin{aligned} \left[\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) \right]_{p_{i,j}} &\approx \frac{1}{h} \left[\left(v \frac{\partial u}{\partial y} \right)_{v_{i+1/2,j}} - \left(v \frac{\partial u}{\partial y} \right)_{v_{i-1/2,j}} \right] \\ &\approx \frac{1}{h} \left[\bar{v}_{i+1/2,j} \left(\frac{u_{i+1/2,j+1} - u_{i+1/2,j-1}}{2h} \right) - \bar{v}_{i-1/2,j} \left(\frac{u_{i-1/2,j+1} - u_{i-1/2,j-1}}{2h} \right) \right] \end{aligned} \quad (30)$$

$$\begin{aligned} \left[\frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) \right]_{p_{i,j}} &\approx \frac{1}{h} \left[\left(u \frac{\partial v}{\partial x} \right)_{v_{i,j+1/2}} - \left(u \frac{\partial v}{\partial x} \right)_{v_{i,j-1/2}} \right] \\ &\approx \frac{1}{h} \left[\bar{u}_{i,j+1/2} \left(\frac{v_{i+1,j+1/2} - v_{i-1,j+1/2}}{2h} \right) - \bar{u}_{i,j-1/2} \left(\frac{v_{i+1,j-1/2} - v_{i-1,j-1/2}}{2h} \right) \right] \end{aligned} \quad (31)$$

$$\begin{aligned} \left[\frac{\partial}{\partial y} \left(v \frac{\partial v}{\partial y} \right) \right]_{p_{i,j}} &\approx \frac{1}{h} \left[\left(v \frac{\partial v}{\partial y} \right)_{v_{i,j+1/2}} - \left(v \frac{\partial v}{\partial y} \right)_{v_{i,j-1/2}} \right] \\ &\approx \frac{1}{h} \left[v_{i,j+1/2} \left(\frac{v_{i,j+3/2} - v_{i,j-1/2}}{2h} \right) - v_{i,j-1/2} \left(\frac{v_{i,j+1/2} - v_{i,j-3/2}}{2h} \right) \right] \end{aligned} \quad (32)$$

$$\left. \frac{\partial u}{\partial x} \right|_{p_{i,j}} \approx \frac{u_{i+1/2,j} - u_{i-1/2,j}}{h}, \quad \left. \frac{\partial v}{\partial y} \right|_{p_{i,j}} \approx \frac{v_{i,j+1/2} - v_{i,j-1/2}}{h}. \quad (33)$$

We substitute (29)-(33) in (28) and use second-order central differences for approximating the derivatives p_{xx} e p_{yy} , so that

$$\begin{aligned} p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} - 4p_{i,j} &= -\frac{1}{2}u_{i+1/2,j} (u_{i+3/2,j} - u_{i-1/2,j}) \\ &+ \frac{1}{2}u_{i-1/2,j} (u_{i+1/2,j} - u_{i-3/2,j}) - \frac{1}{2}\bar{v}_{i+1/2,j} (u_{i+1/2,j+1} - u_{i+1/2,j-1}) \\ &+ \frac{1}{2}\bar{v}_{i-1/2,j} (u_{i-1/2,j+1} - u_{i-1/2,j-1}) - \frac{1}{2}\bar{u}_{i,j+1/2} (v_{i+1,j+1/2} - v_{i-1,j+1/2}) \\ &+ \frac{1}{2}\bar{u}_{i,j-1/2} (v_{i+1,j-1/2} - v_{i-1,j-1/2}) - \frac{1}{2}v_{i,j+1/2} (v_{i,j+3/2} - v_{i,j-1/2}) \\ &+ \frac{1}{2}v_{i,j-1/2} (v_{i,j+1/2} - v_{i,j-3/2}) + \frac{h}{\Delta t} (u_{i+1/2,j} - u_{i-1/2,j} + v_{i,j+1/2} - v_{i,j-1/2}) \end{aligned} \quad (34)$$

$$i = 2, 3, \dots, n-1, \quad j = 2, 3, \dots, m-1.$$

5 Cells Adjacent to the Boundary

The boundary condition (24) is computed at $u_{3/2,j}$ by using a central difference approximation. Thus

$$\begin{aligned} p_{2,j} - p_{1,j} &= -\frac{h}{\Delta t} (u_{3/2,j}^{k+1} - u_{3/2,j}^k) - \frac{1}{2}u_{3/2,j} (u_{5/2,j} - u_{1/2,j}) \\ &- \frac{1}{2}\bar{v}_{3/2,j} (u_{3/2,j+1} - u_{3/2,j-1}) - \frac{v}{h} (-v_{1,j+1/2} + v_{2,j+1/2} \\ &+ v_{1,j-1/2} - v_{2,j-1/2} - u_{3/2,j+1} + 2u_{3/2,j} - u_{3/2,j-1}), \end{aligned} \quad (35)$$

