A NOTE ON COMPACT SURFACES WITH NON ZERO CONSTANT MEAN CURVATURE

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- Trabalho de Pesquisa -
Série A8/OUT/89

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In a classical work on minimal surfaces, Shiffmann studies the uniqueness of the solutions of Plateau's problem for convex curves lying in parallel planes of R^3 ([Sh]). Considering the same problem for non-zero constant mean curvature surfaces, we prove here the following result:

THEOREM. There exists a constant K (aprox. = 0.757...), given as a limit of elliptic integrals, which satisfies de following. Let π_1 and π_2 be two parallel planes in R^3 such that the distance $d(\pi_1,\pi_2)$ between them is less than or equal to K. Let D_1 and D_2 be disks of radius 1/2 with common axis of symmetry, contained in π_1 and π_2, respectively. Given any two curves γ_1 and γ_2 in D_1 and D_2, respectively, let M be a compact, connected, embedded surface with constant mean curvature 1 and boundary $\partial M = \gamma_1 \cup \gamma_2$. If M is contained in a ball of radius 1, then M is contained in a cylinder of radius 1/2. Furthermore, if $d(\pi_1,\pi_2) =$ K, then $\gamma_1 = \partial D_1$, $\gamma_2 = \partial D_2$ and M is a cylinder of radius 1.

1. Idea of the proof.

The proof rests on the following construction. We will consider now just the case $d(\pi_1, \pi_2) = K$, which is the most difficult. Let S be the sphere of radius 1 centered on the origin of R^3. Let C' be the part of a cylinder determined by the conditions:

a) C' is contained in the bounded connected component, say V, of R^3/S^2,

b) the axis of symmetry of C' is the x-axis,

c) the boundary of C' is the union of the circles C_1, C_2 orthogonal to the x-axis, centered on points p_1, p_2 on the x-axis whose distance to the origin is $K/2$.

Denote by D_1 and D_2 the disks with boundaries C_1 and C_2.

We will construct a one parameter family of compact embedded rotational surfaces of constant mean curvature 1, contained in V, with boundaries always in S^2 and which do not intersect the disks D_1, D_2, with the exception of one unique element of the family which is $C'' := C \cap V$, where C is the cylinder with radius $1/2$ centered on the x-axis ($C' \subset C'' \subset C$).

The first element of this family is the whole sphere S^2 and, furthermore, the intersections between the members of the family with the yz-plane are circles centered on the origin, whose radii assume all values of the interval $(0,1]$. Therefore, if M is a compact embedded surface with constant mean curvature 1 contained in V and whose boundary is contained in $D_1 \cup D_2$, we can find an element of the above family which is tangent to M. It will be possible then to apply the tangency principle (see Lemma 1 of [Sc]) to conclude that M is an element of the family. Since C' is the only element having boundary contained in $D_1 \cup D_2$, we obtain $M = C'$.

For constructing the above family, it will be necessary to study in detail some properties of the rotational surfaces of constant mean curvature.

2. *Rotational surfaces of constant mean curvature.*

Such surfaces were first studied by C. Delaunay, who discovered an interesting way to obtain them: take a curve in a half plane obtained by the cut locus of the focus of a conic by rolling this conic through the boundary of the half plane. A rotational surface of constant mean curvature is therefore obtained by rotating this curve around the boundary of the half plane.

We will use this method of construction in our proof. In his paper ([D]), Delaunay makes this construction when the conic is a hyperbola. We will need the case that the conic is an ellipse. This case has been described in detail in Medeiro's Master Dissertation ([M]).

Let C_e be the ellipse in the xy-plane, with eccentricity e, whose foci are the points $f_1 = (0, (1-e)/2, 0)$ and $f_2 = (0, (1+e)/2, 0)$. Let α_e be the cut locus of f_2 when C_e rolls right and left until f_2 meets the semi circle $x^2+y^2 = 1$, $x \geq 0$. We observe that α_1 is the semi circle $x^2+y^2 = 1$ and α_0 is the line segment $y = 0.5$ with $-\sqrt{3}/2 \leq x \leq \sqrt{3}/2$. Let β_e be the cut locus of f_1 obtained in the same way. Clearly, $\beta_0 = \alpha_0$. When $e \longrightarrow 1$, β_e tends to two pieces of circles orthogonal to the x-axis at the origin.

Given $t \in [0,1)$, let R_t be the rotational surfaces obtained by rotating around the x-axis the curve α_{1-2t} if $0 \leq t \leq 1/2$ or β_{2t-1} if $1/2 \leq t < 1$.

R_t is a one parameter family of rotational surfaces of constant mean curvature 1, depending continuously on t. Furthermore, for any t, the boundary of R_t is contained in the sphere S^2, that is, the sphere centered on the origin with radius 1. We also have that the intersection between R_t and the yz-plane is a circle whose radius is exactly $(1-t)$.

Let $x_t > 0$ be the x-coordinate of the point of intersection between the curve α_{1-2t} and the straight line $y = 1/2$ if $0 \leq t < 1/2$ and the x-coordinate of the intersection point between the curve β_{2t-1} and the same line if $1/2 < t < 1$.

