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Instituto de Matemática
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Silvia Lopes
Benjamin Kedem

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Silvia Lopes and Benjamin Kedem[1]

Instituto de Matemática
Universidade Federal do Rio Grande do Sul
Porto Alegre - RS - Brasil

and

Department of Mathematics
University of Maryland
College Park - MD - USA

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Abstract

We will analyze the stationary frequency modulated (FM) process with the additive ambient noise

$$Z_t = Y_t + \varepsilon_t = A\cos(\omega_c\, t + X(t) + \phi) + \varepsilon_t, \quad \text{for } t \in \mathbf{Z}$$

where

$$X(t) = B\sin(\omega_0\, t + \varphi) \tag{1.2}$$

is the sinusoidal modulating process, A and B are constants, $\omega_c, \omega_0 \in [0, \pi]$ are, respectively, the carrier and the modulating frequencies and φ and ϕ are uniformly distributed random variables on $(-\pi, \pi]$ independent of each other and of the noise process $\{\varepsilon_t\}_{t\in\mathbf{Z}}$. We will consider the noise process as being Gaussian and white for simplicity of the exposition. However, the results are similar for any stationary and ergodic process with continuous spectral density function. Here we will estimate the relevant parameters A, B, ω_c and ω_0 by an updating procedure based on *HOC* (*higher order correlations*) sequences in the fine tuning of parametric filters. We will use two different parametric families of time invariant linear filters: *the alpha and complex filters*. Here we alleviate the assumption of Gaussianity for the signal and we prove its stationarity and ergodicity under appropriate conditions.

Abbreviated Title : "Frequency Modulation".

Key words and phrases : Stationary, sinusoidal frequency modulation, spectrum, instantaneous frequency, recursive method, parametric filter, ergodicity.

AMS Subject Classification : Primary 62M10, secondary 62M07.

1. Introduction

In this paper we want to apply the *CM Method* ideas to the frequency modulated (FM) process

$$Z_t = Y_t + \varepsilon_t = A\cos(\omega_c\, t + X(t) + \phi) + \varepsilon_t, \quad \text{for } t \in \mathbf{Z} \tag{1.1}$$

where

$$X(t) = B\sin(\omega_0\, t + \varphi) \tag{1.2}$$

is the sinusoidal modulating process, A and B are constants, $\omega_c, \omega_0 \in [0, \pi]$ are, respectively, the carrier and the modulating frequencies and φ and ϕ are uniformly distributed random variables on $(-\pi, \pi]$ independent of each other and of the noise process $\{\varepsilon_t\}_{t \in \mathbf{Z}}$. We consider the noise process as being Gaussian white noise for simplicity of the exposition, that is, $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, nonetheless for any stationary and ergodic process with continuous spectral density function $f_\varepsilon(\lambda)$ the results follow similarly.

Our goal is to estimate the *instantaneous frequency*, that is, the derivative with respect to the time of the *instantaneous phase* defined as

$$\omega(t) = \frac{d}{dt}\left(\omega_c\, t + B\sin(\omega_0\, t + \varphi) + \phi\right) = \omega_c + B\,\omega_0\cos(\omega_0\, t + \varphi), \quad \text{for } t \in \mathbf{Z} \tag{1.3}$$

under the extra assumption that the modulating signal varies slowly compared to the carrier, that is,

$$\omega_c >> \omega_0.$$

The *instantaneous frequency* varies about the unmodulated carrier frequency ω_c at the rate ω_0 of the modulating signal and with a maximum deviation of $B\,\omega_0$ radians.

We will consider discrete time parameter set $T = \mathbf{Z}$, but we point out that the results of Sections 2 and 3 also apply to the continuous time $T = \mathbf{R}$.

We also assume that

$$-\pi < \omega_c - B\,\omega_0 < \omega_c + B\,\omega_0 < \pi$$

since we want the frequency support to be in $[-\pi, \pi]$. The constant B is called the *modulation index*.

In order to estimate the instantaneous frequency (1.3) we need to estimate the parameters ω_c, ω_0 and B. We will also estimate the parameters A and σ_ε. The novelty here is the updating procedure based on *HOC* (*higher order crossings* or *higher order correlations*, depending on what one chooses to observe, zero-crossing counts or correlations) analysis to produce these estimates. Here we will just consider *higher order correlations*. For the case of finite number of frequencies using this procedure see He and Kedem (1989), Lopes (1991), Kedem and Lopes (1991) and Lopes and Kedem (1991).

The analysis of FM models is much more complicated than the case of finite frequencies (see Kedem and Lopes (1991)). In some sense we have to deal with an infinite and dense set of frequencies in $(-\pi, \pi]$ (see (3.9) and (3.10)).

It is well known that sine-wave modulations enable signals to be transmitted at frequencies much higher than the signal-frequency components.

The notion of sine-wave modulation means that we have available a source of sinusoidal energy with a carrier wave of the form

$$Y(t) = A\cos(\omega_c t + \phi), \quad t \in \mathbf{Z}. \tag{1.4}$$

Any of the parameters A, ω_c or ϕ may be varied in accordance with the modulating signal. In FM systems one modulates the frequency in accordance with some information-bearing signal. An advantage of FM systems over others is that the former provides better protection against interfering signals and noise. However, to obtain this improved response a wider bandwidth is required. We noted previously that increasing the amplitude of the modulating signal it should increase the bandwidth occupied by the FM signal. Increasing the modulated signal amplitude corresponds to increasing the modulation index B. So the bandwidth of the FM wave will depend on B. If the modulation index is zero, the resulting process

$$Z_t = A\cos(\omega_c t + \phi) + \varepsilon_t, \quad t \in \mathbf{Z},$$

is one sinusoid plus noise model. The *CM Method* applied to the multiple frequency version of this model was already pursued in some papers due to Benjamin Kedem and his collaborators. For an application of the *CM Method* under the point of view of fixed points of a certain mapping see Lopes (1991), Kedem and Lopes (1991) and Lopes and Kedem (1991).

When the parameter B in (1.2) is equal to zero we face the case of only one sinusoid with frequency ω_c. In this case it is already known that the updating procedure based on *HOC* analysis works as it can be seen in He and Kedem (1989). When the parameter B in (1.2) is equal to one or it is already known *a priori* there is a simplification in our procedure as we will see at the end of Section 5.3.

A careful analysis in the model (1.1) is needed before showing how to use the higher order correlations.

Let $\{\mathcal{L}_\theta(\cdot)\}_{\theta\in\Theta}$ be a parametric family of time invariant linear filters, where θ is a finite dimensional parameter in the parameter space Θ. Denote by $\{Z_t(\theta)\}_{t\in T}$ the filtered process

$$Z_t(\theta) \equiv \mathcal{L}_\theta(Z)_t,$$

where $\{Z_t\}_{t\in T}$ is the zero mean stationary process given in (1.1).

Then $\{\rho_1(\theta)\}_{\theta\in\Theta}$, defined by

$$\rho_1(\theta) = \frac{\mathcal{R}\{E[Z_t(\theta)\overline{Z_{t+1}(\theta)}]\}}{E|Z_t(\theta)|^2}$$

is a *HOC* family defined from a parametrized first order autocorrelation. Here and elsewhere, a bar denotes complex conjugate and $\mathcal{R}\{z\}$ the real part of z.

Let the updating scheme based on *higher order correlations* given by

$$\alpha_{k+1} = \rho_1(\alpha_k), \quad \text{for } k \in \mathrm{N}, \tag{1.5}$$

be applied to the process (1.1).

We will choose a time series $\{Z_t\}_{t=1}^{N}$ of size N to give estimates for $E[Z_t(\theta)\overline{Z_{t+l}(\theta)}]$, for any $\theta \in \Theta$, when $l = 1$ or 0. Therefore, the estimates of the autocovariance or variance of the process $\{Z_t(\theta)\}_{t\in T}$, for any $\theta \in \Theta$, are

$$\frac{1}{N}\sum_{j=1}^{N-1}[Z_j(\theta) - \overline{Z(\theta)}]\overline{[Z_{j+1}(\theta) - \overline{Z(\theta)}]} \quad \text{or} \quad \frac{1}{N}\sum_{j=1}^{N-1}|Z_j(\theta) - \overline{Z(\theta)}|^2,$$

based on the time series $Z_1, Z_2, \cdots, Z_N$. Here the inner bar denotes the mean average value.

In Section 6 we will show that the estimates are consistent in the situation we are interested.

We will analyze the efect of special filters applied to the time series that will make the updating procedure to converge to values that give us important information. For instance, α_k will converge to ω_c in Section 5.1.

This paper is organized as follows. Section 2 contains the derivation of the autocorrelation function and the spectral measure of the general process $\{Y_t\}_{t\in \mathrm{Z}}$ as in (1.1), considered as the real part of a complex signal. In Section 3 the spectral distribution function of the sinusoidal modulating process as in (1.1) and (1.2) is presented (see also Subba-Rao and Yar (1982)). The first order autocorrelation of the alpha-filtered process $Z_t(\alpha) \equiv \mathcal{L}_\alpha(Z)_t$, where

$$\mathcal{L}_\alpha(\cdot) \equiv (1-\alpha)I + \alpha\,\mathcal{L}_\alpha(\mathcal{B})$$

with I and $\mathcal{B}$, respectively, the identity and the shift operators, is given in Section 4. In Section 5 we present the *instantaneous frequency estimate* based on ideas related to the *CM Method*. This estimate is given in two different ways (the one in Section 5.4 is based on stretches of data). The ergodicity of the stochastic process (1.1), analyzed in Section 6, ensures the strong consistency of the estimator, used in Section 5.1, via Birkhoff Ergodic Theorem.

The content of this paper is part of the Ph.D. dissertation of the first author under the guidance of the second at the University of Maryland.

2. Angle Modulation Processes

The general results of this section will be applied to the specific situation we want to analyze in Section 3.

Let $\{Y_t\}_{t\in T}$ be the angle modulated process

$$Y_t = A\,\cos(\omega_c t + X(t) + \phi), \quad \text{for } \ t \in T, \tag{2.1}$$

where $T = \mathbf{R}$ or $\mathbf{Z}$, A is a constant, $\omega_c \in [0,\pi]$ is the carrier frequency, X_t is the modulating process and the phase $\phi \sim \mathcal{U}((-\pi,\pi])$ is a uniform random variable on $(-\pi,\pi]$ independent of X_t.

The stochastic process $\{Y_t\}_{t\in T}$ is called phase modulation (PM) of the carrier frequency by the input process $\{X_t\}_{t\in T}$. If the input process is itself formed by integrating another random process, say $\{U_t\}_{t\in T}$, then the process $\{Y_t\}_{t\in T}$ is called frequency modulation (FM) of the carrier frequency by the process $\{U_t\}_{t\in T}$. Angle modulation processes are extremely important examples of complex exponential modulation (see Gray and Davisson (1986)). Notice that we can rewrite $\{Y_t\}_{t\in T}$ as the process

$$\begin{aligned} Y_t &= A\cos(\omega_c t + X(t) + \phi) = \frac{A}{2}\{e^{i\omega_c t}e^{i(X(t)+\phi)} + e^{-i\omega_c t}e^{-i(X(t)+\phi)}\} \\ &= \frac{A}{2}\{e^{i\omega_c t}V(t) + e^{-i\omega_c t}\overline{V(t)}\} \end{aligned}$$

where

$$V_t = \exp\{i(X_t + \phi)\}.$$

It is well known that, for all $t, s \in T$ and $t > s$,

- $E(V_t) = E[e^{i(X_t+\phi)}] = E[e^{iX_t}]E[e^{i\phi}] = 0 = E(\overline{V_t})$,

- $R_{VV}(t,s) = E[V_t\overline{V_s}] = E[e^{i(X_t-X_s)}] = \Phi_{X_t-X_s}(1)$, where $\Phi_{X_t - X_s}(1)$ is the characteristic function of the increment of the process X_t between the two sample times.