$$j = 2, 3, \dots, m-1.$$

Similar expressions are obtained by using (24) at $u_{n-1/2,j}$ and computing the boundary condition (25) at $v_{i,3/2}$ and $v_{i,m-1/2}$, that is

$$\begin{aligned} -p_{n,j} + p_{n-1,j} &= \frac{h}{\Delta t} (u_{n-1/2,j}^{k+1} - u_{n-1/2,j}^k) + \frac{1}{2}u_{n-1/2,j} (u_{n+1/2,j} - u_{n-3/2,j}) \\ &+ \frac{1}{2}\bar{v}_{n-1/2,j} (u_{n-1/2,j+1} - u_{n-1/2,j-1}) + \frac{v}{h} (-v_{n-1,j+1/2} + v_{n,j+1/2} \\ &+ v_{n-1,j-1/2} - v_{n,j-1/2} - u_{n-1/2,j+1} + 2u_{n-1/2,j} - u_{n-1/2,j-1}), \end{aligned} \quad (36)$$

$$j = 2, 3, \dots, m-1.$$

$$\begin{aligned}
p_{i,2} - p_{i,1} &= -\frac{h}{\Delta t} (v_{i,3/2}^{k+1} - v_{i,3/2}^k) - \frac{1}{2}\bar{u}_{i,3/2}(v_{i+1,3/2} - v_{i-1,3/2}) \\
&\quad - \frac{1}{2}v_{i,3/2}(v_{i,5/2} - v_{i,1/2}) + \frac{\nu}{h} (v_{i+1,3/2} - 2v_{i,3/2} \\
&\quad + v_{i-1,3/2} + u_{i-1/2,2} - u_{i+1/2,2} - u_{i-1/2,1} + u_{i+1/2,1}) , \\
i &= 2, 3, \dots, n-1 .
\end{aligned} \tag{37}$$

$$\begin{aligned}
-p_{i,m} + p_{i,m-1} &= \frac{h}{\Delta t} (v_{i,m-1/2}^{k+1} - v_{i,m-1/2}^k) + \frac{1}{2}\bar{u}_{i,m-1/2}(v_{i+1,m-1/2} - v_{i-1,m-1/2}) \\
&\quad + \frac{1}{2}v_{i,m-1/2}(v_{i,m+1/2} - v_{i,m-3/2}) - \frac{\nu}{h} (v_{i+1,m-1/2} - 2v_{i,m-1/2} \\
&\quad + v_{i-1,m-1/2} + u_{i-1/2,m} - u_{i+1/2,m} - u_{i-1/2,m-1} + u_{i+1/2,m-1}) , \\
i &= 2, 3, \dots, n-1 .
\end{aligned} \tag{38}$$

The terms $\bar{v}_{i+1/2,j}$ and $\bar{u}_{i,j+1/2}$ in (34)–(38) are defined by (18) and (21).

The addition of terms on both sides of (34)–(38) can be interpreted as a discrete divergence theorem [Alfrink, 1981]. In our case, both add up to zero which tell us that the compatibility equation (11) is exactly satisfied on a staggered grid.

We should observe that the viscous terms in the momentum equations (12), (13) do not appear in the source term for the Poisson equation (23). However, they are present within the Neumann boundary conditions (24), (25). In order to satisfy the compatibility condition (11), the integral of the viscous terms over the boundary must cancel. This is obtained by writing the viscous terms in a convenient way. More precisely, by using the continuity equation (14), we can write $u_{xx} + u_{yy} = -v_{xy} + u_{yy}$ in (24) and $v_{xx} + v_{yy} = v_{xx} - u_{xy}$ in (25). The additional term does not occasionate any trouble on the compatibility condition because the integral of the dilatation over the solution domain vanishes due to global continuity.

6 The Velocity-Pressure Algorithm

We now give an algorithm for integrating the Navier-Stokes equations. First, the pressure is initialized by least-squares from the singular system that arises from the discretization of (23) with the Neuman conditions (24)–(25). Second, the momentum equations (12)–(13) are solved for the velocity field at each time step. Third, the pressure is up-dated from (23)–(25) by giving a special treatment for the interior points that correspond to interior cells and to the adjacent cells in such a way that the compatibility condition is verified. The pressure at interior points of interior cells are computed in a direct manner, that is, by incorporating the already known pressure values at neighbouring points in (34).

