ϵ - DILATIONS

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- Trabalho de Pesquisa -

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1. Introduction. The concept of dilation was introduced and investigated by several important mathematicians. Given probability measures P, Q on the σ-field of Borel subsets of a topological space S, we say that Q is a dilation of P relative to a set K of functons $S \to \mathbb{R}$, and write $P \underset{K}{\prec} Q$, iff $\int f\,dP \leq \int f\,dQ$ for all $f \in K$ (the integrability is assumed). The set of functions K is usually a cone. It is possible that, although Q does not dilate P relatively to K, it nearly does so in some sense, giving rise to what we call an ε-dilation of P. A natural approach is to employ a "distance" of type

$$\delta(P,Q) := \inf\{\varepsilon \geq 0 \mid \int f\,dP \leq \int f\,dQ + \varepsilon L(f),\ f \in K\},$$

where $L(f) \geq 0$ measures the "size" of f. For exemple, suppose (S, d) is a separable metric space and L(f) the Lipschitz constant of f, $L(f) := \inf\{c \in \mathbb{R} \mid |f(s) - f(t)| \leq c\,d(s,t)\}$. Let further $K := \{f \mid L(f) < \infty\}$. Then, provided all the functions in K are P-integrable and Q-integrable, Dudley (1976) proved that $\delta(P,Q)$ is equal to the Wasserštein metric

$$W(P,Q) := \inf \int d(s,t)\,d\mu,$$

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where the infimum is taken over all $\mu \in \mathcal{M}(S^2)$ having marginals P, Q (see Notations below). This result follows from Theorem 9 of Kemperman (1982) too and is often called the Kantarovich-Rubinštein Theorem (1958) because these authors established the special case where S is compact.

We allow any cone of bounded functions which is <u>admissible</u>, i. e., a convex cone of continuous functions containing the constants and beeing invariant under the operation $\vee$. The latter means that $\max(f, g) \in K$ as soon as $f, g \in K$. Initially L(f) will be taken as the oscilation of f. Afterwards, other ϵ-dilations will also be discussed. Theorem 12 is our main result.

2. <u>Notations</u>. In this paper A^c denotes the complement of the set A; $\mathcal{B} = \mathcal{B}(S)$ the σ-field of Borel subsets of a topological space S; C(S) the set of all continuous functions $S \to \mathbb{R}$; $C_b(S)$ and $C_{bb}(S)$ the set of all functions in C(S) which are bounded and bounded from below, respectively; distribution function is abbreviated as d. f.; K' is the set of all $f \in K$ (K is a cone of functions) with $\inf f = 0$ and $\sup f = 1$; $\mathcal{M}(S)$ the set of all probability measures on the σ-field of Borel subsets of S; osc f stands for oscilation of the function f; δ_s represents the Dirac measure at the point s; and, finally, the symbols $\vee$, $\wedge$ have the usual meaning, i. e., they denote the maximum and the minumum operation, respectively, and l. s. c. abbreviates lower semicontinuous.

3. Lemma. If X is a compact topological space and (f_n) is a sequence in $C(X)$ with $f_n \uparrow f \in C(X)$ pointwise, then this convergence is uniform, in particular, $\lim_n \min f_n = \min f$.

Proof. Apply Dini's Theorem to the sequence $(f - f_n)$. ||

The next lemma is essential for the fundamental Theorem . It was suggested by Lemma 4 in [2], to which it reduces when $\varepsilon = 0$.

4. Lemma. Let S be a completely regular Hausdorff topological space and $K \subset C_{bb}(S)$ an admissible cone. Let $P, Q \in \mathcal{M}(S)$ be such that $\int f\,dP \leq \int f\,dQ + \varepsilon \operatorname{osc} f$ for all $f \in K$. Let us fix bounded functions α, β, $\phi_i : S \to \mathbb{R}$, where α and β are Borel measurable and $\phi_i \geq 0$, $i = 1, \ldots, n$. Further let us fix $f_i \in K$, $i = 1, \ldots, n$. Then

(1) $$\inf_{s,t \in S} \left[\alpha(s) + \beta(t) + \sum_{i=1}^{n} (f_i(s) - f_i(t))\phi_i(s)\right] \geq 0$$

implies

(2) $$\int \alpha\,dP + \int \beta\,dQ + \varepsilon \operatorname{osc} \beta \geq 0.$$

Proof. The proof is patterned after that of Lemma 4 in [2]. As in that lemma, the crucial step consists of defining an auxiliary function $\hat{\beta}:\mathbb{R}^n \to \overline{\mathbb{R}}:= [-\infty; +\infty]$ having convenient properties. The Euclidean space $\mathbb{R}^n$ will be equipped with the usual coordinatewise partial ordering. Throughout the rest of the proof we will use the notations $f:= (f_1,\ldots,f_n)$ and $\bar{f}:= (\bar{f}_1,\ldots,\bar{f}_n)$ where $\bar{f}_i$ denotes the Stone extension of f_i (see, for instance, [4], p. 86), $i = 1,\ldots,n$. Also βS will denote the Stone-Čech compactification of S.

From (1) we obtain the inequality

$$(3) \qquad \alpha(s) + \underline{\beta}(t) + \sum_{i=1}^{n} [f_i(s) - \bar{f}_i(t)]\phi_i(s) \geq 0,$$

valid for all $(s,t) \in S\times\beta S$. Here $\underline{\beta} : \beta S \to \mathbb{R}$ is the l. s. c. regularization of β. It is given by $\underline{\lim}_{\substack{s\to t\\ s\in S}} \beta(s):= \underline{\beta}(t)$.

Let $x \in \mathbb{R}^n$ and consider the sequences

$$(4) \qquad (p_1, p_2, \ldots) \in [0;1]^{\infty} \text{ with } p_1 + p_2 + \ldots = 1,$$

$$(5) \qquad (t_1, t_2, \ldots) \in (\beta S)^{\infty} \text{ with } x \leq \sum_j p_j\bar{f}(t_j).$$

Set

$$T_x:= \{\sum_j p_j \underline{\beta}(t_j) \mid (4) \text{ and } (5) \text{ hold}\}$$

and define

$$\hat{\beta}(x):= \inf T_x.$$

It is easy to see that $\hat{\beta}(x)$ is finite on and only on the set $U := \{x \in \mathbb{R}^n \mid x \leq y$ for some $y \in \operatorname{conv} \bar{f}(\beta S)\}$. Here the notation $\operatorname{conv} \bar{f}(\beta S)$ indicates the convex hull of $\bar{f}(\beta S)$. The properties of $\hat{\beta}$ that we are interested in are: (i) $-\alpha \leq \hat{\beta} \circ \bar{f} \leq \beta$ on S, (ii) $\hat{\beta}$ is increasing, (iii) $\hat{\beta}$ is convex, and (iv) $\hat{\beta}$ is l. s. c. . The last one is the more important and it is the Lemma 5 in [2].

Let us prove the property (i). Taking $(p_1, p_2, \ldots) := (1, 0, \ldots)$ and $(t_1, t_2, \ldots) := (t, t, \ldots) \in (\beta S)^{\infty}$, we see that $\underline{\beta}(t) \in T_{\bar{f}(t)}$, hence $\hat{\beta}(\bar{f}(t)) \leq \underline{\beta}(t)$, that is,

$$(6) \qquad \hat{\beta} \circ \bar{f} \leq \underline{\beta} \leq \beta \quad \text{on } S.$$

For the first inequality in (i), fix $s \in S$, set $x := f(s)$ and take sequences (p_j), (t_j) verifying (4) and (5), repectively. In particular

$$(7) \qquad f(s) \leq \sum_j p_j \bar{f}(t_j).$$

Let us apply (3) with $t := t_j$; afterwards, we multiply by p_j and sum over j obtaining

$$\alpha(s) + \sum_j p_j \underline{\beta}(t_j) + \sum_{i=1}^{n} [f_i(s) - \sum_j p_j \bar{f}_i(t_j)] \phi_i(s) \geq 0,$$

which gives, using (7), $\alpha(s) + \sum_j p_j \underline{\beta}(t_j) \geq 0$. This together with the definition of $\hat{\beta}$ yield $\alpha(s) + \hat{\beta} \circ f(s) \geq 0$ so that, by (6),

(8) $\quad -\alpha \leq \hat{\beta}\circ f \leq \beta$ on S.