Our initial goal is to obtain the second order properties of the process $\{Y_t\}_{t\in T}$ in terms of the process $\{V_t\}_{t\in T}$.

Observe that the autocovariance function of the process $\{V_t\}_{t\in T}$ is not, in general, symmetric but it is Hermitian, that is, $R_{VV}(t,s) = \overline{R_{VV}(s,t)}$. Also, observe that for $t \neq s$

$$\bullet\ E[V_t V_s] = E[\exp\{i(X_t + X_s)\}]E[e^{i2\phi}] = 0 = E[\overline{V_t V_s}]. \tag{2.2}$$

Lemma 2.1: *Let the process $\{V_t\}_{t\in T}$ be weakly stationary (e.g., if $\{X_t\}_{t\in T}$ is strictly stationary). Then, for all t, $h \in T$,*

$$\bullet\ E(Y_t) = 0$$

$$\bullet\ R_{YY}(h) = \frac{A^2}{4}\{e^{i\omega_c h} R_{VV}(h) + e^{-i\omega_c h}\overline{R_{VV}(h)}\}. \tag{2.3}$$

Proof:

For any t $\in T$, observe that

$$E(Y_t) = \frac{A}{2}\{e^{i\omega_c t}E(V_t) + e^{-i\omega_c t}E(\overline{V_t})\} = 0.$$

From equalities in (2.2), for any t, h $\in T$,

$$\begin{aligned} R_{YY}(h) &= E[Y_{t+h}\overline{Y_t}] = \frac{A^2}{4}E[(e^{i\omega_c(t+h)}V_{t+h} + e^{-i\omega_c(t+h)}\overline{V_{t+h}}]) \\ &\times (e^{-i\omega_c t}\overline{V_t} + e^{i\omega_c t}V_t)] = \frac{A^2}{4}\{e^{i\omega_c h}R_{VV}(h) + e^{-i(2\omega_c t + \omega_c h)}E[\overline{V_{t+h}V_t}] \\ &+ e^{i(2\omega_c t+\omega_c h)}E[V_{t+h}V_t] + e^{-i\omega_c h}\overline{R_{VV}(h)}\} \\ &= \frac{A^2}{4}\{e^{i\omega_c h}R_{VV}(h) + e^{-i\omega_c h}\overline{R_{VV}(h)}\}. \end{aligned}$$

■

Denote the set of all frequencies by T'. Depending on T being the continuous or discrete time parameter set T' will be, respectively, the real line or the interval $(-\pi, \pi]$.

Lemma 2.2: *Let $\{V_t\}_{t\in T}$ be weakly stationary process with periodic spectral distribution function $F_V(\lambda)$ with period 2π. Then, the spectral measure of the process $\{Y_t\}_{t\in T}$ is given by*

$$dF_Y(\lambda) = \frac{A^2}{4}\{dF_V(\lambda-\omega_c) + dF_V(-\lambda-\omega_c)\}, \quad \textit{for all } \lambda \in T'. \tag{2.4}$$

Proof:

Since the process $\{V_t\}_{t\in T}$ is weakly stationary, we have

$$\begin{aligned}
\int_{T'} e^{i\lambda h}\, dF_Y(\lambda) = R_{YY}(h) &= \frac{A^2}{4}\{e^{i\omega_c h} R_{VV}(h) + e^{-i\omega_c h}\overline{R_{VV}(h)}\} \\
&= \frac{A^2}{4}\{e^{i\omega_c h}\int_{T'} e^{i\omega h}\, dF_V(\omega) + e^{-i\omega_c h}\int_{T'} e^{-i\omega h}\, \overline{dF_V(\omega)}\} \\
&= \frac{A^2}{4}\{\int_{T'} e^{i(\omega+\omega_c)h}\, dF_V(\omega) + \int_{T'} e^{i(-\omega-\omega_c)h}\, \overline{dF_V(\omega)}\} \\
&= \frac{A^2}{4}\{\int_{T'} e^{i\lambda h}\, dF_V(\lambda-\omega_c) + \int_{T'} e^{i\lambda h}\, \overline{dF_V(-\lambda-\omega_c)}\}.
\end{aligned}$$

The last equality comes from changing variables $\lambda = \omega + \omega_c$ for the first integral and $\lambda = -\omega - \omega_c$ for the second one.

Therefore, since

$$\int_{T'} e^{i\lambda h}\, dF_Y(\lambda) = \int_{T'} e^{i\lambda h}\frac{A^2}{4}\{dF_V(\lambda-\omega_c) + \overline{dF_V(-\lambda-\omega_c)}\},$$

the expression (2.4) follows from the uniqueness of the spectral representation of the autocovariance function of the process $\{Y_t\}_{t\in T}$.

■

We mention here that

$$\overline{R_{VV}(h)} = R_{VV}(-h)$$

since the process $\{V_t\}_{t\in T}$ is weakly stationary. Then, $\overline{dF_V(\lambda)} = dF_V(-\lambda)$.

The first term in expression (2.4) is the spectrum of the complex baseband wave V_t shifted by the carrier frequency $+\omega_c$; while the second term is the spectrum of $\overline{V_t}$ shifted by $-\omega_c$ (see Rowe and Prabhu (1975)).

We can see that in order to analyze the real process $\{Y_t\}_{t\in T}$ we only need to analyze the complex process $\{V_t\}_{t\in T}$. This will be done in the next section.

3. Sinusoidal Modulating Process

In this section we will be interested in analyzing the modulated process $\{Y_t\}_{t\in T}$ as in (2.1) when the modulating process X_t is given by

$$X_t = B\,\sin(\omega_0\,t+\varphi), \quad \text{for } t\in T, \tag{3.1}$$

where B is a constant, $\varphi \sim \mathcal{U}((-\pi,\pi])$ and $\omega_0 \in [0,\pi]$. Observe that the random variables φ and ϕ are independent of each other and also independent of the noise process when it is present. Therefore, we are interested in analyzing the following stochastic process

$$Y_t = A\,\cos[\omega_c\,t + B\,\sin(\omega_0\,t+\varphi)+\phi], \quad \text{for t} \in T. \tag{3.2}$$

Our main purpose here is to estimate consistently the real parameters A, B, ω_c and ω_0 based on sample values. This will be explained in Sections 5 and 6 but first we address some preliminaries.

Definition 3.1: The function

$$\omega(t) = \omega_c + B\,\omega_0\,\cos(\omega_0\,t+\varphi), \quad \text{for all } \ t\in T,$$

is called the *instantaneous frequency* of the FM model $\{Y_t\}_{t\in T}$.

The results in Section 2 will be applied here for the input process $\{X_t\}_{t\in T}$ given in (3.1). First we observe that $\{X_t\}_{t\in T}$ is a strictly stationary process.

The assumptions outlined in Section 2 are satisfied for the process (3.2). In fact, we know that the process in (3.1) is weakly stationary and, for the process $\{V_t\}_{t\in T}$ given by

$$V_t = \exp\{i[B\sin(\omega_0 t+\varphi)+\phi]\} \tag{3.3}$$

we have, for any t, h $\in T$,

- $E(V_t) = 0$,

- $E[V_{t+h}V_t] = E[\exp\{iB[\sin(\omega_0(t+h)+\varphi)+\sin(\omega_0 t+\varphi)]\}]E[\exp\{i2\phi\}] = 0$

$$= E[\overline{V_{t+h}V_t}].$$

In the next lemma (see also Subba-Rao and Yar (1982)) we give the autocovariance function at lag $h \in T$ of the process $\{V_t\}_{t\in T}$ and we observe that this is a real and symmetric function of h even though the process V_t itself is complex.

Lemma 3.1: *The autocovariance function at lag $h \in T$ of the process $\{V_t\}_{t\in T}$ is given by*

$$R_{VV}(h) = J_0(2B\sin[\frac{1}{2}\omega_0 h]) = \sum_{n=-\infty}^{\infty} J_n^2(B)\cos(n\omega_0 h) \tag{3.4}$$

where for complex z with $|arg\ z| < \pi$ and for $\nu \in \mathbb{Z}$

$$J_\nu(z) = (\frac{z}{2})^\nu \sum_{k\geq 0}(-1)^k \frac{(\frac{z}{2})^{2k}}{k!\Gamma(\nu+k+1)}$$

is the Bessel function of the first kind of order ν with $J_0(0) \equiv 1$.

Proof:

Observe that, for any $h \in T$,

$$\begin{aligned} R_{VV}(h) = E[V_{t+h}\overline{V_t}] &= E[\exp\{iB\sin[\omega_0(t+h)+\varphi]+i\phi\} \times \exp\{-iB\sin[\omega_0 t+\varphi]-i\phi\}] \\ &= E[\exp\{iB[\sin(\omega_0 t+\omega_0 h+\varphi)-\sin(\omega_0 t+\varphi))]\}]. \end{aligned}$$

In order to show the first equality in expression (3.4) one makes use of the fact that φ is a uniformly distributed random variable on $(-\pi,\pi]$ and also makes use of the formula for the difference of sine at two different arguments and the following formulas (see formulas 13 and 18 in Gradshteyn and Ryzhik page 402)

- $\displaystyle\int_{-\pi}^{\pi}\cos[z\cos(x)]\cos(nx)\,dx = 2\pi\cos(\frac{1}{2}n\pi)J_n(z)$
- $\displaystyle\int_{-\pi}^{\pi}\sin[z\cos(x)]\cos(nx)\,dx = 2\pi\sin(\frac{1}{2}n\pi)J_n(z) \qquad (3.5)$

where $J_n(z)$ is the Bessel function of the first kind of order $n \in \mathbb{Z}$ and z is any real number.

Using the above fact we have that, for any $h \in T$,

$$
\begin{aligned}
R_{VV}(h) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\{i2B\sin(\frac{1}{2}\omega_0 h)\cos[\frac{1}{2}(2\omega_0 t + \omega_0 h + 2x)]\}\, dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[2B\sin(\frac{1}{2}\omega_0 h)\cos(\omega_0 t + \frac{1}{2}\omega_0 h + x)]\, dx \\
&+ \frac{i}{2\pi}\int_{-\pi}^{\pi} \sin[2B\sin(\frac{1}{2}\omega_0 h)\cos(\omega_0 t + \frac{1}{2}\omega_0 h + x)]\, dx.
\end{aligned}
$$

The second integral in the last equality is zero where one makes use of the second formula in expression (3.5) for $n = 0$. For the first integral in this equality consider the change of variables

$$
z = 2B\sin(\frac{1}{2}\omega_0 h) \qquad \text{and} \qquad y = \omega_0 t + \frac{1}{2}\omega_0 h + x
$$

and use the first formula in expression (3.5), to obtain

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[2B\sin(\frac{1}{2}\omega_0 h)\cos(\omega_0 t + \frac{1}{2}\omega_0 h + x)]\, dx &= \frac{1}{2\pi}\int_{\omega_0 t + \frac{1}{2}\omega_0 h - \pi}^{\omega_0 t + \frac{1}{2}\omega_0 h + \pi} \cos[z\cos(y)]\, dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[z\cos(y)]\, dy = J_0(z).
\end{aligned}
$$

Therefore, for any $h \in T$,

$$
R_{VV}(h) = J_0(2B\sin[\frac{1}{2}\omega_0 h]).
$$

Now we want to show the second equality in expression (3.4). See Watson (1962) for the proof of the equality

$$
J_0(\sqrt{\zeta^2 + z^2 - 2\zeta z\cos(\psi)}) = \sum_{n\geq 0} a_n J_n(\zeta) J_n(z)\cos(n\psi) \tag{3.6}
$$

where $\zeta,\ z \in \mathbf{R}$, $\psi \in (-\pi, \pi]$ and

$$
a_n = \begin{cases} 1, & \text{if } n = 0 \\ 2, & \text{if } n \neq 0 \end{cases}.
$$

Using the equality (3.6) for $\zeta = z = B$, $\psi = \omega_0 h$ and the fact that for any $n \in \mathrm{N}$,

$$J_{-n}(z) = \begin{cases} J_n(z), & \text{if n is even} \\ -J_n(z), & \text{if n is odd} \end{cases}$$

we can derive that

$$J_0(2B\sin[\frac{1}{2}\omega_0 h]) = \sum_{n=-\infty}^{\infty} J_n^2(B)\cos(n\omega_0 h).$$

Therefore, the expression (3.4) holds.