6.1 Pressure Initialization

From (34) at the time level $k = 0$, we set

$$\begin{aligned}
p_{i+1,j}^0 + p_{i-1,j}^0 + p_{i,j+1}^0 + p_{i,j-1}^0 - 4p_{i,j}^0 &= -\frac{1}{2}u_{i+1/2,j}^0 (u_{i+3/2,j}^0 - u_{i-1/2,j}^0) \\
&+ \frac{1}{2}u_{i-1/2,j}^0 (u_{i+1/2,j}^0 - u_{i-3/2,j}^0) - \frac{1}{2}\bar{v}_{i+1/2,j}^0 (u_{i+1/2,j+1}^0 - u_{i+1/2,j-1}^0) \\
&+ \frac{1}{2}\bar{v}_{i-1/2,j}^0 (u_{i-1/2,j+1}^0 - u_{i-1/2,j-1}^0) - \frac{1}{2}\bar{u}_{i,j+1/2}^0 (v_{i+1,j+1/2}^0 - v_{i-1,j+1/2}^0) \\
&+ \frac{1}{2}\bar{u}_{i,j-1/2}^0 (v_{i+1,j-1/2}^0 - v_{i-1,j-1/2}^0) - \frac{1}{2}v_{i,j+1/2}^0 (v_{i,j+3/2}^0 - v_{i,j-1/2}^0) \\
&+ \frac{1}{2}v_{i,j-1/2}^0 (v_{i,j+1/2}^0 - v_{i,j-3/2}^0) + \frac{h}{\Delta t} (u_{i+1/2,j}^0 - u_{i-1/2,j}^0 + v_{i,j+1/2}^0 - v_{i,j-1/2}^0) \\
&i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m.
\end{aligned} \tag{39}$$

The Neumann condition at such time level is discretized at the boundary as follows. At $u_{1/2,j}$, we use second-order central differences so that

$$\begin{aligned}
p_{0,j}^0 &= p_{1,j}^0 + \frac{h}{\Delta t} (u_{1/2,j}^1 - u_{1/2,j}^0) + \frac{1}{2}u_{1/2,j}^0 (u_{3/2,j}^0 - u_{-1/2,j}^0) \\
&+ \frac{1}{2}\bar{v}_{1/2,j}^0 (u_{1/2,j+1}^0 - u_{1/2,j-1}^0) + \frac{\nu}{h} (-v_{0,j+1/2}^0 + v_{1,j+1/2}^0 \\
&+ v_{0,j-1/2}^0 - v_{1,j-1/2}^0 - u_{1/2,j+1}^0 + 2u_{1/2,j}^0 - u_{1/2,j-1}^0), \\
&j = 1, 2, \dots, m.
\end{aligned} \tag{40}$$

Similar expressions are obtained by using (24) at $u_{n+1/2,j}$ and computing the boundary condition (25) at $v_{i,1/2}$ and $v_{i,m+1/2}$, given by (41)-(43)

$$\begin{aligned}
p_{n+1,j}^0 &= p_{n,j}^0 - \frac{h}{\Delta t} (u_{n+1/2,j}^1 - u_{n+1/2,j}^0) - \frac{1}{2}u_{n+1/2,j}^0 (u_{n+3/2,j}^0 - u_{n-1/2,j}^0) \\
&- \frac{1}{2}\bar{v}_{n+1/2,j}^0 (u_{n+1/2,j+1}^0 - u_{n+1/2,j-1}^0) - \frac{\nu}{h} (-v_{n,j+1/2}^0 + v_{n+1,j+1/2}^0 \\
&+ v_{n,j-1/2}^0 - v_{n+1,j-1/2}^0 - u_{n+1/2,j+1}^0 + 2u_{n+1/2,j}^0 - u_{n+1/2,j-1}^0), \\
&j = 1, 2, \dots, m.
\end{aligned} \tag{41}$$

$$\begin{aligned}
p_{i,0}^0 &= p_{i,1}^0 + \frac{h}{\Delta t} (v_{i,1/2}^1 - v_{i,1/2}^0) + \frac{1}{2}\bar{u}_{i,1/2}^0 (v_{i+1,1/2}^0 - v_{i-1,1/2}^0) \\
&+ \frac{1}{2}v_{i,1/2}^0 (v_{i,3/2}^0 - v_{i,-1/2}^0) - \frac{\nu}{h} (v_{i+1,1/2}^0 - 2v_{i,1/2}^0 \\
&+ v_{i-1,1/2}^0 + u_{i-1/2,1}^0 - u_{i+1/2,1}^0 - u_{i-1/2,0}^0 + u_{i+1/2,0}^0), \\
&i = 1, 2, \dots, n.
\end{aligned} \tag{42}$$