It follows by the rolling method of construction of the curves $\alpha_{(.)}$ and $\beta_{(.)}$ (see [CH] p.283 $ 42), that the function $t \longrightarrow x_t$ is strictly decreasing. Furthermore, we have:

$$\lim_{t\longrightarrow 1} x_t = \frac{2-\sqrt{3}}{2} \quad \text{and } x_0 = \frac{\sqrt{3}}{2}$$

We need to know the limit of x_t when the curves $\alpha_{(.)}$ tend to the straight line $y = 1/2$, that is, $x_{\frac{1}{2}} := \lim_{t\longrightarrow\frac{1}{2}} x_t$. For a given eccentricity e, the coordinates of the curve α_e are given by:

$$x(\phi) = 0.5 \left\{ \sin\phi + \tan g\phi\, (e^2 - \sin^2\phi)^{\frac{1}{2}} + (1 - e^2) \int_0^{\phi} \frac{d\lambda}{\cos^2\lambda\, (e^2-\sin^2\lambda)^{\frac{1}{2}}} \right\}$$

$$y(\phi) = 0.5 \left\{\cos\phi + (e^2 - \sin^2\phi)^{\frac{1}{2}}\right\}$$

Geometrically, 2ϕ is the angle between the foci rays at the contact point of the ellipse with the x-axis (see [D], or Lemma 1 p. 18 of [M].

Since we want to estimate the point of intersection between α_e and the line $y = 1/2$, we put $y(\phi) = 0.5$, and we obtain:

$$\phi = \arccos(1 - 0.5e^2).$$

We can therefore compute the positive x-coordinate $x_{\frac{1}{2}}$ of the intersection point when $\alpha_{(.)}$ tends to the line $y = 1/2$:

$$x_{\frac{1}{2}} = \lim_{e\longrightarrow 0} x(\arccos(1-0.5e^2))$$

and, from the expression for $x(\phi)$,

$$x_{\frac{1}{2}} = 0.5 \lim_{e \to 0} \int_0^{\arccos(1-0.5e^2)} \frac{d\lambda}{\cos^2\lambda(e^2 - \sin^2\lambda)^{\frac{1}{2}}} = 0.5K,$$

where

$$K = \lim_{e \to 0} \int_0^{\arccos(1-0.5e^2)} \frac{d\lambda}{\cos^2\lambda(e^2 - \sin^2\lambda)^{\frac{1}{2}}}$$

By computacional estimates, we obtain:

$$\boxed{K = 0.757...}$$

We can now conclude the proof of the theorem.

3. *Proof of the theorem.*

Let M be as stated in the theorem. We can assume that M is contained in a ball whose boundary is the sphere S^2 with radius 1 centered on the origin. We use the notations introduced in the previous section.

Since $x_t' > K/2$ for $1/2 < t \leq 1$, the boundaries of the rotational surface R_t do not intersect M. Also, since the y-coordinate of the curves α_t are greater than $1/2$ for $1/2 < t \leq 1$, the boundary of M does not intersect any R_t for these values of t. Furthermore, since R_0 is the sphere S^2, which does not intersect M, it follows from the tangency principle that $R_t \cap M = \emptyset$, $1/2 \leq t \leq 1$, and this implies that M is contained in the cylinder with radius $1/2$ centered on the x-axis, proving the first part of the theorem.

Assume now that $d(\pi_1,\pi_2) = K$. By contradiction, assume that $M \neq C'$. If $\partial M \cap \partial C' = \emptyset$, since $C' \subset R_o = C''$, there exists $\epsilon > 0$ such that $R_t \cap M = \emptyset$ for $1/2 - \epsilon < t \leq 1/2$, and it will be possible to apply the tangency principle to conclude that $M = R_t$ for some $0 < t < 1/2$, contradiction. If $\partial M \cap \partial C' \neq \emptyset$, then the tangency principle in the boundary implies that there exists $\theta > 0$ such that the angle between the tangent planes of C' and M at a common point of their boundary is greater than θ. We choose $t_o < 1/2$ such that if $t_o < t < 1/2$, then the angle between R_t and C' on $R_t \cap C'$ is less than $\theta/2$. Therefore, M does not intersect R_t in a neighbourhood of ∂M, and, by the tangency principle, $M \cap R_t = \emptyset$, $t_o < t \leq 1/2$, and this leads, as above, to a contradiction. Therefore $M = C'$ and the theorem is proved.

4. Remarks.

a) A similar result can be proved in the sphere S^3, using the rotational minimal surfaces of S^3 and applying a similar reasoning. It is possible then to give aa characterization of the Clifford torus (see Theorem 2 of [R]).

b) If $d(\pi_1, \pi_2) > K$ both assertions of the Theorem are false and if $d(\pi_1, \pi_2) \neq K$ the second one is false. The counter-examples are also given by the rotational surfaces of constant mean curvature, as it can easily be seen.

c) Lucio Rodriguez, from geometrical reasons, conjectured that the constant K is equal to $\pi/2$. However, we couldn't get a prove of this.

I want to thank Nubem Medeiros and Lucio Rodriguez for their aid on the realization of this work.

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