■

From Lemma 3.1 one concludes that the process $\{V_t\}_{t\in T}$ is also weakly stationary.

The results in expression (2.3) and Lemma 3.1 give that, for any t, h $\in T$,

- $E(Y_t) = 0$,

- $$R_{YY}(h) = \frac{A^2}{4}\{e^{i\omega_c h}J_0(2B\sin[\frac{1}{2}\omega_0 h]) + e^{-i\omega_c h}J_0(2B\sin[\frac{1}{2}\omega_0 h])\}$$
$$= \frac{A^2}{2}J_0(2B\sin[\frac{1}{2}\omega_0 h])\cos(\omega_c h). \tag{3.7}$$

As expected, this autocovariance function is symmetric. Because $J_0(0) \equiv 1$, we mention that $R_{YY}(0) = \frac{A^2}{2}$. *From this we see that the average power associated with the frequency-modulated carrier is independent of the modulating signal and it is in fact the same as the average power of the unmodulated carrier. This result is true for any modulating signal whose highest frequency component is small compared to the carrier frequency.* See Schwartz (1959).

Since $R_{VV}(h) = R_{VV}(-h)$ we have a symmetric measure for the process $\{V_t\}_{t\in T}$. Then, from the expression (2.4) in Lemma 2.2, the spectral measure of the process $\{Y_t\}_{t\in T}$ is given by

$$dF_Y(\lambda) = \frac{A^2}{4}\{\,dF_V(\lambda - \omega_c) + dF_V(\lambda + \omega_c)\}, \quad \text{for any } \lambda \in T'. \tag{3.8}$$

Therefore, we have above a version of the expression (2.4) and due to the symmetry of the autocovariance function of the process $\{V_t\}_{t\in T}$, we have $dF_V(-\lambda) = dF_V(\lambda)$. Hence the spectrum of $\{Y_t\}_{t\in T}$ is the sum of shifted components of dF_V, where the shifts are determined by the carrier frequency ω_c.

We remark here that the results of the previous sections were proved for T discrete or continuous but they will be used, in the next sections, for the case when the time parameter set T is discrete and the set of all frequencies is $T' = (-\pi, \pi]$.

3.1 - Spectral Distribution Function of the FM Process

We are interested in providing an explicit expression for $dF_Y(\lambda)$. First notice that, from expression (3.7), $R_{YY}(h)$ does not go to 0 when $|h| \to \infty$. We will show below (see expression (3.9)) that F_Y contains jumps (a countable number of them) and this is another way to express the fact that $R_{YY}(h)$ does not go to 0 when $|h| \to \infty$. Therefore, the spectral measure of the process $\{Y_t\}_{t\in \mathbf{Z}}$ has atoms (see Subba-Rao and Yar (1982)). Since $\{Y_t\}_{t\in \mathbf{Z}}$ has mixed spectrum, we can not write $dF_Y(\lambda) = f_Y(\lambda)\, d\lambda$, where f_Y is the spectral density function of Y_t (we could have an equality of this type in a generalized sense for mixed spectrum cases where the generalized function f_Y is allowed to contain some Dirac delta functions).

Here one wants to obtain a representation of the spectral distribution function of the process $\{Y_t\}_{t\in \mathbf{Z}}$ whose support is in $(-\pi, \pi]$. Notice that we are working with discrete time parameter set T and all frequency values are in $(-\pi, \pi]$.

Lemma 3.2: *The spectral distribution function of the process $\{Y_t\}_{t\in \mathbf{Z}}$ is given by*

$$dF_Y(\lambda) = \frac{A^2}{4} \sum_{n=-\infty}^{\infty} J_n^2(B)\{\delta[\lambda - (\omega_c + n\omega_0)] + \delta[\lambda + (\omega_c + n\omega_0)]\}\, d\lambda. \tag{3.9}$$

where $\lambda \pm (\omega_c + n\omega_0)$ is considered modulus 2π, that is, $\lambda \pm (\omega_c + n\omega_0) \in (-\pi, \pi]$, for any $n \in \mathbf{Z}$.

Proof:

Using the second equality of the autocovariance function (in Lemma 3.1) of the process $\{V_t\}_{t\in \mathbf{Z}}$ associated to the process $\{Y_t\}_{t\in \mathbf{Z}}$ and by invoking once more the spectral representation of $R_{VV}(h)$, we have

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{i\lambda h}\, dF_V(\lambda) = R_{VV}(h) &= \sum_{n=-\infty}^{\infty} J_n^2(B)\cos(n\omega_0 h) \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n^2(B)(e^{in\omega_0 h} + e^{-in\omega_0 h}) \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n^2(B)\{\int_{-\pi}^{\pi} e^{i\lambda h}\delta(\lambda - n\omega_0)\, d\lambda + \int_{-\pi}^{\pi} e^{i\lambda h}\delta(\lambda + n\omega_0)\, d\lambda\} \\
&= \int_{-\pi}^{\pi} e^{i\lambda h}\frac{1}{2}\sum_{n=-\infty}^{\infty} J_n^2(B)[\delta(\lambda - n\omega_0) + \delta(\lambda + n\omega_0)]\, d\lambda,
\end{aligned}
$$

where $\delta(\lambda)$ is the Dirac delta function. Then, for any $\lambda \in (-\pi, \pi]$,

$$
dF_V(\lambda) = \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n^2(B)[\delta(\lambda - n\omega_0) + \delta(\lambda + n\omega_0)]\, d\lambda.
$$

From equality (3.8) we obtain the desired expression for the spectral distribution function of the process $\{Y_t\}_{t\in\mathbf{Z}}$.

■

We refer the reader to Lopes (1991) for a different representation of the process in (3.2). *We have shown there that the process $\{Y_t\}_{t\in T}$ is equivalent (with probability equal to 1) to the process*

$$
\widetilde{Y}_t = \sum_{n=-\infty}^{\infty} AJ_n(B)\, \cos[(\omega_c + n\,\omega_0)t + \psi_n], \tag{3.10}
$$

where $AJ_n(B)$ is a constant for each $n \in \mathbf{Z}$, the phases ψ_n are equal to $n\varphi + \phi$ with φ and ϕ independent random variables uniformly distributed on $(-\pi, \pi]$. Moreover, ψ_n are dependent random variables uniformly distributed on $(-\pi, \pi]$.

In order to prove this result we show in Lopes (1991) that $E|Y_t - \widetilde{Y}_t|^2 = 0$.

Notice that, when ω_0 is irrational, from (3.9) or (3.10) one can conclude that there exists a dense set of jumps in the discrete part of the spectral distribution function of the process $\{Y_t\}_{t\in\mathbf{Z}}$.

4. First Order Autocorrelation of the Alpha-Filtered Process

We recall that our main goal is to give an estimate of the instantaneous frequency $\omega(t)$ (see Definition 3.1) for the zero mean stationary process $\{Y_t\}_{t\in\mathbf{Z}}$ given by

$$Y_t = A\cos[\omega_c t + B\,\sin(\omega_0 t + \varphi) + \phi], \quad \text{for } t \in T,$$

where A and B are constants, $\omega_c, \omega_0 \in [0,\pi]$, and the phases φ and $\phi \sim \mathcal{U}((-\pi,\pi])$ are independent of each other and of the process $\{\varepsilon_t\}_{t\in T}$ where the noise process is Gaussian and white with mean zero and variance σ_ε^2.

In order to do that we need to compute the first-order autocorrelation function of the alpha-filtered process $\{Y_t(\alpha)\}_{t\in\mathbf{Z}}$. First one recalls some definitions.

Consider the stochastic process $\{Z_t\}_{t\in\mathbf{Z}}$ as in (1.1).

Definition 4.1: A *parametric family $\mathcal{L}_\theta$ of linear time invariant filters* is defined as the set of filters

$$\{\mathcal{L}_\theta(\cdot)\ ;\ \theta \in \Theta\},$$

where Θ is the parameter space, with impulse response function $\{h_n(\theta)\}_{n=-\infty}^{\infty}$ and transfer function $H(\lambda;\theta)$ obtained from the Fourier Transform of the $h_n(\theta)$, that is,

$$H(\lambda;\theta) = \sum_{n=-\infty}^{\infty} \exp(-in\lambda) h_n(\theta).$$

For this to happen we consider the following matching condition

$$\sum_{n=-\infty}^{\infty} |h_n(\theta)|^2 < \infty$$

and that

$$\int_{T'} |H(\lambda;\theta)|^2\, dF_Z(\lambda) < \infty,$$

where $T' = (-\pi,\pi]$ or $\mathbf{R}$ depending on the process being considered with discrete or continuous time parameter set T.

Let us denote $\{Z_t(\theta)\}_{t\in T}$ the filtered process defined by the convolution

$$Z_t(\theta) \equiv \mathcal{L}_\theta(Z)_t = \sum_{n=-\infty}^{\infty} h_n(\theta) Z_{t-n} = (h_\theta * Z)_t$$

where $*$ denotes convolution.

We shall consider a particular parametric family of linear filters.

Definition 4.2: The *alpha filter* applied to the process $\{Z_t\}_{t\in Z}$ is defined as the time invariant linear transformation

$$Z_t(\alpha) = (1-\alpha)Z_t + \alpha\, Z_{t-1}(\alpha), \quad \text{for} \quad t \in Z$$

where $-1 < \alpha < 1$, with impulse response function (see Kedem and Li (1990))

$$h(n;\alpha) = \begin{cases} (1-\alpha)\alpha^n, & \text{for n } \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

The corresponding squared gain function is given by

$$|H(\lambda;\alpha)|^2 = \frac{(1-\alpha)^2}{1-2\alpha\cos(\lambda)+\alpha^2}, \quad -1<\alpha<1 \quad \text{and} \quad -\pi<\lambda\leq\pi. \tag{4.1}$$

One can write the squared gain function above in a more convenient way as

$$\begin{aligned}
|H_\alpha(\lambda)|^2 &= \frac{(1-\alpha)^2}{1-2\alpha\cos(\lambda)+\alpha^2} = \frac{(1-\alpha)^2}{(e^{i\lambda}-\alpha)(e^{-i\lambda}-\alpha)} \\
&= (1-\alpha)^2\{\frac{\frac{i}{2\sin(\lambda)}}{e^{i\lambda}-\alpha} + \frac{\frac{-i}{2\sin(\lambda)}}{e^{-i\lambda}-\alpha}\} \\
&= (1-\alpha)^2\{\frac{i}{2\sin(\lambda)e^{i\lambda}[1-\alpha e^{-i\lambda}]} - \frac{i}{2\sin(\lambda)e^{-i\lambda}[1-\alpha e^{i\lambda}]}\} \\
&= (1-\alpha)^2\{\frac{ie^{-i\lambda}}{2\sin(\lambda)[1-\alpha e^{-i\lambda}]} - \frac{ie^{i\lambda}}{2\sin(\lambda)[1-\alpha e^{i\lambda}]}\} \\
&= (1-\alpha)^2\frac{i}{2\sin(\lambda)}\{e^{-i\lambda}\sum_{m\geq 0}(\alpha e^{-i\lambda})^m - e^{i\lambda}\sum_{m\geq 0}(\alpha e^{i\lambda})^m\} \\
&= (1-\alpha)^2\sum_{m\geq 0}\frac{i}{2\sin(\lambda)}\{e^{-i(m+1)\lambda} - e^{i(m+1)\lambda}\}\alpha^m \\
&= (1-\alpha)^2\sum_{m\geq 0}\frac{\sin[(m+1)\lambda]}{\sin(\lambda)}\alpha^m.
\end{aligned} \tag{4.2}$$

The expression (4.2) will be used later in Lemma 4.1.