$$\begin{aligned}
p_{i,m+1}^0 &= p_{i,m}^0 - \frac{h}{\Delta t} (v_{i,m+1/2}^1 - v_{i,m+1/2}^0) - \frac{1}{2} \bar{u}_{i,m+1/2}^0 (v_{i+1,m+1/2}^0 - v_{i-1,m+1/2}^0) \\
&\quad - \frac{1}{2} v_{i,m+1/2}^0 (v_{i,m+3/2}^0 - v_{i,m-1/2}^0) + \frac{\nu}{h} (v_{i+1,m+1/2}^0 - 2v_{i,m+1/2}^0 \\
&\quad + v_{i-1,m+1/2}^0 + u_{i-1/2,m+1}^0 - u_{i+1/2,m+1}^0 - u_{i-1/2,m}^0 + u_{i+1/2,m}^0) , \\
&\quad i = 1, 2, \dots, n .
\end{aligned} \tag{43}$$

The above system, when written in matrix terms, turns out

$$A p_0 = b \tag{44}$$

where A is the singular matrix

$$A = \begin{bmatrix} S_1 & I & & & & & & \\ I & S_2 & I & & & & & \\ & I & S_2 & I & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & I & S_2 & I & \\ & & & & & I & S_1 & \end{bmatrix}_{(m \times n) \times (m \times n)}$$

with

$$S_1 = \begin{bmatrix} -2 & 1 & & & & & \\ 1 & -3 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -3 & 1 & \\ & & & & 1 & -2 & \end{bmatrix}_{n \times n}, \quad S_2 = \begin{bmatrix} -3 & 1 & & & & & \\ 1 & -4 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 1 & \\ & & & & 1 & -3 & \end{bmatrix}_{n \times n}$$

and I is the Identity matrix of order n .

At time $k = 0$, the vector p_0 contains all associated values of the pressure at interior points, that is,

$$p_0 = [p_{1,1}^0 \ p_{2,1}^0 \ \dots \ p_{n,1}^0 \ p_{1,2}^0 \ p_{2,2}^0 \ \dots \ p_{n,2}^0 \ \dots \ p_{1,m}^0 \ p_{2,m}^0 \ \dots \ p_{n,m}^0]^T .$$

The vector b contains all values $u_{i+1/2,j}^0, v_{i,j+1/2}^0$ from the right hand side of (39)-(43), which are given initial values, and this has the particular form,

$$b = [0 \ \dots \ 0 \ b_{mn-n+1} \ 0 \ \dots \ 0 \ b_{mn}]_{m \times n}^T$$

where

$$b_{mn-n+1} = \frac{2\nu}{h} \quad \text{and} \quad b_{mn} = -\frac{2\nu}{h}$$

Hence, b is a non-zero vector.

The singular system (44) is then solved by least-squares.

6.2 Pressure Equation

Once the pressure is initialized, the interior pressure values $p_{i,j}$ at time $t + \Delta t$ are computed with the following criteria (Fig. 3):

1. At the interior points corresponding to adjacent boundary cells we employ (15)–(18), which are obtained from (35)–(38):