That $\hat{\beta}$ is increasing is immediate: if x, y $\in \mathbb{R}^n$ with $x \leq y$, then $T_y \subset T_x$. Therefore $\hat{\beta}(x) = \inf T_x \leq \inf T_y = \hat{\beta}(y)$.

The convexity is just as easy: let p, q $\in [0;1]$ with $p + q = 1$, x, y $\in \mathbb{R}^n$ and

$$\sum_j p_j \underline{\beta}(t_j) \in T_x, \quad \sum_j q_j \underline{\beta}(t_j) \in T_y.$$

Therefore it is readily seen that

$$\left[\sum_j p\, p_j \underline{\beta}(t_j) + \sum_j q\, q_j \underline{\beta}(t_j)\right] \in T_{px+qy},$$

hence

$$\hat{\beta}(px + qy) \leq p \sum_j p_j \underline{\beta}(t_j) + q \sum_j q_j \underline{\beta}(t_j),$$

which produces

$$\hat{\beta}(px + qy) \leq p \inf T_x + q \inf T_y = p\hat{\beta}(x) + q\hat{\beta}(y),$$

so $\hat{\beta}$ is convex indeed.

It is known that a convex l. s. c. function like $\hat{\beta}$ restricted to U, which is a convex set with non-empty interior, is the limit of an increasing sequence $(h_{(\nu)})$ of functions

$$h_{(\nu)} := h_1 \vee \dots \vee h_\nu ,$$

where, for $i = 1, \dots, \nu$, h_i is the restriction to U of an affine function on $\mathbb{R}^n$ given by

$$h_i(x) := \langle A_i , x \rangle + a_i , \quad A_i \in \mathbb{R}^n, \quad a_i \in \mathbb{R}.$$

Here $\langle \cdot , \cdot \rangle$ is the usual inner product. Since $\hat{\beta}$ is increasing, we can suppose that all the h_i's are increasing, equivalently, that $A_i \geq 0$. As K contains the constants, the linear combinations $h_i \circ f \in K$, thus also $h_{(\nu)} \circ f \in K$ for all $\nu \in \mathbb{N}$, because K is invariant under the operation $\vee$, so that

$$\int h_{(\nu)} \circ f \, dP \leq \int h_{(\nu)} \circ f \, dQ + \epsilon \operatorname{osc}(h_{(\nu)} \circ f) \text{ for all } n \in \mathbb{N}.$$

Therefore by the Monotone Convergence Theorem (each $h_{(\nu)}$ being bounded and thus integrable relatively to both P and Q)

$$\int \hat{\beta} \circ f \, dP \leq \int \hat{\beta} \circ f \, dQ + \epsilon \lim \operatorname{osc}(h_{(\nu)} \circ f).$$

It is obvious that $\sup h_{(\nu)} \circ f \leq \sup \hat{\beta} \circ f$. Further $\lim h_{(\nu)} \circ f =$

$\inf \beta \circ f$ by Lemma 3. Thus $\overline{\lim}\, \mathrm{osc}\,(h_{(\nu)} \circ f) \leq \mathrm{osc}\,(\beta \circ f) \leq \mathrm{osc}\, \beta$. Putting all together, one arrives at the inequality

$$(9) \qquad \int \hat{\beta} \circ f \, dP \leq \int \hat{\beta} \circ f \, dQ + \varepsilon \, \mathrm{osc}\, \beta .$$

Finally, using (8) and (9), we conclude that

$$\begin{aligned} \int \alpha dP + \int \beta dQ &\geq \int \alpha dP + \int \hat{\beta} \circ f dQ \\ &\geq \int (\alpha + \hat{\beta} \circ f)\, dP - \varepsilon \, \mathrm{osc}\, \beta \\ &\geq - \varepsilon \, \mathrm{osc}\, \beta . \quad \| \end{aligned}$$

Let $P, Q \in \mathcal{M}(S)$. We will describe the property $\int f dP \leq \int f dQ + \varepsilon \, \mathrm{osc}\, f$, for all f in a subset L of $C_b(S)$, also by saying that Q is an <u>ε-dilation</u> of P relative to L.

The following theorem supplies an equivalent definition of ε-dilation relative to an admissible cone $K \subset C(S)$ for the case that S is compact. It says that a necessary and sufficient condition for Q to be an ε-dilation of P relative to K is that one can find a probability measures $\lambda \in \mathcal{M}(S^2)$ that satisfies

$$(10) \qquad \int (f(s) - f(t))\, \phi(s)\, \lambda(ds, dt) \leq 0 \text{ for all } f \in K, \ \phi \in C^+(S)$$

and whose first marginal is P and second marginal is "ε-close" to Q.

(c) $\|P - Q\| \leq 2\varepsilon$.

Proof. We will show that (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (a).

(a) $\rightarrow$ (b): Since the indicator function 1_A of an open set $A \subset S$ is l. s. c., it is the pointwise limit of an increasing sequence of non-negative functions in $C_b(S)$. So (a) implies through the Monotone Convergence Theorem that $P(A) \le Q(A) + \varepsilon$ for all open sets $A \subset S$. Now (b) follows by regularity of P.

(b) $\rightarrow$ (c): Let $\mu := (P + Q)/2$ and consider $f := \frac{dP}{d\mu}$, $g := \frac{dQ}{d\mu}$, the Rodon-Nikodyn derivatives. We have, using (b),

$$\|P - Q\| = \int |f - g|\, d\mu \le 2\varepsilon.$$

(c) $\rightarrow$ (a): Let μ, f and g be as in the proof of (b) $\rightarrow$ (c), $\alpha \in C_b(S)$ and $c := (\sup \alpha + \inf \alpha)/2$. Therefore $2\|\alpha + c\| = \operatorname{osc}\alpha$ and

$$\int \alpha dP - \int \alpha dQ = \int (\alpha + c)(f - g)\, d\mu \le \|\alpha + c\| \int |f - g| d\mu$$

$$= \|\alpha + c\| \cdot \|P - Q\| \le \varepsilon \operatorname{osc} \alpha. \quad ||$$

7. Definitions. In view of Theorem 5 and Lemma 6 it becomes natural to study the five quantities $\varepsilon_i(P,Q)$, $i = 1,\ldots,5$, denined as follows.

Let S be a standard space, $K \subset C_b(S)$ an admissible cone and P, $Q \in \mathcal{M}(S)$. By "$\prec$" we will mean "$\prec_K$". Let us define

$E_1 := \{ \epsilon \geq 0 \mid \int f dP \leq \int f dQ + \epsilon \operatorname{osc} f \text{ for all } f \in K \}$,

$E_2 := \{ \epsilon \geq 0 \mid$ there exists $Q' \in \mathcal{M}(S)$ with $P \prec Q'$ and $\|Q' - Q\| \leq 2\epsilon \}$,

$E_3 := \{ \epsilon \geq 0 \mid$ there exists $P' \in \mathcal{M}(S)$ with $P' \prec Q$ and $\|P' - P\| \leq 2\epsilon \}$,

$E_4 := \{ \epsilon \geq 0 \mid$ there exist $P', Q' \in \mathcal{M}(S)$ with $P' \prec Q'$, $\|P' - P\| \leq 2\epsilon$ and $\|Q' - Q\| \leq 2\epsilon \}$,

$E_5 := \{ \epsilon \geq 0 \mid$ there exist $P', Q' \in \mathcal{M}(S)$ with $P' \prec Q'$ and $\|P' - P\| + \|Q' - Q\| = 2\epsilon \}$.