From now on we do not consider the noise process and consider the zero mean stationary process $\{Y_t\}_{t\in Z}$ given by (3.2) with autocovariance $R_{YY}(h)$ given by (3.7) and generalized spectral density $f_Y(\lambda)$ (it contains Dirac delta functions corresponding to the points of jump in the spectral distribution function $dF_Y(\lambda)$ of Y) given by (3.9).

Definition 4.3: The *autocovariance at lag h* of the filtered process $\{Y_t(\alpha)\}_{t\in Z}$ is defined as

$$E[Y_t(\alpha)Y_{t+h}(\alpha)] = \int_{-\pi}^{\pi} e^{i\lambda h}|H_\alpha(\lambda)|^2\,dF_Y(\lambda) = \int_{-\pi}^{\pi} \cos(\lambda h)|H_\alpha(\lambda)|^2\,dF_Y(\lambda)$$

where $\{Y_t(\alpha)\}_{t\in Z}$ is a real process and $dF_{Y(\alpha)}(\lambda) = |H_\alpha(\lambda)|^2\,dF_Y(\lambda)$ is symmetric and $|H_\alpha(\lambda)|^2$ is given by the expression (4.2).

Definition 4.4: The *first-order autocorrelation function* of the filtered process $\{Y_t(\alpha)\}_{t\in Z}$ is defined as

$$\rho_1^Y(\alpha) = \frac{E[Y_t(\alpha)Y_{t+1}(\alpha)]}{E[Y_t^2(\alpha)]} = \frac{\int_{-\pi}^{\pi}\cos(\lambda)|H_\alpha(\lambda)|^2\,dF_Y(\lambda)}{\int_{-\pi}^{\pi}|H_\alpha(\lambda)|^2\,dF_Y(\lambda)}. \tag{4.3}$$

where $|H_\alpha(\lambda)|^2$ is given by (4.2).

In what follows we derive a convenient representation for $\rho_1^Y(\alpha)$.

Using the expression (3.9), the autocovariance at lag 1 and the variance of the filtered process $\{Y_t(\alpha)\}_{t\in Z}$ (respectively, h=1 and h=0 in Definition 4.3) are given by

- $E[Y_t(\alpha)Y_{t+1}(\alpha)] = \int_{-\pi}^{\pi}\cos(\lambda)|H_\alpha(\lambda)|^2\,dF_Y(\lambda)$

$$= \int_{-\pi}^{\pi}\cos(\lambda)|H_\alpha(\lambda)|^2\frac{A^2}{4}\sum_{n=-\infty}^{\infty}J_n^2(B)[\delta(\lambda-(\omega_c+n\omega_0))+\delta(\lambda+(\omega_c+n\omega_0))]$$

$$= \frac{A^2}{2}\sum_{n=-\infty}^{\infty}J_n^2(B)|H_\alpha(\omega_c+n\omega_0)|^2\cos(\omega_c+n\omega_0) \tag{4.4}$$

and

- $E[Y_t^2(\alpha)] = \int_{-\pi}^{\pi}|H_\alpha(\lambda)|^2\,dF_Y(\lambda)$

$$= \int_{-\pi}^{\pi}|H_\alpha(\lambda)|^2\frac{A^2}{4}\sum_{n=-\infty}^{\infty}J_n^2(B)[\delta(\lambda-(\omega_c+n\omega_0))+\delta(\lambda+(\omega_c+n\omega_0))]$$

$$= \frac{A^2}{2}\sum_{n=-\infty}^{\infty}J_n^2(B)|H_\alpha(\omega_c+n\omega_0)|^2. \tag{4.5}$$

The last equality in expressions (4.4) and (4.5) was obtained from the symmetry of the cosine function in $(-\pi, \pi]$ with the square gain function $|H_\alpha(\lambda)|^2$ given by (4.2).

Therefore, for any $\alpha \in (-1,1)$ and any $t \in Z$,

- $E[Y_t(\alpha)Y_{t+1}(\alpha)] = \dfrac{A^2}{2} \displaystyle\sum_{n=-\infty}^{\infty} J_n^2(B)|H_\alpha(\omega_c + n\omega_0)|^2 \cos(\omega_c + n\omega_0)$

- $E[Y_t^2(\alpha)] = \dfrac{A^2}{2} \displaystyle\sum_{n=-\infty}^{\infty} J_n^2(B)|H_\alpha(\omega_c + n\omega_0)|^2. \qquad (4.6)$

Our purpose in the end of this section is to write the above expressions as power series in α. This is done in the next lemma.

Lemma 4.1: *For any $\alpha \in (-1,1)$, the autocorrelation at lag 1 of the process $\{Y_t(\alpha)\}_{t\in T}$ is given by*

$$\rho_1^Y(\alpha) = \frac{E[Y_t(\alpha)Y_{t+1}(\alpha)]}{E[Y_t^2(\alpha)]}$$

$$= \frac{(1-\alpha)^2\frac{A^2}{2}\{\frac{1+\alpha^2}{1-\alpha^2}\sum_{j\geq 0} J_0(2B\sin[\frac{1}{2}(j+1)\omega_0])\cos[(j+1)\omega_c]\alpha^j + \frac{\alpha}{1-\alpha^2}\}}{(1-\alpha)^2\frac{A^2}{2}\{\frac{2}{1-\alpha^2}\sum_{j\geq 0} J_0(2B\sin[\frac{1}{2}j\omega_0])\cos(j\omega_c)\alpha^j - \frac{1}{1-\alpha^2}\}}. \qquad (4.7)$$

Proof:

The proof of this lemma involves extremely long computations and we refer the reader to Lopes (1991) for it.

■

We need the above expression for an error estimation at the end of Section 5 (see (5.12)).

5. Estimation Based on Sample Autocorrelation

In this section we consider the stochastic process $\{Z_t\}_{t\in Z}$ given by (1.1) and (1.2) with $\omega_0 << \omega_c$, that is,

$$Z_t = Y_t + \varepsilon_t = A\, \cos[\omega_c\, t + B\, \sin(\omega_0\, t + \varphi) + \phi] + \varepsilon_t, \quad \text{for } t \in T, \tag{5.1}$$

where A and B are constants, ω_c, $\omega_0 \in [0, \pi]$ with $\omega_0 << \omega_c$, and the phases φ and $\phi \sim \mathcal{U}((-\pi, \pi])$ are independent of each other and of the process $\{\varepsilon_t\}_{t\in T}$ where the noise process is Gaussian and white with mean zero and variance σ_ε^2. Later we will make some assumptions related to the modulation index B.

We want to estimate the parameters of the model (5.1) through the sample autocorrelation based on a time series $\{Z_t\}_{t=1}^N$ of N observations. The parameters are the amplitudes A and B, the carrier frequency ω_c, the modulating frequency ω_0 and the variance σ_ε^2 of the noise. The estimates of the above parameters are everything we need in order to estimate the instantaneous frequency of the signal Y_t.

The estimation of the parameters is based on the autocovariance function at lag $h \in T$ and variance of the time series $\{Z_t\}_{t=1}^N$ and, therefore, we need to know if the estimator is consistent. This is the reason for the considerations about ergodicity in Section 6.

We shall consider another parametric family of linear filters different from the *alpha filter* introduced previously. Denote $\theta(\alpha) = \cos^{-1}(\alpha)$.

Definition 5.1: The *complex filter* applied to the process $\{Z_t\}_{t\in \mathrm{Z}}$ is defined by the transformation

$$Z_t(\alpha, M) = (1 + e^{i\theta(\alpha)}\mathcal{B})^M Z_t, \quad \text{for } \ t \in \mathrm{Z}, \ \ -1 < \alpha < 1 \ \text{ and } \ -\pi < \theta(\alpha) < \pi,$$

where M is a positive integer and $\mathcal{B}$ is the *shift operator* $\mathcal{B}Z_t = Z_{t-1}$. We think of $\theta(\alpha)$ as the *"center of the filter"*.

Clearly,

$$Z_t(\alpha, M) = \sum_{n=0}^{M} \binom{M}{n} e^{i\theta(\alpha)n} Z_{t-n}, \quad \text{for } \ t \in \mathrm{Z}, \ -\pi < \theta(\alpha) < \pi \ \text{ and } \ M \in \mathrm{N} - \{0\} \tag{5.2}$$

and the impulse response function is

$$h(n; \alpha, M) = \begin{cases} \binom{M}{n} e^{i\theta(\alpha)n}, & \text{for } \ 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}.$$

The transfer function is

$$H(\lambda; \alpha, M) = (1 + e^{i(\theta(\alpha) - \lambda)})^M, \quad \text{for } \ -\pi < \lambda \le \pi$$

and the corresponding square gain function is

$$|H(\lambda;\theta(\alpha),M)|^2 = 4^M \cos^{2M}\left(\frac{\lambda-\theta(\alpha)}{2}\right), \quad \text{for } -\pi < \lambda, \theta \leq \pi \text{ and } -1 < \alpha < 1. \quad (5.3)$$

For properties of the *complex filter* see Lopes (1991) and Lopes and Kedem (1991).

Let $\{Z_t\}_{t=1}^{N+M}$ be a time series of length $N+M$ obtained from the process (5.1) and $Z_t(\alpha, M)$ the correspondent complex-filtered time series version. We consider M fixed (in fact, M=1 will be used in Section 5.1). For each variable $\alpha \in (-1,1)$, the first order autocorrelation function is given by

$$\rho_1^Z(\alpha) = \rho_1^Z(\alpha, M) = \frac{\mathcal{R}\{E[Z_t(\alpha,M)\overline{Z_{t+1}(\alpha,M)}]\}}{E|Z_t(\alpha,M)|^2}.$$

Let $\alpha_{k+1} = \rho_1(\alpha_k)$, for $k \in \mathrm{N}$, be the updating scheme applied to the process (5.1).

The *complex filter* is used in Lopes and Kedem (1991) for the updating procedure based on correlations for the case of finite number of frequencies but not for FM models.

5.1 - Carrier Frequency Estimation

In this section we want to obtain the estimate of the carrier frequency ω_c. It will be based on the *CM Method* using the *complex filter* (see Definition 5.1). Here we will consider the situation of a narrow band signal where the modulation index B is less than $\frac{\pi}{2}$. Figure 1 show the graph of J_n, for any $z \in \mathrm{R}$ and $n = 0, 1, \cdots, 4$. Notice that $J_0(z) > J_n(z)$, for all $n \in \mathrm{N} - \{0\}$, when $z \in (0, \frac{\pi}{2})$ (see Figure 1). So, among all possible frequencies $\omega_c + n\omega_0$, for $n \in \mathrm{N}$, of the process $\{Y_t\}_{t \in T}$, ω_c is the one with the largest amplitude (see expression (3.9)).

Here we will need to iterate the map $\rho_1(\cdot)$. If we apply the complex filter with $M=1$ to the time series $\{Z_t\}_{t=1}^{N}$, then the updated α in the iterative procedure will converge to ω_c. We explain now why this happens.

When there exists a finite number of frequencies the iterative procedure applied to the complex filter has a tendency to converge to the frequency closest to the initial value α_0. The reason is that the weight in the weighted average (see Kedem and Lopes (1991) and Lopes (1991)) is larger for the frequency closest to the initial value α_0.

The situation when we have an infinite number of frequencies (see (3.9)) is different. There exists a dense set of frequencies. Therefore, the closest one to the initial value α_0 does not exist. In this way, the iterative procedure has simply a tendency to converge to the frequency with largest amplitude (the weighted average makes the iterative procedure to converge to the frequency with the largest amplitude). So, we can estimate ω_c, since this is the frequency with largest amplitude $\frac{A^2 J_0(B)}{8}$ (see (3.9)) when the modulation index

B is less than $\frac{\pi}{2}$ (observe that $J_0(z) > J_n(z)$, for all $n \in \mathrm{N} - \{0\}$ when $z \in (0, \frac{\pi}{2})$ as one can see in Figure 1). Notice that increasing M it attenuates the difference. Therefore, it is better to take M small (say, less than 10). The value $M = 1$ is optimal.