$$\begin{aligned}
 p_{1,j}^{k+1} &= p_{2,j}^k + \frac{h}{\Delta t} (u_{3/2,j}^{k+1} - u_{3/2,j}^k) + \frac{1}{2} u_{3/2,j}^{k+1} (u_{5/2,j}^{k+1} - u_{1/2,j}^{k+1}) \\
 &\quad + \frac{1}{2} \bar{v}_{3/2,j}^{k+1} (u_{3/2,j+1}^{k+1} - u_{3/2,j-1}^{k+1}) + \frac{\nu}{h} (-v_{1,j+1/2}^{k+1} + v_{2,j+1/2}^{k+1} \\
 &\quad + v_{1,j-1/2}^{k+1} - v_{2,j-1/2}^{k+1} - u_{3/2,j+1}^{k+1} + 2u_{3/2,j}^{k+1} - u_{3/2,j-1}^{k+1}), \\
 &\quad j = 2, 3, \dots, m-1.
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 p_{n,j}^{k+1} &= p_{n-1,j}^k - \frac{h}{\Delta t} (u_{n-1/2,j}^{k+1} - u_{n-1/2,j}^k) - \frac{1}{2} u_{n-1/2,j}^{k+1} (u_{n+1/2,j}^{k+1} - u_{n-3/2,j}^{k+1}) \\
 &\quad - \frac{1}{2} \bar{v}_{n-1/2,j}^{k+1} (u_{n-1/2,j+1}^{k+1} - u_{n-1/2,j-1}^{k+1}) - \frac{\nu}{h} (-v_{n-1,j+1/2}^{k+1} + v_{n,j+1/2}^{k+1} \\
 &\quad + v_{n-1,j-1/2}^{k+1} - v_{n,j-1/2}^{k+1} - u_{n-1/2,j+1}^{k+1} + 2u_{n-1/2,j}^{k+1} - u_{n-1/2,j-1}^{k+1}), \\
 &\quad j = 2, 3, \dots, m-1.
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 p_{i,1}^{k+1} &= p_{i,2}^k + \frac{h}{\Delta t} (v_{i,3/2}^{k+1} - v_{i,3/2}^k) + \frac{1}{2} \bar{u}_{i,3/2}^{k+1} (v_{i+1,3/2}^{k+1} - v_{i-1,3/2}^{k+1}) \\
 &\quad + \frac{1}{2} v_{i,3/2}^{k+1} (v_{i,5/2}^{k+1} - v_{i,1/2}^{k+1}) - \frac{\nu}{h} (v_{i+1,3/2}^{k+1} - 2v_{i,3/2}^{k+1} \\
 &\quad + v_{i-1,3/2}^{k+1} + u_{i-1/2,2}^{k+1} - u_{i+1/2,2}^{k+1} - u_{i-1/2,1}^{k+1} + u_{i+1/2,1}^{k+1}), \\
 &\quad i = 2, 3, \dots, n-1.
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 p_{i,m}^{k+1} &= p_{i,m-1}^k - \frac{h}{\Delta t} (v_{i,m-1/2}^{k+1} - v_{i,m-1/2}^k) - \frac{1}{2} \bar{u}_{i,m-1/2}^{k+1} (v_{i+1,m-1/2}^{k+1} - v_{i-1,m-1/2}^{k+1}) \\
 &\quad - \frac{1}{2} v_{i,m-1/2}^{k+1} (v_{i,m+1/2}^{k+1} - v_{i,m-3/2}^{k+1}) + \frac{\nu}{h} (v_{i+1,m-1/2}^{k+1} - 2v_{i,m-1/2}^{k+1} \\
 &\quad + v_{i-1,m-1/2}^{k+1} + u_{i-1/2,m}^{k+1} - u_{i+1/2,m}^{k+1} - u_{i-1/2,m-1}^{k+1} + u_{i+1/2,m-1}^{k+1}), \\
 &\quad i = 2, 3, \dots, n-1.
 \end{aligned} \tag{18}$$

2. At interior points of the interior cells, we employ (34) to compute the pressure values at each time level by incorporating previous values of the velocity and pressure fields. This modification

lead us to

$$\begin{aligned}
p_{i,j}^{k+1} &= \frac{1}{4}(p_{i+1,j}^k + p_{i-1,j}^{k+1} + p_{i,j+1}^k + p_{i,j-1}^{k+1}) + \frac{1}{8}u_{i+1/2,j}^{k+1} (u_{i+3/2,j}^{k+1} - u_{i-1/2,j}^{k+1}) \\
&- \frac{1}{8}u_{i-1/2,j}^{k+1} (u_{i+1/2,j}^{k+1} - u_{i-3/2,j}^{k+1}) + \frac{1}{8}\bar{v}_{i+1/2,j}^{k+1} (u_{i+1/2,j+1}^{k+1} - u_{i+1/2,j-1}^{k+1}) \\
&- \frac{1}{8}\bar{v}_{i-1/2,j}^{k+1} (u_{i-1/2,j+1}^{k+1} - u_{i-1/2,j-1}^{k+1}) + \frac{1}{8}\bar{u}_{i,j+1/2}^{k+1} (v_{i+1,j+1/2}^{k+1} - v_{i-1,j+1/2}^{k+1}) \\
&- \frac{1}{8}\bar{u}_{i,j-1/2}^{k+1} (v_{i+1,j-1/2}^{k+1} - v_{i-1,j-1/2}^{k+1}) + \frac{1}{8}v_{i,j+1/2}^{k+1} (v_{i,j+3/2}^{k+1} - v_{i,j-1/2}^{k+1}) \\
&- \frac{1}{8}v_{i,j-1/2}^{k+1} (v_{i,j+1/2}^{k+1} - v_{i,j-3/2}^{k+1}) - \frac{h}{4\Delta t} (u_{i+1/2,j}^{k+1} - u_{i-1/2,j}^{k+1} + v_{i,j+1/2}^{k+1} - v_{i,j-1/2}^{k+1}) \\
&i = 2, 3, \dots, n-1, \quad j = 2, 3, \dots, m-1.
\end{aligned} \tag{49}$$