Now we define

(13) $\epsilon_i(P,Q) := \inf E_i$, $i = 1, \ldots, 5$.

It is trivial that $E_2 \subset E_1$ and that $(E_2 \cup E_3) \subset E_5 \subset E_4$. Therefore $\epsilon_1 \leq \epsilon_2$ and $\min\{\epsilon_2, \epsilon_3\} \geq \epsilon_5 \geq \epsilon_4$.

8. <u>Theorem</u>. Suppose that S is compact. Then $\epsilon_4(P,Q) \leq \epsilon_1(P,Q) = \epsilon_2(P,Q) \leq \epsilon_3(P,Q)$.

<u>Proof</u>. It suffices to show that $E_1 \subset E_2$ and $E_3 \subset E_2$. The first

inclusion follows at once from Theorem 5 taking Q' as the second marginal of the measure λ in that theorem. For the other inclusion, let $\varepsilon \in E_3$. This means that there exists $P' \in \mathcal{M}(S)$, such that,

(14) $\int f dP' \leq \int f dQ$, for all $f \in K$

and

(15) $\|P' - P\| \leq 2\varepsilon$.

By Lemma 6 the inequality (15) can be expressed in the form

(16) $\int \alpha dP \leq \int \alpha dP' + \varepsilon \operatorname{osc} \alpha$, for all $\alpha \in C(S)$.

The relations (14) and (15) give $\int f dP \leq \int f dQ + \varepsilon \operatorname{osc} f$ for all $f \in K$. Thus $\varepsilon \in E_1$. ||

9. Remarks. (i) Later on it will be seen that $\varepsilon_5 = \varepsilon_1$ and that the inequalities in Theorem 8 are frequently strict.

(ii) If $P \prec Q$, then $\varepsilon_i(P,Q) = 0$, $i = 1, \ldots, 5$.

(iii) We always have $0 \leq \varepsilon_i(P,Q) \leq 1$, $i = 1,\ldots,5$.

(iv) Obviously

(17) $$\varepsilon_1(P,Q) = \sup_{\substack{\operatorname{osc} f \leq 1 \\ f \in K}} \left[\int f dP - \int f dQ\right].$$

(v) Theorem 5 is false for non-compact standard spaces. For such spaces the condition $\int f dP \leq \int f dQ + \varepsilon \operatorname{osc} f$ for all $f \in K$ is (obviously) necessary but no longer sufficient for (10), (11) and (12). To see that the named condition fails to be sufficient, consider $S := [0;1)$, take $P := \delta_x$ and $Q := \delta_y$ with $0 < y < x < 1$ and let K consist of all increasing convex functions on S. One can show that $\varepsilon_1(P,Q) = (x-y)/(1-y)$ and that there is no $Q' \in \mathcal{M}(S)$ dilating P with $\|Q' - Q\| \leq 2\varepsilon$. This contradicts Theorem 8, specifically, it contradicts the inclusion $E_1 \subset E_2$ thus Theorem 5. ||

10. Example. Let $S := [a;b] \subset \mathbb{R}$, K the cone of convex increasing continuous functions $S \to \mathbb{R}$ and $P, Q \in \mathcal{M}(S)$. We want to compute $\varepsilon_1(P,Q)$. For that goal we need to take into account only the functions in K of the form $s \longmapsto (s-c)^+ := (s-c) \vee 0$, where c is a constant, because those functions (together with the constants) span a cone dense in K. Here we implicitly also use that $\operatorname{osc}(f+g) = \operatorname{osc} f + \operatorname{osc} g$ when f, g are increasing. Hence, by (17),

$$(1 \quad \varepsilon_1(P,Q) = \sup_{a \leq c \leq b} \frac{1}{b-c} \left[\int (s-c)^+ dP - \int (s-c)^+ dQ \right].$$

As a special illustration take $[a;b] = [0;1]$, P(ds) the Lebesgue measure, and let Q be the discrete proba-

bility measure defined by $Q(\{1/2^n\}) := 1/2^n$, $n = 1, 2, \ldots$.
Then (1 8) becomes

$$\varepsilon_1(P,Q) = \sup_{0\le c\le \frac{1}{2}} \frac{1}{1-c} \left[\int_c^1 (s-c)\,ds - \int_{[c;\frac{1}{2}]} (s-c)\,Q(ds)\right]$$

$$= \sup_{0\le c\le \frac{1}{2}} g(c),$$

where $g(c) := \frac{1}{1-c}[\frac{1}{2}(1-c)^2 - \frac{1}{3}(1-4^{-m}) + c(1-2^{-m})]$ and m is the largest integer with $c \le 1/2^m$. Note that it is only necessary to use c in the interval $[0;\frac{1}{2}]$ because for $c > \frac{1}{2}$ the value $[1/(1-c)]\int(s-c)^+ds = 1/2$, while $\int(s-c)^+Q(ds) = 0$. Now using the derivative $g'(c)$ one easily shows that $c = \frac{1}{4}$ and $c = \frac{1}{2}$ are the unique points of maximum of g. By computing one can see that $g(\frac{1}{4}) < g(\frac{1}{2}) = 1/4$. Thus $\varepsilon_1(P,Q) = 1/4$. ||

From (1 7) it follows immediately that ε_1 satisfies the triangle inequality. But ε_1 is not symmetric. The mapping $(P,Q) \longmapsto \delta_1(P,Q) := \varepsilon_1(P,Q) + \varepsilon_1(Q,P)$ is a pseudo-metric on $\mathcal{M}(S)$, in fact a metric when K is a determining class for $\mathcal{M}(S)$ (for instance, S a convex compact metrizable subset of a topological vector space and $K \subset C(S)$ the cone of convex functions). It is not difficult to prove that a sequence (P_n) in $\mathcal{M}(S)$ converges with respect to δ_1, i. e., $\delta_1(P_n,P) \to 0$ for some $P \in \mathcal{M}(S)$, iff the sequence of linear functional $f \longmapsto \int f dP_n$ converges uniformly on $K \bigcap \{f \in C(S) \mid \|f\| = 1\}$. As a consequence, if K is a determining class for $\mathcal{M}(S)$, then the δ_1-topology on $\mathcal{M}(S)$ is finer than the weak topology.

Neither ε_3 nor ε_4 satisfy the triangle inequality as Example 11 and 20 will show. On the other hand it is easy to see that $\varepsilon_4(P,R) \leq 2[\varepsilon_4(P,Q) + \varepsilon_4(Q,R)]$.

11. Example. A case where $\varepsilon_3(P,R) > \varepsilon_3(P,Q) + \varepsilon_3(Q,R)$. Let $S := [a;b]$, $K \subset C(S)$ be the cone of all convex functions and $a \leq x < y \leq b$. Put $z := (1-\alpha)x + \alpha y$ with $0 < \alpha < 1$, so that $x \neq z \neq y$. Consider

$$P := \delta_z,\ Q := (1-\alpha)\,\delta_x + \alpha\delta_y,\ R := \delta_x.$$

For each $f \in K$, $f(z) \leq (1-\alpha)f(x) + \alpha f(y)$, so that $P \prec Q$, hence $\varepsilon_3(P,Q) = 0$. Since $P' \prec R := \delta_x$ requires $P' = \delta_x$ and since $P = \delta_z$ with $z \neq x$, it follows that $\varepsilon_3(P,R) = \|\delta_x - \delta_z\| / 2 = 1$. On the other hand $\varepsilon_3(Q,R) \leq \|Q - R\| / 2 = \alpha$. ||

Probably there is no easy formula for computing the value ε_i, $i = 1, \ldots, 5$, but next theorem and corollary are an important step in this direction.

