Densities in large permutations and parameter testing

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ABSTRACT

A classical theorem of Erdős, Lovász and Spencer asserts that the densities of connected subgraphs in large graphs are independent. We prove an analogue of this theorem for permutations and we then apply the methods used in the proof to give an example of a finitely approximable permutation parameter that is not finitely forcible. The latter answers a question posed by two of the authors and Moreira and Sampaio.

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1. Introduction

Computer science applications that involve large networks form one of the main motivations to develop methods for the analysis of large graphs. The theory of graph limits, which emerged in a series of papers by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [4,6,5,18], gives analytic tools to cope with problems related to large graphs. It also provides an analytic view of many standard concepts, e.g. the regularity method [19] or property testing algorithms [14,20]. In this paper, we focus

on another type of discrete objects, permutations, and we give permutation counterparts of some of classical results on large graphs. It is worth noting that not all results on large graphs have permutation analogues and vice versa as demonstrated, for example, by the finite forcibility of graphons and permutons [9] (vaguely speaking, finite forcibility means that a global structure is determined by finitely many substructure densities).

Both our main results are related to the dependence of possible densities of (small) substructures. In the case of graphs, Erdős, Lovász and Spencer [8] considered three notions of substructure densities: the subgraph density, the induced subgraph density and the homomorphism density. They showed that these types of densities in a large graph are strongly related and that the densities of connected graphs are independent in the sense that none of the densities can be expressed as a function of the others. The result has a natural formulation in the language of graph limits, which are called graphons: the body of possible densities of any k connected graphs in graphons, which is a subset of $[0, 1]^k$, has a non-empty interior (in particular, it is full dimensional).

Our first result asserts that the analogous statement is also true for permutations. As in the case of graphs, it is natural to cast our result in terms of permutation limits, called permutons. The theory of permutation limits was initiated in [12,15] (also see [21]) and successfully applied e.g. in [14,17]. To state our first result, we use the notion of a *indecomposable* permutation, which is an analogue of graph connectivity in the sense that an indecomposable permutation cannot be split into *independent parts*. Let T^q be the body of possible densities of indecomposable permutations of order at most q in a permuton (a precise definition and further details can be found in Section 2.1). Our first result says that T^q has a non-empty interior for every q. In particular, it contains $B(\mathbf{w}, \varepsilon)$, for some vector $\mathbf{w}$ and some $\varepsilon > 0$, where $B(\mathbf{w}, \varepsilon)$ denotes the ball of radius ε around $\mathbf{w}$.

Theorem 1. *For every integer $q \geq 2$, there exist a vector $\mathbf{w} \in T^q$ and $\varepsilon > 0$ such that $B(\mathbf{w}, \varepsilon) \subseteq T^q$.*

Our second result is related to algorithms for large permutations. Such algorithms are counterparts of extensively studied graph property testing, see e.g. [2,3,10,11,22]. In the case of permutations, two of the authors and Moreira and Sampaio [13,14] established that every hereditary permutation property is testable with respect to the rectangular distance and two of the other authors [16] strengthened the result to testing with respect to Kendall's tau distance. In addition to property testing, a related notion of parameter testing was also considered in [14] where testable bounded permutation parameters were characterized.

However, the interplay between testing and the finite forcibility of permutation parameters was not fully understood in [14]. In particular, the authors asked [14, Question 5.5] whether there exists a testable bounded permutation parameter that is not finitely forcible. Our second result gives a positive answer to this question.

Theorem 2. *There exists a bounded permutation parameter f that is finitely approximable but not finitely forcible.*

Informally speaking, we utilize the methods used in the proof of Theorem 1 to construct a permutation parameter that oscillates on indecomposable permutations, with bounded amplitude, so that the parameter is testable though it fails to be finitely forcible.

2. Preliminaries

In this section, we introduce the notions used throughout the paper. Most of our notions are standard but we include all of them for the convenience of the reader.