The up dating (49) of the pressure field at points of the interior cells (49) can be written in matrix terms as

$$R p_{t+\Delta t} = -D \mathbb{N}(\bar{u}_{t+\Delta t}) \tag{50}$$

where R is a non-singular matrix of the type

$$R = \begin{bmatrix} R_1 & R_2 & & & \\ R_2 & R_1 & R_2 & & \\ & R_2 & R_1 & \ddots & \\ & & \ddots & \ddots & R_2 \\ & & & R_2 & R_1 \end{bmatrix}_{((m-2) \times (n-2)) \times ((m-2) \times (n-2))}$$

with

$$R_1 = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & & 1 & -4 \end{bmatrix}_{(n-2) \times (n-2)} \quad \text{c} \quad R_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}_{(n-2) \times (n-2)}.$$

The term $D \mathbb{N}(\bar{u}_{t+\Delta t})$ in (50) contains all the values $u_{i+1/2,j}^{k+1}, v_{i,j+1/2}^{k+1}$ of the right hand side of (49).

Thus, the computation of the pressure field at interior points of the grid can be visualized as

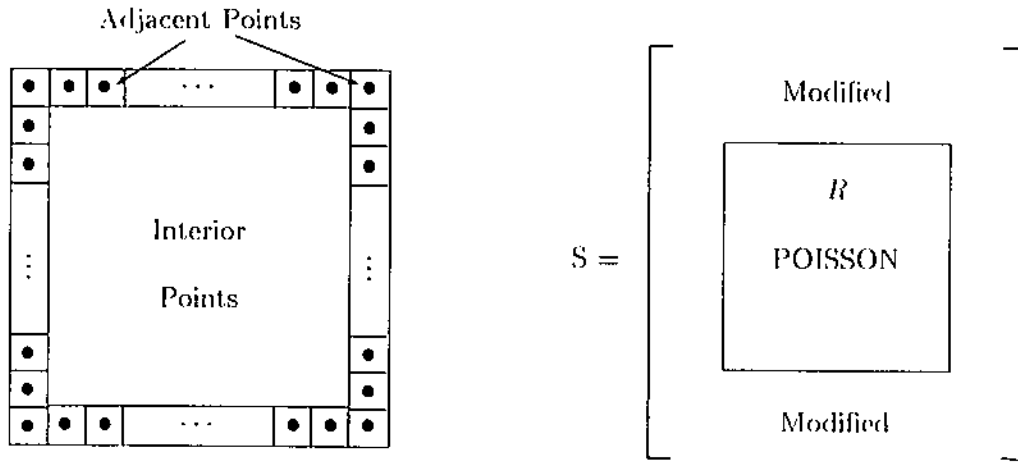


Figure 3: Computation of the pressure at interior points

6.3 Velocity-Pressure Algorithm

The algorithm for solving an incompressible viscous flow with prescribed Neuman condition for the pressure is as follows.

1. Introduction of the initial velocity components $u_{i+1/2,j}^0, v_{i,j+1/2}^0$ at time $t_0 = 0$, corresponding to level $k = 0$, and the boundary conditions for the velocity field.
2. Initialization of the pressure by solving (39)-(43) through least-squares, that is, to solve a singular linear system of the type

$$A p_0 = b.$$

3. Computation of the velocity field $u_{i+1/2,j}^{k+1}$ and $v_{i,j+1/2}^{k+1}$ by using (16)-(18) e (19) (21).
4. Computation of the pressure p at level time $k + 1$ through (45)-(49).
5. Up-dating of the pressure and velocity field by setting $p_{t+\Delta t}$ instead of p_0 and $\bar{u}_{t+\Delta t}$ for \bar{u}_0 .
6. To perform steps (4)-(6) for $k = 1, 2, \dots$
7. End the calculations.

We should emphasize that the pressure values at interior points are obtained without any iteration method.

7 Simulations

Numerical simulations were carried out for the cavity problem for a broad range of Reynolds numbers and geometric ratios $A = \frac{\text{height}}{\text{base}}$. The Figure 4 show the velocity and pressure fields for $Re = 100, 400, 1000, 5000$ and 10000 on a square grid ($A = 1$) with $\Delta x = 0.01$ and time steps $\Delta t = 0.001, 0.002$. This values meet the stability criteria $\Delta t/h < 1$ and $\Delta t \leq h^2/4\nu$ as suggested by

[Roach, 1982; Casulli, 1988], among others.