12. Theorem. Let S be a compact space, $K \subset C(S)$ an admissible cone, $P, Q \in \mathcal{M}(S)$ and $u, v \geq 0$ constants. Then there exist

$P', Q' \in \mathcal{M}(S)$, such that,

(19) $\quad \|P' - P\| \leq 2u, \quad \|Q' - Q\| \leq 2v, \quad P' \underset{K}{\prec} Q'$

if and only if, for all $f \in K$ with $\inf f = 0$ and all $c \in \mathbb{R}$ with $0 < c \leq \sup f$,

(20) $\quad \int f \wedge c \, dP \leq \int f \, dQ + uc + v \sup f.$

<u>Proof</u>. By the very definition of ε_2, (19) is equivalent to the existence of $P' \in \mathcal{M}(S)$, such that,

(21) $\quad \|P' - P\| \leq 2u, \quad \varepsilon_2(P', Q) \leq v.$

By Lemma 6 and the equality $\varepsilon_2 = \varepsilon_1$, condition (21) on P' is equivalent to

(22) $$\begin{aligned} &\int \alpha \, dP' \leq \int \alpha \, dP + u \operatorname{osc} \alpha, \text{ for all } \alpha \in C(S) \\ &\int f \, dP' \leq \int f \, dQ + v \operatorname{osc} f, \text{ for all } f \in K. \end{aligned}$$

Since $C(S)$ and K are cones, Theorem A.2 (see Appendix) tells us that a $P' \in \mathcal{M}(S)$ satisfying (22) exists iff, for all $f_j \in K$, $\alpha_i \in C(S)$, and $m, n \in \mathbb{N}$, we have that

(23) $\quad \inf \left(\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} f_j \right) \geq 0$

implies

$$(2.4) \qquad \sum_{i=1}^{m} (\int \alpha_i dP + u \operatorname{osc} \alpha_i) + \sum_{j=1}^{n} (\int f_j dQ + v \operatorname{osc} f_j) \geq 0.$$

Letting $\alpha := \Sigma\alpha_i$ and $f := \Sigma f_j$, then $\alpha \in C(S)$ and $f \in K$, since the cones $C(S)$ and K are convex. As $\operatorname{osc}\alpha \leq \Sigma \operatorname{osc}\alpha_i$ and $\operatorname{osc} f \leq \Sigma \operatorname{osc} f_j$, it suffices to establish the implication

$$(2.5) \qquad \alpha \in C(S),\ f \in K,\ \inf(\alpha + f) \geq 0 \Rightarrow \int \alpha dP + \int f dQ + u \operatorname{osc}\alpha + v \operatorname{osc} f \geq 0.$$

Introducing $h := \alpha + f$, this is equivalent to the requirement that

$$(2.6) \qquad \int f dP - \int f dQ \leq \int h dP + u \operatorname{osc}(f - h) + v \operatorname{osc} f, \text{ if } f \in K,\ h \in C^+(S).$$

Given $f \in K$, we want to choose $h \in C^+(S)$ so as to minimize the right hand side of (2.6). Put $a := \inf(f - h)$ and $c := \sup(f - h)$ so that $\operatorname{osc}(f - h) = c - a$ and $a \leq f - h \leq c$, or $f - c \leq h \leq f - a$. As $h \geq 0$, setting $h_o := (f - c)^+ := (f - c) \vee 0$, we have $f - c \leq h_o \leq h \leq f - a$. Further $f - c \leq h_o \leq f - a$, or $a \leq f - h_o \leq c$, which shows that $\operatorname{osc}(f - h_o) \leq c - a = \operatorname{osc}(f-h)$. Since $0 \leq h_o = (f - c)^+ \leq h$ and $\operatorname{osc}(f - h_o) \leq \operatorname{osc}(f - h)$, it is clear from (2.6) that it suffices to consider only functions of the form $h := (f - c)^+$, where c is a constant. Observing that

$f - (f - c)^{+} = f \wedge c$, (2.6) is equivalent to

(2.7) $\quad \int f \wedge c \, dP - \int f \, dQ \le u \operatorname{osc}(f \wedge c) + v \operatorname{osc} f$, for all $f \in K$, $c \in \mathbb{R}$

Let us show that in (2.7) we only need

(2.8) $\quad \inf f < c \le \sup f$.

For, the choice $c > \sup f$ is the same as the choice $c = \sup f$ because in both cases $f \wedge c = f$. If $c \le \inf f$, then $\int f \wedge c \, dP = c$ and $\int f dQ \ge \inf f \ge c$ so that (2.7) is always true.

Since K contains the constants we can take always $\inf f = 0$, in which case $\operatorname{osc} f = \sup f$. Thus the proof will be complete if we show that $\operatorname{osc}(f \wedge c) = c$. Indeed, by (2.8) $\inf(f \wedge c) = \inf f = 0$ and $\sup(f \wedge c) = 0$. ||

Besides using only functions $f \in K$ with $\inf f = 0$ in (2.0) one may also assume without loss of generality that $\sup f = 1$. Hence (2.0), thus also (1.9), is equivalent to

(2.9) $\quad tu + v \ge \phi(t)$, for all $0 \le t \le 1$.

Here

$$\phi(t) := \sup \left\{ \int f \wedge t \, dP - \int f \, dQ \mid f \in K, \ \inf f = 0, \ \sup f = 1 \right\}.$$

The set of relations (2 9) represents a family $(H_t)_{t\in[0;1]}$ of closed half planes. The intersection

$$A := A(P,Q,K) := \left(\bigcap_{t\in[0;1]} H_t\right) \cap \{(u,v) \in \mathbb{R}^2 \mid u \geq 0,\ v \geq 0\}$$

is a closed convex subset of $\mathbb{R}^2$. The pairs $(u,v) \in A$ are precisely the pairs for which there exist $P', Q' \in \mathcal{M}(S)$ satisfying (1 9)

Considering the definitions of $\varepsilon_i(P,Q)$ it is clear that

$$\varepsilon_2(P,Q) = \inf\{v \mid (0,v) \in A\},$$
$$\varepsilon_3(P,Q) = \inf\{u \mid (u,0) \in A\},$$
$$\varepsilon_4(P,Q) = \inf\{u \mid (u,u) \in A\},$$
$$\varepsilon_5(P,Q) = \inf\{u+v \mid (u,v) \in A\}.$$

The geometric meaning of $\varepsilon_1 = \varepsilon_2$, ε_3, ε_4 and ε_5 is clear. So putting all together we have the situation described in Fig. 1.

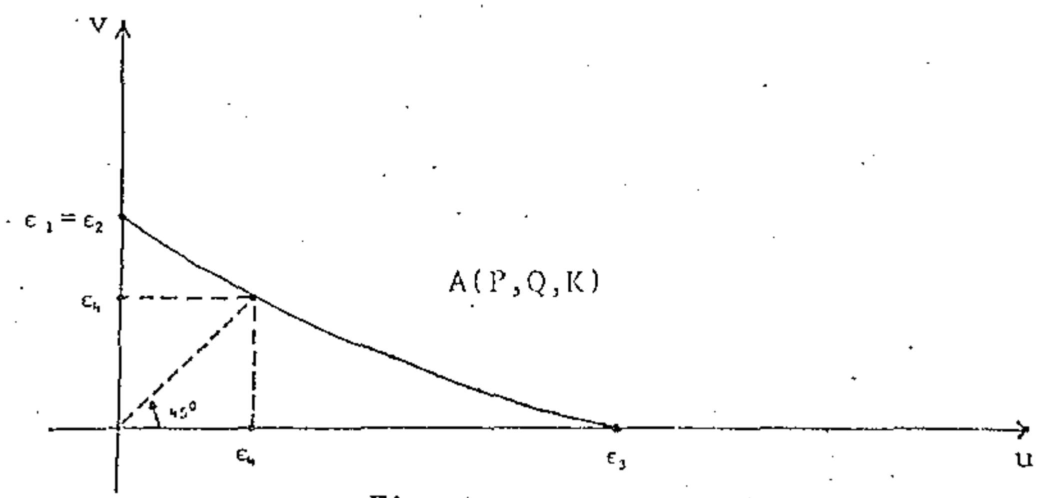

Fig. 1

The only thing that is not clear is how ε_5 fits into the picture. In fact one has:

13. Corollary. $\varepsilon_5 = \varepsilon_2$.

Proof. The function $t \mapsto \phi(t)$ in (29) is increasing. Hence $\varepsilon_2(P,Q) = \phi(1)$. Therefore taking $t = 1$ in (29) all points $(u,v) \in A$ satisfy

$$u + v \geq \varepsilon_2(P,Q).$$

The equality sign is attained at $(0, \varepsilon_2(P,Q))$. This proves that $\varepsilon_5 = \varepsilon_2$. ||

Before going further, we will present some illustrations. We will consistently use the notation

$$K' := \{f \in K \mid \inf f = 0,\ \sup f = 1\},$$

where K is a given cone of functions.