Let $\{Z_t(\alpha, M)\}_{t \in \mathrm{Z}}$ be the complex-filtered version of the process (5.1) where $\alpha \in (-1, 1)$ and $M \in \mathrm{N} - \{0\}$ are the filter parameters. The first order autocorrelation function of the complex-filtered stochastic process $\{Z_t(\alpha, M)\}_{t \in \mathrm{Z}}$ is given by

$$\rho_1^Z(\alpha, M) = \frac{\mathcal{R}\{E[Z_t(\alpha, M)\overline{Z_{t+1}(\alpha, M)}]\}}{E|Z_t(\alpha, M)|^2}$$

$$= \frac{4^{M-1}A^2 \sum_{n=-\infty}^{\infty} J_n^2(B)\{\cos^{2M}(\frac{\cos^{-1}(\alpha)-\theta_n}{2}) + \cos^{2M}(\frac{\cos^{-1}(\alpha)+\theta_n}{2})\}\cos(\theta_n) + \alpha\,\rho_0^{\varepsilon}(\alpha, M)}{4^{M-1}A^2 \sum_{n=-\infty}^{\infty} J_n^2(B)\{\cos^{2M}(\frac{\cos^{-1}(\alpha)-\theta_n}{2}) + \cos^{2M}(\frac{\cos^{-1}(\alpha)+\theta_n}{2})\} + \rho_0^{\varepsilon}(\alpha, M)}, \tag{5.4}$$

where $\theta_n = \omega_c + n\omega_0$, for any $n \in \mathrm{Z}$.

Let $\{Z_t\}_{t=1}^{N+M}$ be a time series of length N+M obtained from the process (5.1). Let $Z_t(\alpha, M)$ be the correspondent complex-filtered time series version with $\alpha \in (-1, 1)$ and $M \in \mathrm{N} - \{0\}$ being the filter parameters. We consider a time series Gaussian white noise $\{\varepsilon_t\}_{t=1}^{N}$. In the simulations we take N=3000. As we said before we use this time series to estimate $\rho_1^Z(\alpha, M)$.

See Table 5.1 and Table 5.2 for the above updating procedure applied to the estimation of the carrier frequency. From Table 5.1 we observe that when M is equal to 1 (the first possible value for the parameter $M \in \mathrm{N} - \{0\}$) it does not matter what the size of the time series is. The procedure works fine. If B is small, we always converge to ω_c in the case when the signal-to-noise ratio is sufficiently large. These tables show that our method works for solving the problem of finding the carrier frequency ω_c.

We observe that when M is large, even though B is small and the signal-to-noise ratio is large, the iterations $\alpha_{k+1} = \rho_1^Z(\alpha_k, M)$ do not converge to ω_c.

5.2 - Modulating Frequency Estimation

Now we want to estimate the modulating frequency ω_0. First we apply a low-pass filter to the time series $\{Z_t\}_{t=1}^{N}$ in order to filter out the carrier frequency ω_c, supposing we have its estimated value. The spectral distribution function of the resulting time series (we will use the notation $\{\widetilde{\widetilde{Z_t}}\}_{t=1}^{N}$ for this time series) will have now at $\omega_c + \omega_0$ the highest amplitude and the value of this amplitude is given by $\frac{A^2 J_1^2(B)}{8}$ (see (3.9)) and note that now $J_1(B) > J_n(B)$, for $n \geq 2$ and $B \in (0, \frac{\pi}{2})$ as one can see in Figure 1). Then we apply the iterative procedure again to the resulting time series $\{\widetilde{\widetilde{Z_t}}\}_{t=1}^{N}$ using the complex filter (see Definition 5.1) with M=1. The convergence of this procedure for the time series $\{\widetilde{\widetilde{Z_t}}\}_{t=1}^{N}$ will give us the consistent estimate of $\omega_c + \omega_0$ by the same reasoning as in Section

5.1 when we wanted to find the frequency ω_c for the time series $\{Z_t\}_{t=1}^N$. Since we already know the estimate $\hat{\omega}_c$, we can obtain $\hat{\omega}_0$. This solves the problem of finding the modulating frequency ω_0.

5.3 - Estimation of the Remaining Parameters

In order to estimate the true parameters A, B and σ_ε, we need three pieces of information involving them. This means that we need three equations involving A, B and σ_ε. After we obtain these three equations, then with the help of a numerical method we will be able to get the estimates $\hat{A}$, $\hat{B}$ and $\hat{\sigma}_\varepsilon$. For this section we consider the alpha-filtered version of the stochastic process $\{Z_t(\alpha)\}_{t\in T}$ (see Section 4). Let $\{Z_t(\alpha)\}_{t=1}^N$ be an alpha-filtered time series with length N.

From the expression (4.8) and with Gaussian white noise, we have

$$\rho_1^Z(\alpha) = \frac{E[Z_t(\alpha)Z_{t+1}(\alpha)]}{E[Z_t^2(\alpha)]} = \frac{E[Y_t(\alpha)Y_{t+1}(\alpha)] + E[\varepsilon_t(\alpha)\varepsilon_{t+1}(\alpha)]}{E[Y_t^2(\alpha)] + E[\varepsilon_t^2(\alpha)]}$$

$$= \frac{\frac{A^2}{2}(\frac{1-\alpha}{1+\alpha})\{(1+\alpha^2)\sum_{j\geq 0} J_0(2B\sin[\frac{1}{2}(j+1)\omega_0])\cos[(j+1)\omega_c]\alpha^j + \alpha\} + \sigma_\varepsilon^2\frac{\alpha(1-\alpha)}{1+\alpha}}{\frac{A^2}{2}(\frac{1-\alpha}{1+\alpha})\{2\sum_{j\geq 0} J_0(2B\sin[\frac{1}{2}j\omega_0])\cos(j\omega_c)\alpha^j - 1\} + \sigma_\varepsilon^2\frac{1-\alpha}{1+\alpha}}. \tag{5.5}$$

Two informations can be obtained from the autocovariance and variance of the stochastic process $\{Z_t(\alpha)\}_{t\in Z}$ when $\alpha = 0$, that is, when no filter is applied. From the numerator and denominator of the expression (5.5) above we have

$$E[Z_t(0)Z_{t+1}(0)] = \frac{A^2}{2}J_0(2B\sin[\frac{1}{2}\omega_0])\cos(\omega_c) \tag{5.6}$$

and

$$E[Z_t^2(0)] = \frac{A^2}{2} + \sigma_\varepsilon^2. \tag{5.7}$$

Therefore, by taking samples of $E[Z_t(0)Z_{t+1}(0)]$ and $E[Z_t^2(0)]$, we can find the respective values $\hat{E}_1$ and $\hat{E}_2$. Finally, we have the following two equations

$$\frac{\hat{A}^2}{2} J_0(2\hat{B}\sin[\frac{1}{2}\hat{\omega}_0])\cos(\hat{\omega}_c) = \hat{E}_1 \tag{5.8}$$

and

$$\frac{\hat{A}^2}{2} + \hat{\sigma}_\varepsilon^2 = \hat{E}_2. \tag{5.9}$$

The third equation is obtained from the expression $E[Z_t(\alpha)Z_{t+1}(\alpha)]$, given as the numerator of equation in (5.5), with a fixed value α (say, $\alpha = \frac{1}{2}$). Suppose that in this problem one stipulates a tolerance value ϵ (for instance, $\epsilon = 10^{-4}$) for the error. Now we can choose k large enough such that

$$\frac{A^2}{2}\frac{(1-\frac{1}{2})}{(1+\frac{1}{2})}\{(1+\frac{1}{2}^2)\sum_{j\geq k} J_0(2B)(\frac{1}{2})^j\} < \epsilon.$$

Note that in order to find k above we need estimates for A^2 (from (5.9) $\hat{A}^2 < \hat{E}_2\, 2$) and for B (it is bounded by π; see Section 5.1). We can also choose a smaller α (for instance, $\alpha = 0.1$) to have a smaller k. Therefore, by taking care of the tail part, the following expression

$$A^2\{\frac{5}{4}\sum_{j=0}^{k} J_0(2B\sin[\frac{1}{2}(j+1)\omega_0])\cos[(j+1)\omega_c](\frac{1}{2})^j + \frac{1}{2}\} + \sigma_\varepsilon^2 \tag{5.10}$$

is a good approximation, up to an error ϵ, to

$$A^2\{\frac{5}{4}\sum_{j\geq 0} J_0(2B\sin[\frac{1}{2}(j+1)\omega_0])\cos[(j+1)\omega_c](\frac{1}{2})^j + \frac{1}{2}\} + \sigma_\varepsilon^2.$$

Now $E[Z_t(\frac{1}{2})Z_{t+1}(\frac{1}{2})]$ is approximately equal to the expression (5.10). Hence, by taking samples of $E[Z_t(\frac{1}{2})Z_{t+1}(\frac{1}{2})]$ we can find the value $\hat{E}_3$. The third equation based on expression (5.10) is given by

$$\hat{A}^2\{\frac{5}{4}\sum_{j=0}^{k} J_0(2\hat{B}\sin[\frac{1}{2}(j+1)\hat{\omega}_0])\cos[(j+1)\hat{\omega}_c](\frac{1}{2})^j + \frac{1}{2}\} + \hat{\sigma}_\varepsilon^2 = \hat{E}_3. \tag{5.11}$$

Finally, the system of three equations and three unknowns is given by the expressions (5.8), (5.9) and (5.11). Now we solve this nonlinear system by a numerical method and we obtain the estimates $\hat{A}$, $\hat{B}$ and $\hat{\sigma_\varepsilon^2}$. Recall that we already know the estimates $\hat{\omega}_c$ and $\hat{\omega}_0$ from Sections 5.1 and 5.2, respectively. Note that if B is known *a priori* the estimation will be simpler since B is the most difficult parameter to estimate. In any case we are able to obtain the five parameters A, B, ω_c, ω_0 and σ_ε^2. Therefore, we can estimate the instantaneous frequency (see Definition 3.1) by

$$\hat{\omega}_c + \hat{B}\,\hat{\omega}_0\,\sin(\hat{\omega}_0\,t).$$

5.4 - Instantaneous Frequency Detection Based on Stretches of Data

In this section we want to estimate, by another method, the instantaneous frequency (see Definition 3.1) for the sinusoidal frequency modulated model contaminated by an independent additive Gaussian white noise component (see expression (5.1)). The main goal is to estimate the instantaneous frequency by *the Contraction Mapping (CM) Method* (see Kedem (1990)) based on sample autocorrelation and demodulate the baseband signal $\{Y_t\}_{t\in T}$. The novelty here is to apply *CM Method* stretch by stretch (see Yakowitz (1990) and Kedem and Yakowitz (1990)). The analysis is based on a single time series $\{Z_t\}_{t=1}^N$ of N observations. This time series is divided into several nonoverlapping stretches of data, each stretch containing the same number of observations. The *CM Method* is applied to each stretch using the *alpha filter*.

We need the following assumptions.

Assumptions :

(1). *The model is considered in the discrete time where* $t \in T = \mathbf{Z}$.
(2). *The random variables* ϕ *and* φ *are uniformly distributed on* $(-\pi, \pi]$ *independent of each other and of the process* $\{\varepsilon_t\}_{t\in T}$.
(3). *The modulating signal varies slowly compared to the carrier* ($\omega_c >> \omega_0$).

The reason for *assumption (3)* will be explained later on in the conclusion, in Section 5.5.

Now we will outline the *CM Method* applied to the time series $\{Z_t\}_{t=1}^N$. Our main goal is to estimate the instantaneous frequency $\omega(t)$ (see expression (1.3)). In a simulated model we consider A $= \sqrt{2}$, B $= 238$, $\omega_c = 0.942$, $\omega_0 = 0.00126$, $\phi = \varphi = 0$ and $N = 10{,}000$ observations. We recall, from assumption (3), that $\omega_0 << \omega_c$ and, since we want two complete oscillations, we consider $\omega_0\,10{,}000 = 4\pi$. We divide the time series $\{Z_t\}_{t=1}^N$ into $N_1 = 20$ nonoverlapping stretches with $N_2 = 500$ observations each and we consider Gaussian white noise with $\sigma_\varepsilon^2 = 1.0$.