The proposed algorithm was directly extended for a 3D cavity. The simulations were carried out with a second-order Adams-Bashforth method. The figure 5 exhibits the simulations for $Re = 400$ with a $60 \times 60 \times 60$ grid and $\Delta t = 0.01$.

8 Conclusions

An algorithm has been developed for the numerical solution of the incompressible Navier-Stokes with central differences in primitive variables and the Neuman boundary condition for the pressure on a staggered grid.

This algorithm was tested with the cavity problem for several Reynolds numbers. It has been observed the aparition of the central vortex and the recirculation with secondary and terciary eddies. As the Reynolds number increases, the central vortex moves toward the geometrical center of the cavity as shown before by Burggraf, 1966; Ghia et al., 1982; Schreiber e Keller, 1983, etc..

The matrix formulation allows to follow the influence of the Neuman conditions for the pressure when integrating the velocity and pressure fields at interior points. The time integration can be also performed by other methods.

Simulations for a 3D cavity were carried out with an Adams-Bashforth method and noticed that the proposed algorithm substantially diminishes the time for the computations.

We can also consider, within Casulli's unified formulation, up-wind and semi-lagrangean methods for the spatial discretization.

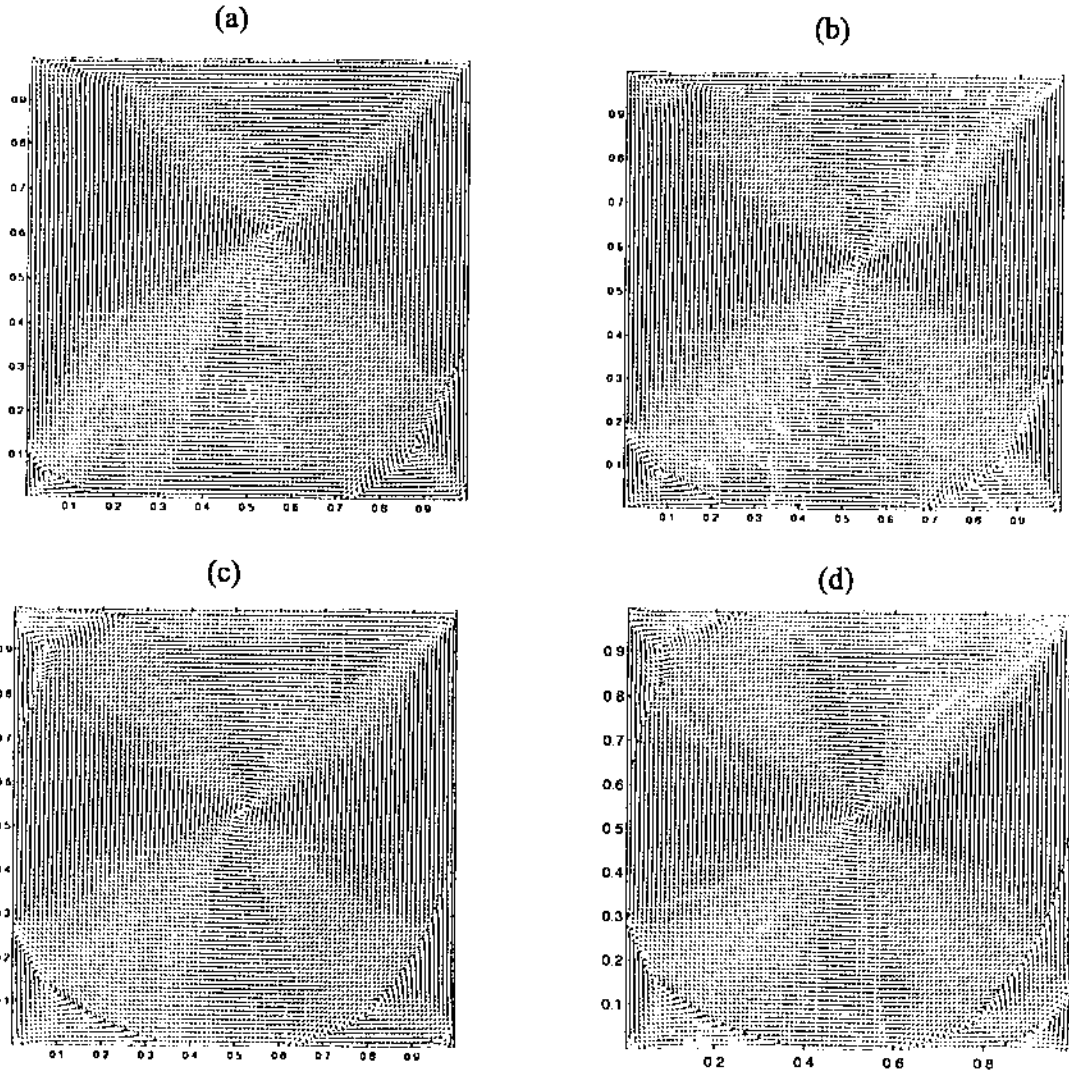


Figure 4: Normalized Velocity Field of Square Cavity: (a) $Re = 400$; (b) $Re = 1000$; (c) $Re = 5000$ and (d) $Re = 10000$.