14. Example. We will study in detail the case $S := [0;1]$, $K :=$ cone of all convex increasing continuous functions $S \to \mathbb{R}$, thus $\inf f = f(0)$ and $\sup f = f(1)$ for each $f \in K'$. Let further $P \in \mathcal{M}(S)$ be arbitrary and $Q := \delta_0$, the Dirac measure at 0. Note

that, for each $f \in K'$, we have $f(0) = 0$, $f(1) = 1$ and $f(t) \leq f^*(t) := t$, where $f^* \in K'$.

The function ϕ in (2 9) is given by

$$\begin{aligned}\phi(t) &= \sup_{f\in K'} \left(\int f\wedge t \, dP - \int f \, d\delta_o\right) \\ &= \sup_{f\in K'} \int f\wedge t \, dP = \int f^*\wedge t \, dP = \int_{[0;t]} s \, dP + t \int_{(t;1]} dP \\ &= tF(t) - \int_o^t F(s) \, ds + t(F(1) - F(t)) \\ &= t - \int_o^t F(s) \, ds.\end{aligned}$$

Here F denotes the distribution function (d. f.) of P and we have integrated by parts. Therefore (2 9) in this case reads

(3 0) $$tu + v \geq \phi(t) = t - \int_o^t F(s) \, ds, \text{ for all } t \in [0;1].$$

We observe that, letting X be a random variable whose distribution is P, then $t = 1$ in (3 0) leads to $u + v \geq E[X]$.

(i) Let us consider the case in which P is supported by $\{x_1, \ldots, x_n\}$, $0 < x_1 < \ldots < x_n$. Let $P(\{x_j\}) = p_j$. Of course $p_1 + \ldots + p_n = 1$. Here

$$F(s) = \sum_{i=1}^{n} p_i 1_{[x_i;\infty)}(s).$$

Hence

$$t - \phi(t) = \int_0^t F(s)\,ds = \sum_{i=1}^{n} p_i (t - x_i)^+ .$$

The line $tu + v = \phi(t) = t$ rotates about the point $T_o := (1,0)$ when t increases from 0 to x_1. Similarly, the line $tu + v = \phi(t) = t - p_1(t - x_1) - \ldots - p_j(t - x_j)$ rotates about the point

$$T_j := (1 - p_1 - \ldots - p_j,\ p_1 x_1 + \ldots + p_j x_j)$$

when t increases from x_j to x_{j+1}, this for for $j = 1, \ldots, n$. In particular for $j = n$ the line rotates about $T_n := (0, E[X])$. The point T_j is also the intersection of the two lines with $t = x_j$ and $t = x_{j+1}$. Hence the region $A(P,\delta_o,K)$ looks like in Fig. 2, and we see that its lower boundary is polygonal.

We conclude that $\epsilon_1(P,\delta_o) = E[X]$. It was obvious from the beginning that $\epsilon_3(P,\delta_o) = 1$, because δ_o only dilates itself relatively to K (see definition (13)). The value $\epsilon_4(P, \delta_o)$ cannot be given by a simple formula.

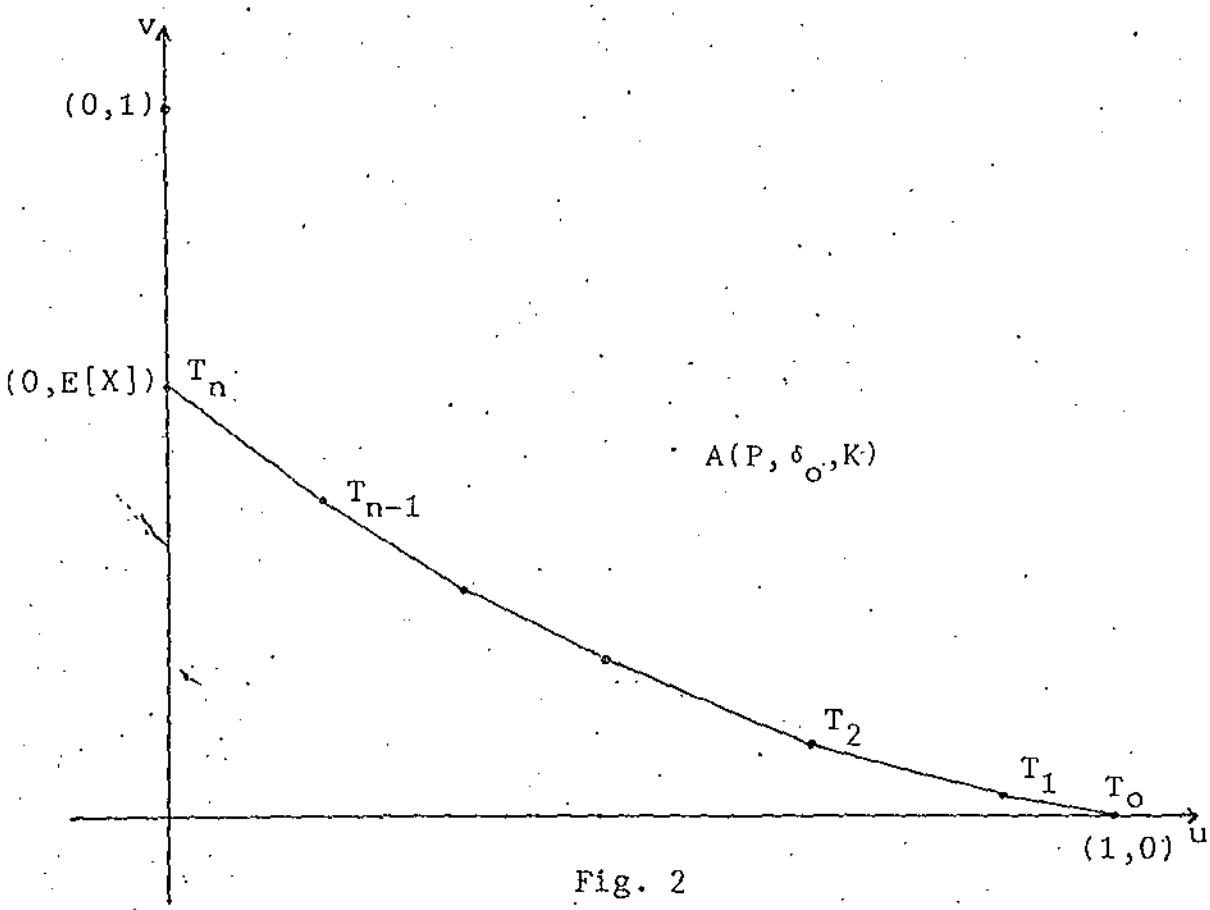

Fig. 2

(ii) Assume now P has no atoms. Therefore F in (3 0) is continuous. Hence we obtain from (3 0) that the part of the lower boundary of $A(P,\delta_o,K)$ not contained in the coordinate axes is a smooth curve (envelope) with parametric equations