Remark 1: We will denote the left hand side of each stretch by $T_i = 500\,i$, for $0 \le i \le 20$.

In each stretch $[T_i, T_{i+1}]$, for $0 \le i \le 20$, with initial condition $\alpha_0 = 0.5$, we calculate the first order autocorrelation of the alpha-filtered time series and we obtain

$$\rho_1(\alpha_0) = \frac{E[Z_t(\alpha_0)Z_{t+1}(\alpha_0)]}{E[Z_t^2(\alpha_0)]}.$$

Remark 2: In practice, we obtain $\hat{\rho}_1(\alpha_0)$ based on a time series of $N_2 = 500$ observations.

With $\alpha_1 = \rho_1(\alpha_0)$ as the updated filter parameter we estimate again the first order autocorrelation of the alpha-filtered time series and we obtain

$$\rho_1(\alpha_1) = \frac{E[Z_t(\alpha_1)Z_{t+1}(\alpha_1)]}{E[Z_t^2(\alpha_1)]}.$$

In an analogous way we obtain α_2 from α_1 with the same updating procedure. In each stretch we obtain, by induction, the sequence

$$\alpha_{k+1} = \rho_1(\alpha_k), \text{ for } k = 0, 1, \cdots, 15.$$

For our purposes to iterate sixteen (16) times is good enough.

We take α_{16} as the best estimate for the instantaneous frequency in that stretch. Notice that the final value α_{16} can change from stretch to stretch (the instantaneous frequency is not a constant function of t). The polygonal line in Figure 2 represents the plot of the best estimate $\hat{\alpha}_{16}$ in each stretch. The graph of the instantaneous frequency

$$\omega(t) = 0.942 + 238\,(0.00126)\,\cos(0.00126\,t)$$

and its estimated by the method presented here are shown in Figure 2.

Another example is given in Figure 3 where we estimate the instantaneous frequency

$$\omega(t) = 0.5 + 500\,(0.00094)\,\cos(0.00094\,t)$$

by the method presented here. We consider a simulated model with $A = \sqrt{2}$, $B = 500$, $\omega_c = 0.5$, $\omega_0 = 0.00094$, $\phi = \varphi = 0$ and $N = 20,000$ observations. We divide the time series $\{Z_t\}_{t=1}^{N}$ into $N_1 = 40$ nonoverlapping stretches with $N_2 = 500$ observations and we consider Gaussian white noise with $\sigma_\varepsilon = 1.0$.

Now we state a useful result to justify our above procedure.

Theorem 5.1: *Consider $\{Z_t\}_{t\in T}$ a real-valued zero-mean stationary process given by*

$$Z_t = Y_t + \varepsilon_t$$

where the signal $\{Y_t\}_{t\in T}$ has spectral measure as a sum of an infinite number of Dirac delta *functions at frequencies ω_j with amplitudes C_j, for $j \in \mathrm{N} - \{0\}$, and it is uncorrelated with the Gaussian white noise component $\{\varepsilon_t\}_{t\in T}$ with $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$. Let $\{Z_t(\alpha)\}_{t\in T}$ be the alpha-filtered process. Let the first-order autocorrelation of Z_t be*

$$\rho_1^Z = \frac{\int_{-\pi}^{\pi} e^{i\lambda} F_Z(d\lambda)}{\int_{-\pi}^{\pi} F_Z(d\lambda)}$$

where $T'=(-\pi, \pi]$ is the frequency domain and $F_Z(d\lambda)$ is the spectral measure associated with $\{Z_t\}_{t\in T}$.
Suppose also that the support of the discrete spectral measure $F_Y(d\lambda)$ is in the interval $[\omega_a, \omega_b] \subseteq [0, \pi]$.
Then, for any $\alpha \in (-1, 1)$, given $\delta > 0$ there exists $N \in \mathrm{N}$ such that for $n \geq N$

$$\rho_1^n(\alpha) \in [\cos(\omega_b) - \delta, \cos(\omega_a) + \delta]$$

where $\rho_1(\alpha)$ is the first-order autocorrelation function of the alpha-filtered process $\{Z_t(\alpha)\}_{t\in T}$.

The theorem says that any $\alpha \in (-1, 1)$ will be attracted by iterations of the mapping ρ_1 to the support of the discrete part of the spectral distribution function of the process $\{Z_t\}_{t\in T}$.

We point out that the claim of Theorem 5.1 is true for a general filter as long as the first-order autocorrelation function of the process $\{Z_t(\theta)\}_{t\in T}$, for any $\theta \in \Theta$, is a convex combination of θ and the cosine of the frequencies (as in (5.12) below).

For the case of finite number of frequencies we refer to Theorem 1 in Kedem and Li (1989).

The proof of the above theorem will be given after some lemmas. The first lemma requires that the process $\{Z_t\}_{t\in T}$ shall satisfy all assumptions outlined in Theorem 5.1.

Note that the atoms of the discrete spectral distribution function $dF_Y(\lambda)$ are denoted by ω_j, for $j \in \mathrm{N} - \{0\}$. Therefore, all ω_j satisfy

$$\omega_a < \omega_j < \omega_b.$$

Lemma 5.1: *Let $\{Z_t\}_{t\in T}$ be a stochastic process as in Theorem 5.1. Let $\{Z_t(\alpha)\}_{t\in T}$ be the alpha-filtered process. Then, the first-order autocorrelation function of the process $\{Z_t(\alpha)\}_{t\in T}$ is given by*

$$\begin{aligned}\rho_1(\alpha) &= \frac{\sum_{j\geq 1} C_j |H_\alpha(\omega_j)|^2 \cos(\omega_j) + \sigma_\varepsilon^2 \alpha \frac{1-\alpha}{1+\alpha}}{\sum_{m\geq 1} C_m |H_\alpha(\omega_m)|^2 + \sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}} \\ &= \sum_{j\geq 1} A_j(\alpha) \cos(\omega_j) + A_0(\alpha)\alpha \qquad (5.12)\end{aligned}$$

where

$$A_j(\alpha) = \frac{C_j |H_\alpha(\omega_j)|^2}{\sum_{m\geq 1} C_m |H_\alpha(\omega_m)|^2 + \sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}} \geq 0, \quad \textit{for all } j \geq 1,$$

and

$$A_0(\alpha) = \frac{\sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}}{\sum_{m\geq 1} C_m |H_\alpha(\omega_m)|^2 + \sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}} \geq 0$$

with $|H_\alpha(\lambda)|^2$ *given by expression (4.2) and the frequencies* ω_j*, for* $j \in \mathrm{N} - \{0\}$*, are the atoms of the spectral distribution function of the process* $\{Z_t\}_{t\in T}$*.*

Proof:

From the assumptions of this lemma (also of the Theorem 5.1) we know that the spectral measure of the process Z_t is given by

$$F_Z(d\lambda) = F_Y(d\lambda) + F_\varepsilon(d\lambda) = \sum_{j\geq 1} C_j \delta(\lambda - \omega_j) + \frac{\sigma_\varepsilon^2}{2\pi}$$

where $C_j \geq 0$ are constants and $\omega_j \in [0, \pi]$. Consider $T' = (-\pi, \pi]$ the frequency domain. Then, from the definition of the alpha filter, we have

$$E[Z_t(\alpha) Z_{t+1}(\alpha)] = \int_{-\pi}^{\pi} \cos(\lambda) |H_\alpha(\lambda)|^2 F_Y(d\lambda) + \int_{-\pi}^{\pi} \cos(\lambda) |H_\alpha(\lambda)|^2 F_\varepsilon(d\lambda)$$

$$= \sum_{j\geq 1} C_j |H_\alpha(\omega_j)|^2 \cos(\omega_j) + \sigma_\varepsilon^2 \, \alpha \frac{1-\alpha}{1+\alpha}.$$

Therefore, we have

$$\rho_1(\alpha) = \frac{\sum_{j\geq 1} C_j |H_\alpha(\omega_j)|^2 \cos(\omega_j) + \sigma_\varepsilon^2 \alpha \frac{1-\alpha}{1+\alpha}}{\sum_{m\geq 1} C_m |H_\alpha(\omega_m)|^2 + \sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}}$$

$$= \sum_{j\geq 1} A_j(\alpha) \cos(\omega_j) + A_0(\alpha)\alpha$$

where

$$A_j(\alpha) = \frac{C_j |H_\alpha(\omega_j)|^2}{\sum_{m\geq 1} C_m |H_\alpha(\omega_m)|^2 + \sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}} \geq 0, \quad \text{for all } j \geq 1,$$

and

$$A_0(\alpha) = \frac{\sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}}{\sum_{m\geq 1} C_m |H_\alpha(\omega_m)|^2 + \sigma_\varepsilon^2 \frac{1-\alpha}{1+\alpha}} \geq 0.$$

■

Remark 3: Note that for any $\alpha \in (-1,1)$, $\sum_{j\geq 0} A_j(\alpha) = 1$ where $A_j(\alpha)$, for $j \geq 0$, is defined as above. Therefore, the expression (5.12) shows that, for any $\alpha \in (-1,1)$, $\rho_1(\alpha)$ is a weighted average of $\cos(\omega_j)$, $j \geq 1$, and α. This property will be essential for our reasoning in what follows.

Lemma 5.2: *Let Ω' be the interval $[\cos(\omega_b), \cos(\omega_a)] = [c,d]$. Then, $\rho_1(\Omega') \subset \Omega'$ and, in particular, $\rho_1(\cos(\omega_a)) \in \Omega'$ and $\rho_1(\cos(\omega_b)) \in \Omega'$.*

Proof:

Suppose $\alpha_0 \geq c = \cos(\omega_b)$. Since $\cos(\omega_b) \leq \cos(\omega_j)$ for all $j \in \mathrm{N} - \{0\}$ such that $\omega_j \in \Omega = [\omega_a, \omega_b]$, then

$$\rho_1(\alpha_0) = \sum_{j\geq 1} A_j \cos(\omega_j) + A_0\alpha_0 \geq (\sum_{j\geq 1} A_j + A_0)\cos(\omega_b) = \cos(\omega_b) = c.$$

Therefore, if $\alpha_0 \geq c$ then $\alpha_1 = \rho_1(\alpha_0) \geq c$.

Similarly, if $\alpha_0 \leq d = \cos(\omega_a)$ then $\alpha_1 = \rho_1(\alpha_0) \leq d$. Therefore, $\rho_1(\Omega') \subset \Omega'$. In particular,

$$\cos(\omega_b) \leq \sum_{j\geq 1} A_j \cos(\omega_j) + A_0 \cos(\omega_b) = \rho_1(\cos(\omega_b)) \leq (\sum_{j\geq 1} A_j + A_0)\cos(\omega_a) = \cos(\omega_a),$$

i.e., $\rho_1(\cos(\omega_b)) \in \Omega'$. Similarly, $\rho_1(\cos(\omega_a)) \in \Omega'$.

■

Remark 4: Notice that, in the situation of this Lemma 5.2 (that is, when $\alpha_0 \in [c,d]$), we can not say that $\alpha_1 > \alpha_0$ or $\alpha_1 < \alpha_0$.

Lemma 5.3: *There exists a constant $K \in (0,1)$ such that for all $\alpha_0 \in (-1,1)$*

- *if* $\alpha_0 < c = \cos(\omega_b)$ *then* $\dfrac{\rho_1(\alpha_0) - \cos(\omega_b)}{\alpha_0 - \cos(\omega_b)} < K < 1.$
- *if* $\alpha_0 > d = \cos(\omega_a)$ *then* $\dfrac{\rho_1(\alpha_0) - \cos(\omega_a)}{\alpha_0 - \cos(\omega_a)} < K < 1.$ (5.13)

Proof:

The proof will be given only for the first part. For the second part, it follows in a similar way.