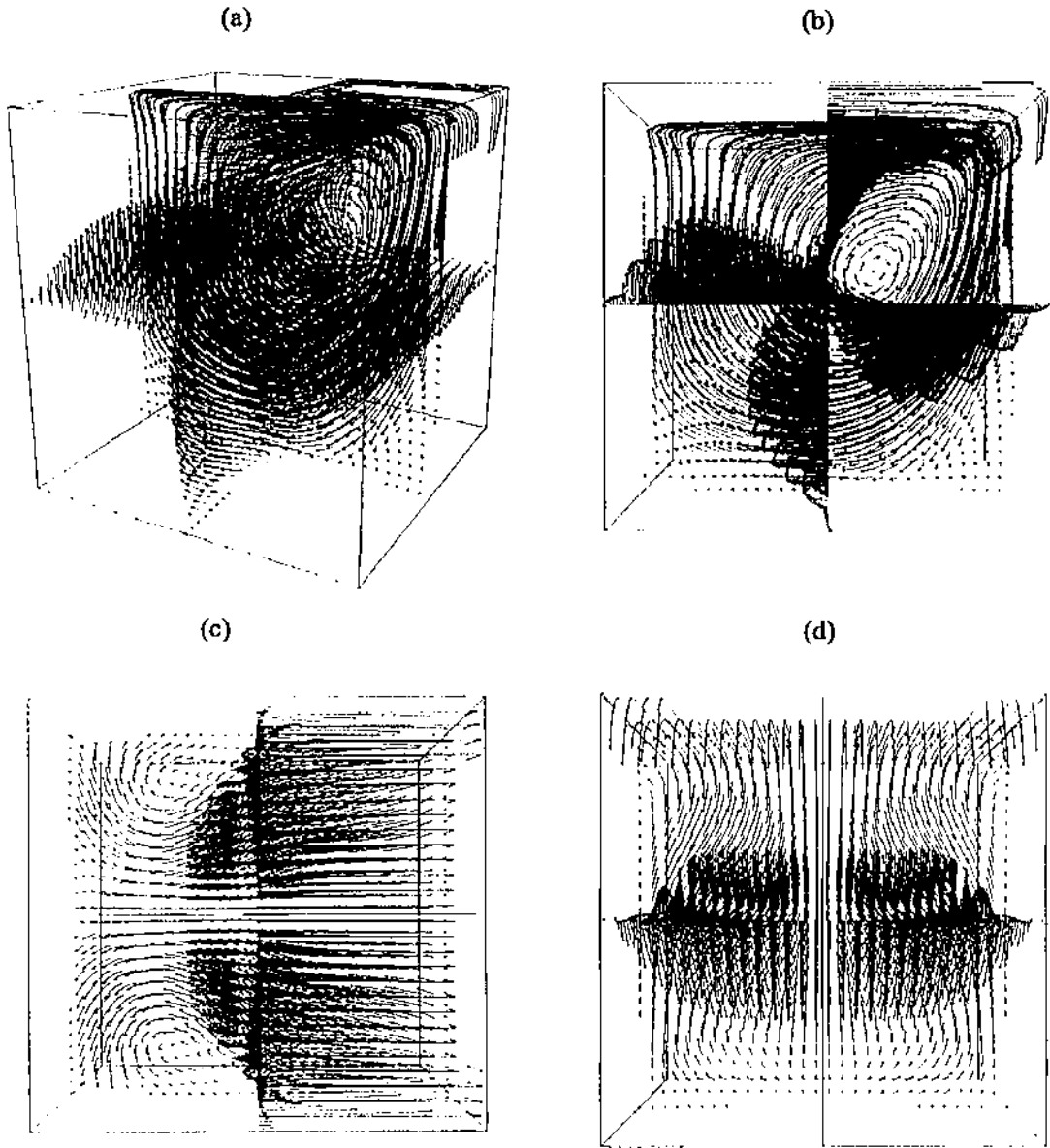


Figure 5: Tridimensional Cavity at $Re = 400$ (a) Perspective View; (b) y-z plane view ; (c) x-z plane view (d) x-y plane view.

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