(3 1)
$$u = 1 - F(t)$$
$$v = tF(t) - \int_o^t F(s)ds, \quad t \in [0;1].$$

Letting $u = 0$, the first equation gives $F(t) = 1$, which has a solution $t = 1$ (not necessarily unique). Substituting these values for $F(t)$ and t in the second equation of (3 1), we arrive to

$$(3\,2) \qquad \varepsilon_1(P,\delta_o) = 1 - \int_o^1 F(s)\,ds = E[X].$$

It is obvious that $\varepsilon_3(P,\delta_o) = 1$. Finally (solving for $v = u$ in (3 1)),

$$(3\,3) \qquad \varepsilon_4(P,\delta_o) = 1 - F(t_o),$$

where $t_o \in [0;1]$ is a solution of the equation

$$(3\,4) \qquad \int_o^t F(s)\,ds = (t+1)F(t) - 1.$$

(iii) Let us specialize (ii) taking for P a measure $P_n \in \mathcal{M}([0;1])$ given by

$$P_n(B) := \int_B (n+1)s^n\,ds, \quad n \in \{0, 1, \dots\}.$$

The d. f. of P_n is the function F given by $F(s) = P_n(-\infty;s]$. Hence, by (3 2),

$$\varepsilon_1(P_n,\delta_o) = 1 - \int_o^1 F(s)\,ds = \frac{n+1}{n+2}.$$

Eliminating the parameter t in (3 1) we get

(3 5) $$v = \frac{n+1}{n+2}(1 + u)^{(n+2)/(n+1)}, \quad u \in [0;1],$$

which is the Cartesian equation of the lower boundary of $A(P_n, \delta_o, K)$. Taking $v = u$ in (3 5) we see that $\varepsilon_4 = \varepsilon_4(P_n, \delta_o)$ is implicitly given by

$$\varepsilon_4 = \frac{n+1}{n+2}(1 + \varepsilon_4)^{(n+2)/(n+1)}.$$

For $n = 0$, $P_n = P_o = ds$ is the Lebesgue measure on $[0;1]$ and (35) represents a convex parabola with vertix $(1,0)$ - see Fig. 3. ||

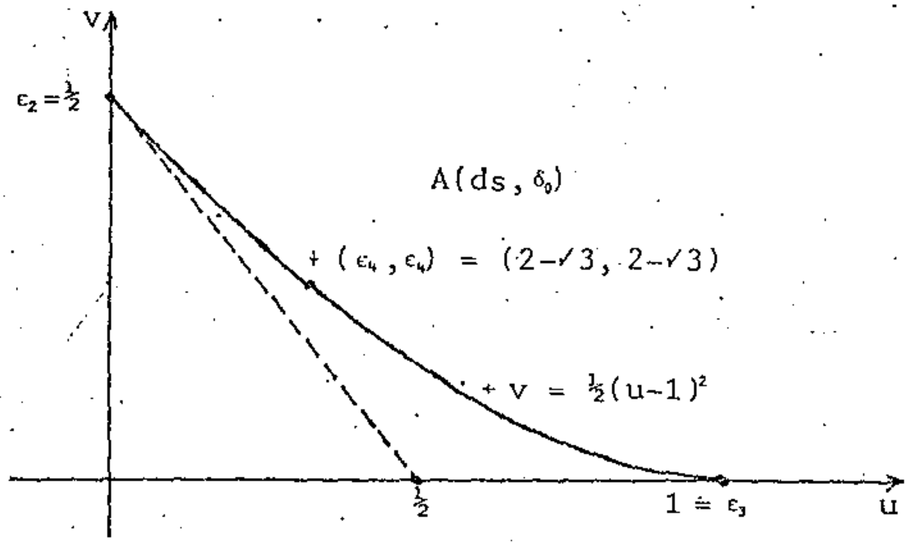

Fig. 3

15. Example. Let $S := [0;1]$, $P := ds$ (Lebesgue measure), $Q := \delta_x$, $x \in (0;1)$, and $K \subset C(S)$ the cone of convex increasing functions. This example generalize the Example 14 (iii). Here with some work we obtain

$\varepsilon_1(ds, \delta_x) = (1-x)/2$, $\varepsilon_3(ds, \delta_x) = 1-x$, $\varepsilon_4(ds, \delta_x) = (1-x)(2-\sqrt{3})$,

valid for all $x \in [0;1]$. ||

Previous calculations with $Q = \delta_0$ were easy because K' contained a largest element f^* while $f(0) = 0$ for all $f \in K'$. More general: let S be a compact space with a partial order, and K the cone of all continuous increasing functions that assume their minimum at every point of $U := \operatorname{supp} Q$, the support of Q. Note that such cone K is not only invariant under the operation $\vee$ but also under $\wedge$. Letting $P \in \mathcal{M}(S)$ be arbitrary, we have as $\phi(t)$ in (2 9)

$$\phi(t) = \int t \wedge 1_{U^c}(s)\, P(ds) = tP(U^c),$$

which leads to $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2\varepsilon_4 = \varepsilon_5 = P(U^c)$ – see Fig. 4.

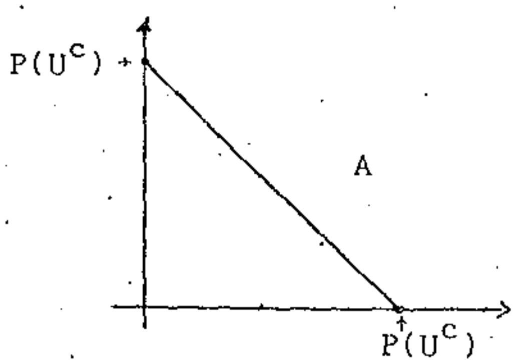

Fig.4

The above expression for $\phi(t)$ was possible because K' is filtering from right (see [1], p. 145), i. e., given $f, g \in K'$, there exists $h \in K'$ with $f, g \leq h$. In general, if S is a compact space with a partial ordering, $K \subset C(S)$ an admissible cone, such that, K' is filtering from the right, and $Q \in \mathcal{M}(S)$ is such that each $f \in K'$ assumes its minimum at every point of supp Q, then (2 9) takes the form

$$(36) \qquad tu + v \geq \phi(t) = t - \int_0^t F(s)\,ds, \quad t \in [0;1],$$

where F is the P-distribution function of $s \mapsto \sup_{f \in K'} f(s) := f^*(s)$.

It is true in general that the right slope r of the lower boundary of a region A(P,Q) at $(0,\epsilon_1)$ is given by the formula $r = -\inf\{t \in [0;1] \mid \phi \text{ is constant on } [t;1]\}$, where ϕ is as in (29). Now, if $\phi(t)$ is the right hand side in (36), then, as it is easy to see, the formula for r specializes to

$$(37) \qquad r = -\inf\{t \in [0;1] \mid F(t) = 1\}.$$

Similarly, it is true in general that the left slope ℓ of the lower boundary of A(P,Q) at $(\epsilon_3,0)$ is obtained by the formula $\ell = -\sup\{t_1 \in [0;1] \mid \phi(t)/t \text{ is constant on } (0;t_1]\}$, which, in the situation of (36), becomes

$$(38) \qquad \ell = -\sup\{t_1 \in [0;1] \mid F \text{ is constant on } [0;t_1]\}.$$

16. Example. Let us reconsider Example 14 (iii). In that example f^* is given by $f^*(s) = s$, whose P_n-distribution function F is given by $F(s) = s^{n+1}$ if $s \in [0;1]$. Hence formulas (37) and (38) yield $r = -1$ and $\ell = 0$, respectively, for all n - see Fig. 3. ||

17. Example. Let S be the interval [0;1], $K \subset C(S)$ the cone

of convex increasing functions, $P \in \mathcal{M}(S)$ the measure with density $\frac{1}{b-a} 1_{[a;b]}(s)\,ds$ where $0 \leq a < b \leq 1$, and $Q := \delta_o$. The corresponding d. f. F is given by $F(s) := (s-a)/(b-a)$ if $s \in [a;b]$. Hence here $r = -b$ and $\ell = -a$, which shows that the right slope of the lower boundary of A at $(0, \epsilon_1)$ can be any number in $[-1;0)$ and its left slope at $(\epsilon_3;0)$ any number in $(-1;0]$. We observe also that here $\epsilon_1(P,\delta_o) = 1 - \int_0^1 F(s)\,ds = (a+b)/2$, so that ϵ_1 can be close to 0 or 1.