Suppose $\alpha_0 < c = \cos(\omega_b)$. Since $\omega_j \leq \omega_b$, for all $j \geq 1$, by monotonicity of the function $\cos(x)$ on $[0, \pi]$, we have

$$\begin{aligned}\frac{\rho_1(\alpha_0) - \cos(\omega_b)}{\alpha_0 - \cos(\omega_b)} &= \frac{\sum_{j\geq 1} A_j \cos(\omega_j) + A_0\alpha_0 - \cos(\omega_b)}{\alpha_0 - \cos(\omega_b)} \\ &= \frac{\sum_{j\geq 1} A_j[\cos(\omega_j) - \cos(\omega_b)]}{\alpha_0 - \cos(\omega_b)} + A_0 \leq A_0.\end{aligned}$$

Notice that the first term in the last sum is nonnegative. Therefore, there exists a constant $K = A_0 < 1$ such that $\alpha_1 - \cos(\omega_b) < K(\alpha_0 - \cos(\omega_b))$.

■

Remark 5: Notice that in Lemma 5.3, when $\alpha_0 \notin [c, d]$ and $\alpha_0 < c$, from expression (5.11), we have $\rho_1(\alpha_0) = \alpha_1 > \alpha_0$ (that is, the sequence is monotone). When $\alpha_0 \notin [c, d]$ and $\alpha_0 > d$ then $\rho_1(\alpha_0) = \alpha_1 < \alpha_0$. This is different from the situation in Remark 4.

Lemma 5.4: *ρ_1 is a map onto $(-1, 1)$.*

Proof:

From Lemma 5.1, $\rho_1(\alpha)$ is a weighted average of $\cos(\omega_j)$, for $j \geq 1$, and α. Let α be in $(-1, 1)$. Since

$$-1 < \sum_{j\geq 1} A_j \cos(\omega_j) + A_0\alpha = \rho_1(\alpha) < (\sum_{j\geq 1} A_j + A_0)(1) = 1,$$

$\rho_1(-1) = -1$ and $\rho_1(1) = 1$ we conclude that ρ_1 is a map onto $(-1, 1)$.

■

Now we will give the proof of Theorem 5.1.

Proof of Theorem 5.1:

Let α be in $(-1, 1)$ and let δ be greater than 0. Suppose $\alpha < c = \cos(\omega_b)$. If $\rho_1(\alpha) \in \Omega' = [c, d]$, for all $n \in \mathbf{Z}$, then from Lemma 5.1

$$\rho_1^n(\alpha) \in \Omega' \subset \Omega'_\delta$$

and the result follows. If there exists $m \in \mathbf{N}$ such that $\rho_1^m(\alpha) \in \Omega'$ then

$$\rho_1^n(\alpha) \in \Omega' \subset \Omega'_\delta, \quad \text{for all } n \geq m.$$

Again the result follows for $N = m$.

Now suppose $\rho_1^m(\alpha) \notin \Omega'$ for all $m \in \mathrm{N}$. From Lemma 5.2, the sequence $\{\rho_1^m(\alpha)\}_{m\geq 1}$ is monotone and bounded. Notice that its limit, say α^*, is equal to

$$\alpha^* = \sup \rho_1^m(\alpha).$$

If $\alpha^* = c$ then, from Lemma 5.1,

$$\rho_1(\alpha^*) \in \quad Int(\Omega) = (c, d)$$

and, therefore, there exists $l \in \mathrm{N}$ such that $\rho_1^l(\alpha) \in \mathrm{Int}(\ \Omega)$. This is a contradiction. If $\alpha^* < c$ then, from Lemma 5.2 (see Remark 4),

$$\rho_1(\alpha^*) > \alpha^*$$

and, therefore, there exists $l \in \mathrm{N}$ such that $\rho_1^l(\alpha) > \alpha^*$. This is again a contradiction. Therefore, there exists $N \in \mathrm{N}$, such that for all $n \geq N$, $\rho_1^n(\alpha) \in \Omega'_\delta$.

The proof is similar when $\alpha > d = \cos(\omega_a)$.

■

Now we shall apply the above theorem in each stretch $[T_i, T_{i+1}]$, for $0 \leq i \leq 20$. Let $i_0 \in \{0, \ldots, 20\}$ be fixed and consider the stretch $[T_{i_0}, T_{i_0+1}]$ in order to fix the ideas. The justification of why *CM Method* works well for the FM model is the following: recall that in each stretch we consider $\hat{\rho}_1^{16}(\alpha_0)$ with $\alpha_0 = 0.5$ (in fact α_0 could be any point in (-1,1)) as a good approximation for the instantaneous frequency.

Notice that, if $\omega_c >> \omega_0$ then the instantaneous frequency has a small interval range. For the fixed stretch $[T_{i_0}, T_{i_0+1}]$ we apply the *CM Method* and, heuristically speaking, the time series does not know what happens in the other stretches. Therefore, if the stretch $[T_{i_0}, T_{i_0+1}]$ is relatively small then in this interval the instantaneous frequency is like a constant function equal to $\cos(\omega_1)$ and we are facing the case where the model looks very much like the mixed spectrum model with one frequency ω_1 and $\cos(\omega_1)$ is in the interval

$$\Omega' = [\omega(T_{i_0}), \omega(T_{i_0+1})].$$

In fact, a more correct model would be the one whose discrete part of the spectral measure is an infinite sum of Dirac delta functions on frequencies (see (3.10)) that are in the interval $\Omega' = [\omega(T_{i_0}), \omega(T_{i_0+1})]$ which has very small length. These frequencies should be of the form $\omega_c + n\,\omega_0$, $n \in \mathrm{Z}$, as we had before in expression (3.9), but contained in Ω'.

Now using Theorem 5.1 for $\Omega' = [\omega(T_{i_0}), \omega(T_{i_0+1})]$ we have that for any $\alpha \in (-1, 1)$, there exists $N \in \mathrm{N}$ such that $\rho_1^n(\alpha) \in \Omega'$ for all $n \geq N$. In this way, we can estimate a value in the range of the instantaneous frequency in the stretch $[T_{i_0}, T_{i_0+1}]$. This justifies the very good performance of *CM Method* (see also Li and Kedem (1991)). See Figures 2 and 3 where the instantaneous frequency is very well estimated in both examples.

5.5 - Conclusion

From the above we conclude that, for any initial condition $\alpha_0 \in (-1, 1)$, the iterated sequence $\alpha_{k+1} = \rho_1(\alpha_k)$, for some large k, will reach the interval of possible frequencies in the stretch and will stay there.

Notice that if $\omega_c >> \omega_0$ then the instantaneous frequency has a small interval range and we are able to obtain a good approximated value for this interval with the iterative procedure. In the case where $\omega_c < \omega_0$, the instantaneous frequency will oscillate very much and the method will not give any useful information. This explain the reason for *assumption (3)*.

If the stretch interval $[T_{i_0}, T_{i_0+1}]$ is small then the interval Ω' will also be small. In this case we have a situation where the model in this stretch looks very much like the mixed model with one frequency ω_1 and $\cos(\omega_1) \in \Omega'$ or several ones in the interval $[\cos(\omega_a, \cos(\omega_b)]$. By Theorem 5.1, using the *alpha filter*, we will have that large iterates of *any* initial condition α_0 will hit, in a finite number of steps, the region $\Omega' = [\cos(\omega_a), \cos(\omega_b)]$.

After we obtain the graphic by the method of stretches (see Figure 2, for instance) we can estimate the parameters ω_c, B and ω_0 in the following way:

(1). *ω_0 is determined by the distance between peaks of the graph in the x-axis.*
(2). *B is determined as half of the distance between the highest and the lowest points in the y-axis.*
(3). *ω_c is the mean value of the highest and lowest points in the y-axis.*

6. Ergodicity of the Stochastic Process

In this section, we want to analyze the ergodic properties that are necessary for the justification of taking the empirical autocovariance and variance as estimates for the autocovariance and variance of the process $\{Y_t\}_{t\in\mathbb{Z}}$ in the frequency modulated (FM) model given by

$$Y_t = A\cos[\omega_c t + B\sin(\omega_0 t + \varphi) + \phi], \quad \text{for all } t \in \mathbb{Z}. \tag{6.1}$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of Borel sets and $\mathbf{P}$ is a probability function on Ω. Consider $\mathcal{T}$ a transformation defined from Ω to itself, so that $\mathcal{T}$ is measurable and also measurably invertible.

Definition 6.1: We say that $\mathbf{P}$ is an *invariant measure for* $\mathcal{T}$ or $\mathcal{T}$ *is measure-preserving* if $\mathbf{P}(\mathcal{T}^{-1}(A)) = \mathbf{P}(A)$, for any Borel set $A \in \mathcal{F}$.

Definition 6.2: We say that $\mathbf{P}$ is *ergodic for* $\mathcal{T}$, if for any Borel set A such that $\mathcal{T}^{-1}(A) = A$, we have that $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$.

A very important result is the Birkhoff Ergodic Theorem (see Skorokhod (1989)). We next state this theorem.

Birkhoff Ergodic Theorem: *Suppose V is an integrable random variable on Ω, $\mathbf{P}$ is a probability invariant measure on Ω and $\mathcal{T}$ is a measurable transformation on Ω. Let $\mathcal{G}$ be the smallest σ-algebra of sets in $\mathcal{F}$ with respect to which all random variables W with $W(\mathcal{T}^t(\omega)) = W(\omega)$ for $\mathbf{P}$-almost all ω and for $t > 0$ are measurable. Then,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} V(\mathcal{T}^t(\omega)) = E(V/\mathcal{G})(\omega) \qquad \mathbf{P}\text{ - } a.s..$$

When $\mathbf{P}$ is ergodic (that is, $\mathcal{G}$ is trivial) then $E(V/\mathcal{G})$ reduces to $E(V) =$ constant and the above result essentially says that for the typical trajectory with respect to $\mathbf{P}$, time average of V converge to spatial average of V.

In terms of stochastic processes, we are considering in the above setting the stationary process $X_t(\omega) = V(\mathcal{T}^t(\omega))$, $\omega \in \Omega$ and $t \in \mathbf{Z}$. This is the standard way to transfer results from transformations with invariant measures to stationary processes (we refer to Lamperti (1977), chapter 5 for further details). Basically, one has to consider on the space $\Omega^{\mathbf{N}}$, the product measure generated by $\mathbf{P}$ on Ω and the above defined stochastic process X_t. We remark here that $\mathbf{P}$ will be a product measure in the case of independent and identical distributed coordinates.

Remark 1: Suppose that $\int V(\omega)\mathbf{P}(d\omega) = 0$. Then, in this case, if the probability is ergodic, the autocovariance at lag k

$$\int V(\omega)V(\mathcal{T}^k(\omega))\,\mathbf{P}(d\omega)$$

can be obtained as the almost-sure limit of the mean

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} V(\mathcal{T}^t(\omega))V(\mathcal{T}^{t+k}(\omega)), \quad \text{for } k \geq 0.$$

In this way, we can say that the sample autocovariance (the case k=1) and variance (the case k=0) are consistent estimators.

In our case we will need to consider $\Omega = (-\pi, \pi] \times (-\pi, \pi]$ and the mapping

$$\mathcal{T} : (-\pi, \pi] \times (-\pi, \pi] \to (-\pi, \pi] \times (-\pi, \pi]$$

given by $\mathcal{T}(\phi, \varphi) = (\omega_c + \phi, \omega_0 + \varphi)$.

We consider the probability measure $\mathbf{P}$ as the product measure obtained from the Lebesgue measure on $(-\pi, \pi]^2$. Without loss of generality, we will suppose that ω_c and ω_0 are irrational and rationally independent. In this case, the probability $\mathbf{P}$ is *ergodic with respect to the above defined map $\mathcal{T}$*.