For instance, letting $a = 0$ and $b = \frac{1}{2}$, we calculate the function ϕ in (36) by

$$\phi(t) = \begin{cases} t - t^2, & \text{if } t \in [0;\frac{1}{2}] \\ \frac{1}{4}, & \text{if } t \in [\frac{1}{2};1]. \end{cases}$$

The system of inequalities $tu + v \geq t - t^2$, $t \in [0;\frac{1}{2}]$, determines $A(P,Q)$. The lower boundary of the latter is the envelope of the family of lines $tu + v = t - t^2$, $t \in [0;\frac{1}{2}]$, which has as Cartesian representation $v = (\frac{1}{4})(1-u)^2$, $u \in [0;1]$ - see Fig. 5.

Similarly, if $a = \frac{1}{2}$, $b = 1$, then $v = (\frac{1}{4})u^2 - u + 3/4$, $u \in [0;1]$, instead - see Fig. 6. ||

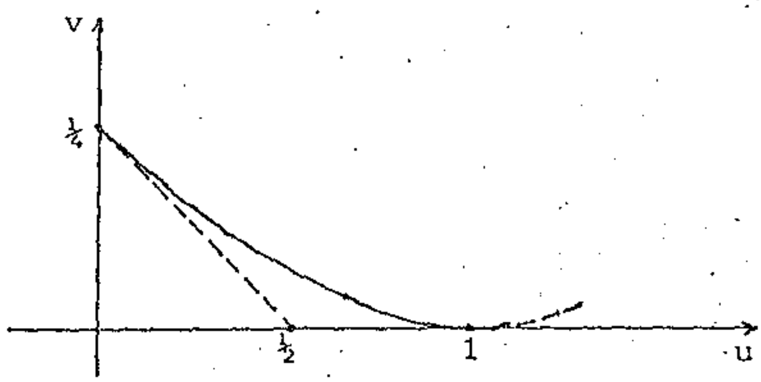

Fig. 5

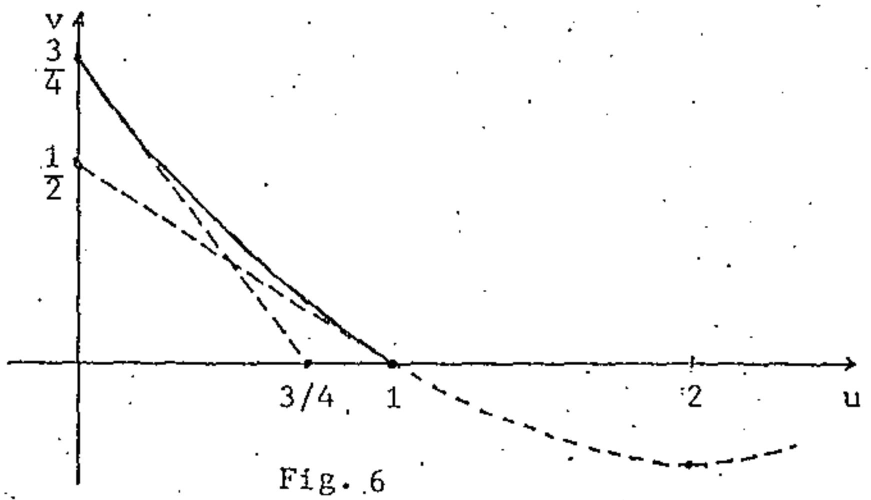

Fig. 6

18. Example. Let $S \subset \mathbb{R}^n$ be a compact convex set and let $P_n(A) := |A|/|S|$ be the normalized Lebesgue measure on S. Let $y \in \mathrm{int}(S)$ and $K_y := \{f \in C(S) \mid f \text{ is convex}, \inf f = f(y)\}$. Also let $Q := {}'_y$. Here there exists the largest element $f^* := f^*_y$ of $K' := K'_y$. Its graph is the "lateral" boundary of the solid cone in $\mathbb{R}^{n+1}$ with vertex $(y,0)$ and base $\{(s,1) \in \mathbb{R}^{n+1} \mid s \in S\}$. For $z \in [0;1]$, let S_z be the part of the hyperplane $s_{n+1} = z$ (we call s_i the i^{th} coordinate of a point $s \in \mathbb{R}^{n+1}$) inside the graph of f^*_y. Therefore

$$P_n(\pi(S_z)) = z^n,$$

where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection. This implies that the d. f. F of f^*_y relative to the probability space (S,P_n) is given by $F(t) = t^n$, if $t \in [0;1]$, which is independent of y or the shape of S. Since, by (3 6), $A(P_n, \delta_y, K_y)$ depends only on F, the conclusions of Example 1 4 (iii) also hold for the present situation. ||

19. Measures P_t', Q_t' Realizing the Boundary of A(P,Q). As was already observed, A(P,Q) is a closed subset of $\mathbb{R}^2$. This means that, for each point (u,v) on the boundary of A(P,Q), one can attain both equality signs in (1 9) by a suitable choice of P' and Q'. Let us now give an example where P', Q' can be explicitly described.

Let S be a compact space and $K \subset C(S)$ an admissible cone. Suppose K' possesses a largest element f*. Choose $P \in \mathcal{M}(S)$ and let F be the P-distribution function of f*. Suppose there is a unique point y in S with f*(y) = 0 and a unique point y' in S with f*(y') = 1. (Example: let S be a compact space with a partial ordering, a least element y and a greatest element y', and let $K \subset C(S)$ be the cone of all convex increasing functions.). Choose $Q = \delta_y$. The parametric equations for the lower portion of the boundary of $A(P, \delta_y, K)$ are, by (31) (we are also assuming that P has no atom),

$$u(t) = 1 - F(t)$$
$$v(t) = tF(t) - \int_0^t F(s)\,ds, \quad t \in [0;1].$$

Define P_t', $Q_t' \in \mathcal{M}(S)$ by

$$P'_t(E) := P[E \cap (f^* \le t)] + u(t)\delta_y(E),$$

$$Q'_t(E) := v(t)\delta_{y'}(E) + (1 - v(t))\delta_y(E).$$

Certainly $\|P'_t - P\| = 2u(t)$ and $\|Q'_t - Q\| = 2v(t)$. Moreover, given $f \in K'$,

$$\int f\, dP'_t \le \int f^*\, dP'_t = \int_{[0;t]} s\, dF(s) + uf^*(y)$$

$$= \int_{[0;t]} s\, dF(s) = tF(t) - \int_o^t F(s)\, ds = v(t),$$

and

$$\int f\, dQ'_t = v(t)f(y') + [1 - v(t)]f(y) = v(t).$$

Thus $\int f\, dP'_t \le \int f\, dQ'_t$. This proves that $P'_t \prec Q'_t$.