Now we consider the random variable $V(\phi, \varphi)$ given by

$$V(\phi, \varphi) = A\cos(\phi + B\ \sin(\varphi)).$$

We can now applied the Ergodic Theorem to the variable V. Note that

$$V(\mathcal{T}^t(\phi, \varphi)) = A\cos(\omega_c t\ + B\ \sin(\omega_0\, t + \varphi) + \phi).$$

Therefore, by the Ergodic Theorem, we can use the samples

$$\frac{1}{N}\sum_{t=0}^{N-1} V(\mathcal{T}^t(\phi, \varphi))V(\mathcal{T}^{t+1}(\phi, \varphi))$$

and

$$\frac{1}{N}\sum_{t=0}^{N-1}[V(\mathcal{T}^t(\phi, \varphi))]^2$$

as consistent estimators for the autocovariance

$$\int V(\phi, \varphi)V(\mathcal{T}(\phi, \varphi))\,\mathbf{P}(d(\phi, \varphi))$$

and the variance

$$\int [V(\phi, \varphi)]^2\,\mathbf{P}(d(\phi, \varphi)).$$

If we introduce an additive Gaussian white noise $\{\varepsilon_t\}_{t\in\mathrm{Z}}$ to the process (6.1), then from the Ergodic Theorem, the fact that the noise has mean zero and variance σ_ε^2, and the fact that the random variable V is uniformily bounded, we conclude that the empirical autocovariance and variance are consistent estimators for the autocovariance and variance of the process $Z_t = Y_t + \varepsilon_t$.

This concludes the considerations about consistency for the estimates of the variance and autocovariance of the FM model.

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Figure 1: Bessel function $J_n(z)$, $z \in \mathbf{R}$, for $n = 0, 1, \cdots, 4$.

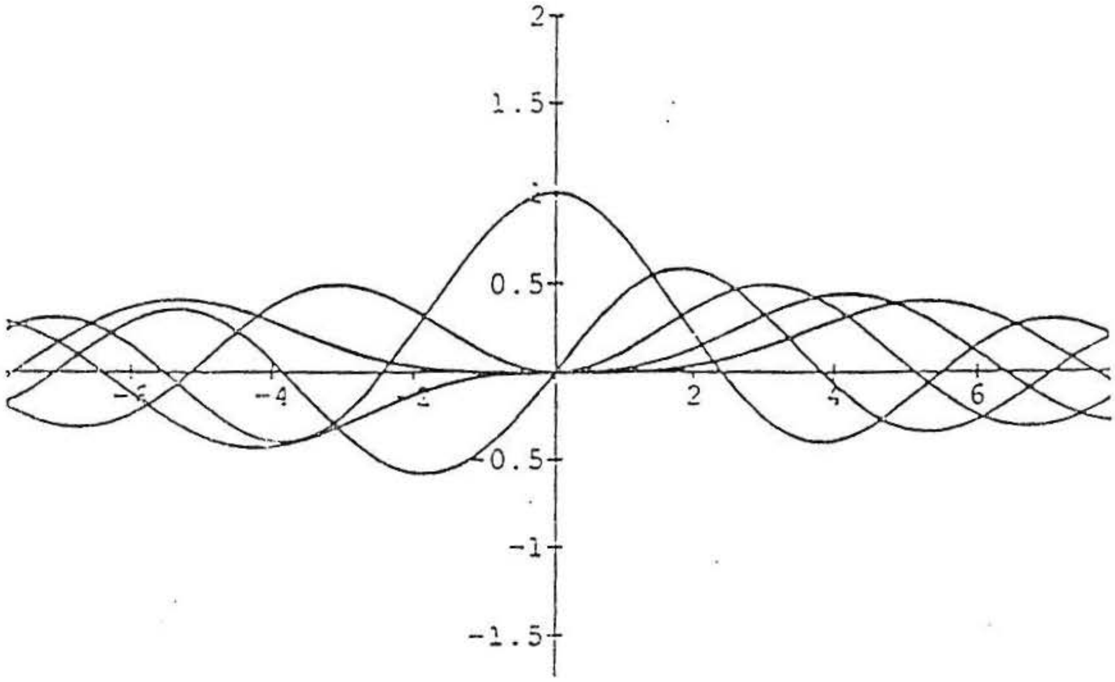


Table 5.1: Carrier Frequency Estimation. $M = 1$, $N = 3,000$, $\theta_0 \in (-\pi, \pi]$ is the algorithm initial value and $SNR = 20\log_{10}\left(\frac{\text{std. signal}}{\text{std. noise}}\right) dB$. Number of iterations = 5.

Convergence	A	B	σ_ϵ	$SNR(dB)$	ω_c	ω_0	θ_0	$\hat{\omega}_c$
yes	5.0	0.30	0.1	30.97	0.8	0.07	0.5	0.80084
yes	10.0	0.01	0.3	27.45	0.942	0.00126	0.8	0.94350
yes	10.0	0.01	0.3	27.45	0.942	0.00126	0.3	0.94350
yes	6.0	0.01	0.3	23.01	0.942	0.00126	0.5	0.94580
yes	6.0	0.10	0.3	23.01	2.5	0.8	0.5	2.49089
yes	6.0	0.01	0.3	23.01	2.0	0.5	0.5	1.99794
yes	4.0	0.01	0.3	19.49	2.0	0.5	0.5	1.99528
yes	3.0	0.30	0.4	14.49	1.5	0.3	0.5	1.50362
yes	2.0	0.20	0.4	10.97	1.6	0.4	0.5	1.59875
no	2.0	0.01	1.0	3.01	2.5	0.8	0.5	2.14349
no	2.0	1.00	1.0	3.01	2.5	0.8	0.5	2.05060
no	$\sqrt{2}$	0.30	1.0	0.0	2.5	0.8	0.5	1.98686
no	$\sqrt{2}$	0.30	1.0	0.0	2.5	0.8	2.0	1.98686
no	1.0	0.30	1.0	-3.01	2.5	0.8	2.0	1.84851

Table 5.2: Carrier Frequency Estimation. $M = 5$, $N = 3,000$, $\theta_0 \in (-\pi, \pi]$ is the algorithm initial value and $SNR = 20\log_{10}\left(\frac{\text{std. signal}}{\text{std. noise}}\right) dB$.

A	B	σ_ϵ	$SNR(dB)$	ω_c	ω_0	θ_0	$\hat{\omega}_c$
5	0.30	0.1	30.97	0.8	0.07	0.5	0.80135
6	0.10	0.3	23.01	2.5	0.80	0.5	2.48944
4	0.01	0.3	19.49	2.0	0.50	0.5	1.99483
2	0.20	0.4	10.97	1.6	0.40	1.0	1.58843
2	1.00	1.0	3.01	2.5	0.80	0.5	2.04778
$\sqrt{2}$	0.30	1.0	0.0	2.5	0.80	2.0	1.98619
1	0.30	1.0	-3.01	2.5	0.80	2.0	1.84938

Figure 2: The Instantaneous Frequency $\omega(t) = 0.942 + 238(0.001256)\cos(0.001256\,t)$ and its estimated by the method based on stretches of data.
$N = 10,000$, $N_1 = 20$ nonoverlapping stretches with $N_2 = 500$ observations. Gaussian white noise with $\sigma_\varepsilon = 1.0$. $A = \sqrt{2}$ and $\phi = \varphi = 0$.

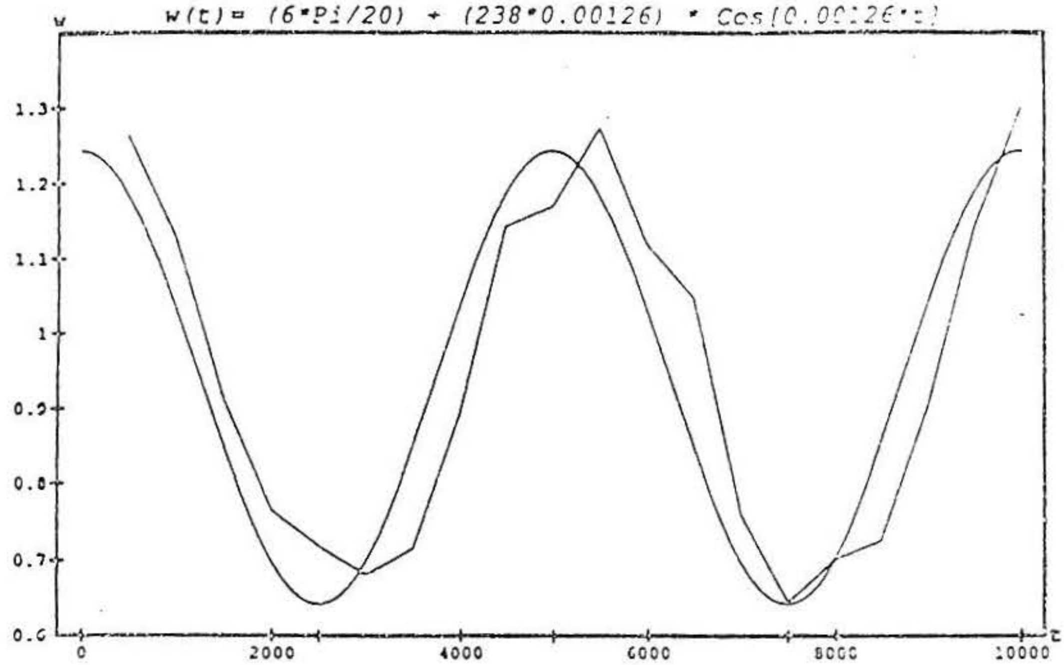
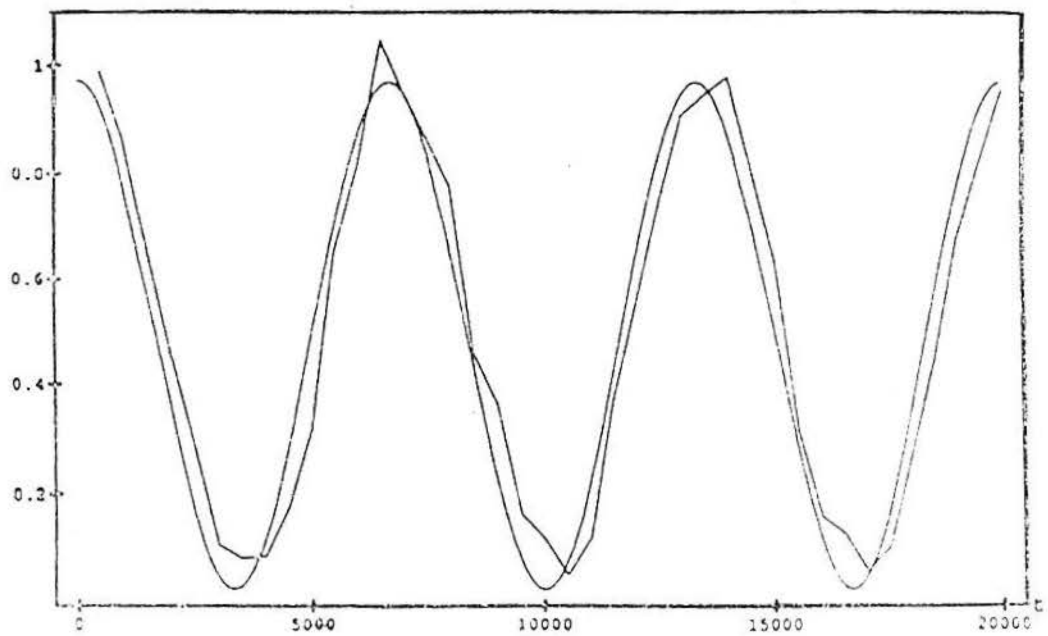


Figure 3: The Instantaneous Frequency $\omega(t) = 0.5 + 500(0.00094)\cos(0.00094\,t)$ and its estimated by the method based on stretches of data.
$N = 20,000$, $N_1 = 40$ nonoverlapping stretches with $N_2 = 500$ observations. Gaussian white noise with $\sigma_\varepsilon = 1.0$. $A = \sqrt{2}$ and $\phi = \varphi = 0$.



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