20. <u>The Triangle Inequality Fails for ε_4</u>. Let $S := [0;1]$, $K \subset C(S)$ be the cone of decreasing convex functions and $Q := p\delta_o + q\delta_1$ with $p + q = 1$. We want to show that, for convenient values of p, q,

(39) $$\varepsilon_4(\delta_{\frac{1}{2}}, \delta_1) > \varepsilon_4(\delta_{\frac{1}{2}}, Q) + \varepsilon_4(Q, \delta_1).$$

Let first compute $\varepsilon_4(\delta_{\frac{1}{2}}, \delta_1)$. Here (36) applies. The function $s \mapsto -s+1$ is the largest element in K and its

$\delta_{\frac{1}{2}}$-distribution function is $F = 1_{[\frac{1}{2};\infty)}$. Using (3 6) we obtain the following family of half planes

$$tu + v \geq \begin{cases} t, & \text{if } t \leq \frac{1}{2} \\ \frac{1}{2}, & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Thus $w + 2v = 1$ is the equation of the lower boundary of $A(\delta_{\frac{1}{2}},\delta_1)$. Letting $v = u$ in that equation, we conclude that

$$\varepsilon_4(\delta_{\frac{1}{2}},\delta_1) = 1/3.$$

Next, consider $\varepsilon_4(\delta_{\frac{1}{2}},Q)$. Here it is easier going back to (2 9). We have

$$\phi(t) = \sup_{f\in K'} [\int (f\wedge t)\, d\delta_{\frac{1}{2}} - \int f\, dQ]$$

$$= \begin{cases} t - p, & t \leq \frac{1}{2} \\ \frac{1}{2} - p, & \text{if } t \geq \frac{1}{2}. \end{cases}$$

The equation of the important part of the lower boundary of $A(\delta_{\frac{1}{2}},Q)$ is $u + 2v = 1 - 2p$, from which, letting $v = u$, we obtain

(4 0) $$\varepsilon_4(\delta_{\frac{1}{2}},Q) = \begin{cases} 0, & \text{if } p \geq \frac{1}{2} \\ (1 - 2p)/3, & \text{if } p \leq \frac{1}{2}. \end{cases}$$

As to $\varepsilon_4(Q,\delta_1)$, here again (3 6) applies. The Q-distribution function F of $s \longmapsto -s+1$ has values $F(s) = 0$ if $s < 0$, $f(s) = q$ if $0 \leq s < 1$ and $F(s) = 1$ if $s \geq 1$. By (3 6)

$$tu + v \geq t - \int_0^t F(s)\,ds = t - qt = pt, \quad t \in [0;1].$$

So the part of the lower boundary of $A(Q,\delta_1)$ we are interested in is given by $u + v = p$, $u \in [0;p]$, showing that

(41) $\epsilon_4(Q, \delta_1) = p/2.$

Adding (40) and (41) we obtain

$$\epsilon_4(\delta_{\frac{1}{2}},Q) + \epsilon_4(Q,\delta_1) = \begin{cases} p/2, & \text{if } p \geq \frac{1}{2} \\ 1/3 - p/6, & \text{if } p \leq \frac{1}{2}. \end{cases}$$

Since $\epsilon_4(\delta_{\frac{1}{2}},\delta_1) = 1/3$, this shows that (39) obtains whenever $0 < p < 2/3$. ||

When we dealt with cones both invariant under max and min operation, the corresponding picture, Fig. 4, was very peculiar. In particular $\epsilon_2 = \epsilon_3 = 2\epsilon_4$ in that situation. Let us show that this is always so whenever the cone has the mentioned property through the following proposition.

21. <u>Proposition</u>. Let S be a compact space, $K \subset C(S)$ an admissible cone which is invariant under the operation $\wedge$ and let $P, Q \in \mathcal{M}(S)$. Then the portion of the boundary of $A(P,Q,K)$ not contained in the u-axis is a line segment with slope -1.

In particular $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2\varepsilon_4 = \varepsilon_5$ at (P, Q).

Proof. The lower boundary of $A(P, Q, K)$ has slope ≤ 1 (in absolute value). But so has the corresponding set $A(P, Q, -K)$, where $-K := \{f \mid -f \in K\}$. Since $A(P,Q,-K)$ is simply the reflexion $\{(v, u) \mid (u, v) \in A(P, Q, K)\}$ of $A(P, Q, K)$, the lower boundary of the latter is a straight line of slope -1. ||

Before ending this article it is worthwile to make the following

22. Remark. Let S be a compact space, $K \subset C(S)$ an admissible cone and $P, Q \in \mathcal{M}(S)$. Using the definition of $\varepsilon_1(P, Q)$ and Theorems 8 and 12, we have

$$\varepsilon_i(P, Q) = \sup_f [\int f\, dP - \int f\, dQ], \quad i = 1, 2, 5;$$

$$\varepsilon_3(P, Q) = \sup_{f,t} [\tfrac{1}{t}\int f\wedge t\, dP - \tfrac{1}{t}\int f\, dQ];$$

$$\varepsilon_4 = \sup_{f,t} [\tfrac{1}{1+t}\int f\wedge t\, dP - \tfrac{1}{1+t}\int f\, dQ];$$

where f runs over K' and t over $(0;1)$. It follows that, endowing $\mathcal{M}(S)$ with the weak topology, the function $(P, Q) \mapsto \varepsilon_i(P, Q)$, $i = 1, \ldots, 5$, is l. s. c. and convex. it is easy to produce examples showing that those functions are not (weakly) continuous. ||

TWO MOMENT THEOREMS

Here we are going to state two moment theorems, Theorem A.1 and Theorem A.2 below, which are basic tools for this paper. In fact they were used several times. As stated below they are particular cases of Theorem 5 and Theorem 7 in [7], respectively. A more general result for Polish spaces of the second theorem can be found in [6]. Below J will be any index set.

A.1. Theorem. Let S be a compact topological space. For each $j \in J$ let $h_j : S \to \mathbb{R}$ be a l. s. c. function and $\eta_j \in \mathbb{R}$. Then there exists $P \in \mathcal{M}(S)$, such that, $\int h_j dP \leq \eta_j$ for each $j \in J$ if and only if

$$\inf_{s \in S} \sum_{j \in J} b_j h_j(s) \geq 0 \quad \Rightarrow \quad \sum_{j \in J} b_j \eta_j \geq 0,$$

for each choice of the family $(b_j)_{j \in J}$ of non-negative constants all but finitely many equal to zero.

Let S_1, S_2 be metric spaces and $P_i \in \mathcal{M}(S_i)$, $i = 1, 2$. Let $\{(h_j, \eta_j)\}_{j \in J}$ be a family of pairs where $h_j: S_1 \times S_2 \to \mathbb{R}$ is a l. s. c. function and $\eta_j \in \mathbb{R}$. Next let K_i be a convex cone of bounded below l. s. c. P_i-integrable functions $\alpha_i: S_i \to \mathbb{R}$ and containing the bounded l. s. c. functions, this for $i = 1, 2$. In addition, suppose that, for each $j \in J$, there exists $\phi_{ji} \in K_i$, $i = 1, 2$, such that, the l. s. c. function $(s,t) \longmapsto h_j(s,t) + \phi_{j1}(s) + \phi_{j2}(t)$ on $S_1 \times S_2$ is bounded from below. We have the

A.2. <u>Theorem</u>. There exists $\lambda \in \mathcal{M}(S_1 \times S_2)$ with marginals P_1, P_2, such that,

$$\int h_j(s,t)\, \lambda(ds,dt) \leq \eta_j \text{ for all } j \in J$$

if and only if

$$\alpha_1(s) + \alpha_2(t) + \sum_{j \in J} b_j h_j(s,t) \geq 0 \text{ for all } (s,t) \in S_1 \times S_2$$

implies

$$\int \alpha_1 dP_1 + \int \alpha_2 dP_2 + \sum_{j \in J} b_j \eta_j \geq 0$$

for each family $(b_j)_{j \in J}$ of non-negative constants all but finitely many equal to zero and each choice of the $\alpha_i \in K_i$, $i=1, 2$.

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