2.1. Permutations

A *permutation of order n* is a bijective mapping from $[n]$ to $[n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$. The order of a permutation σ is denoted by $|\sigma|$. We say a permutation is *non-trivial* if it has order greater than 1. We denote by S_n the set of all permutations of order n and let $\mathfrak{S} = \bigcup_{n \in \mathbb{N}} S_n$. An *inversion* of a permutation σ is a pair (i, j), $i, j \in [|\sigma|]$, such that $i < j$ and $\sigma(i) > \sigma(j)$. An *interval I* in $[m]$ is a

set of integers of the form $\{k \mid a \le k \le b\}$ for some $a, b \in [m]$. An interval I is *proper* if $a < b$ and $I \ne [m]$.

We say that a permutation σ of order n is *indecomposable* if there is no $1 \le m < n$ such that $\sigma([m]) = [m]$. Note that

$$\Pr_{\sigma \in S_n}(\sigma \text{ is not indecomposable}) \le \frac{\sum_{m=1}^{n-1} m!(n-m)!}{n!} = \sum_{m=1}^{n-1} \binom{n}{m}^{-1}$$

$$\le \frac{2}{n} + \sum_{m=2}^{n-2} \binom{n}{m}^{-1} \le \frac{2}{n} + (n-3)\frac{2}{n(n-1)}. \tag{1}$$

Thus, $\lim_{n \to \infty} \Pr_{\sigma \in S_n}(\sigma \text{ is indecomposable}) = 1$.

We say that a permutation σ is *simple* if it does not map any proper interval onto an interval. For example the permutation $(\sigma(1), \ldots, \sigma(4)) = (2, 4, 1, 3)$ is simple.

Albert, Atkinson and Klazar [1] showed that a random permutation is simple with a probability bounded away from zero. Specifically, they proved the following.

$$\lim_{n \to \infty} \mathbb{P}_{\sigma \in S_n}(\sigma \text{ is simple}) = e^{-2}. \tag{2}$$

Let π be a permutation of order k and σ a permutation of order n. We introduce three ways in which π can appear in σ: as a subpermutation, through a monomorphism and through a homomorphism. We say that π is a *subpermutation* of σ if there exists a strictly increasing function $f : [k] \to [n]$, such that $\pi(i) > \pi(j)$ if and only if $\sigma(f(i)) > \sigma(f(j))$ for every $i, j \in [k]$. We then say that $f([k])$ *induces* a subpermutation π in σ. Let $\mathrm{Occ}(\pi, \sigma)$ be the set of all such functions f from $[k]$ into $[n]$ and let $\Lambda(\pi, \sigma) = |\mathrm{Occ}(\pi, \sigma)|$. The *density* of π in σ is defined as

$$t(\pi, \sigma) = \begin{cases} \Lambda(\pi, \sigma)\binom{n}{k}^{-1} & \text{if } k \le n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A non-decreasing function $f : [k] \to [n]$ is a *homomorphism* of π to σ if $\sigma(f(i)) > \sigma(f(j))$ for every $i, j \in [k]$ such that $i < j$ and $\pi(i) > \pi(j)$, that is, f preserves inversions. A *monomorphism* is a homomorphism that is injective.

Let $\mathrm{Hom}(\pi, \sigma)$ and $\mathrm{Mon}(\pi, \sigma)$ be the sets of homomorphisms and monomorphisms of π to σ, respectively, and let $\Lambda_{\mathrm{hom}}(\pi, \sigma)$ and $\Lambda_{\mathrm{mon}}(\pi, \sigma)$ denote the sizes of the respective sets. Note that $\mathrm{Occ}(\pi, \sigma) \subseteq \mathrm{Mon}(\pi, \sigma) \subseteq \mathrm{Hom}(\pi, \sigma)$. The *homomorphism density* t_{hom} and *monomorphism density* t_{mon} are defined as follows:

$$t_{\mathrm{mon}}(\pi, \sigma) = \begin{cases} \Lambda_{\mathrm{mon}}(\pi, \sigma)\binom{n}{k}^{-1} & \text{if } k \le n \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

$$t_{\mathrm{hom}}(\pi, \sigma) = \Lambda_{\mathrm{hom}}(\pi, \sigma)\binom{n+k-1}{k}^{-1}.$$

The three densities that we have just introduced are analogues of the induced subgraph density, homomorphism density and subgraph density for graphs studied in [8].

Let q be an integer and let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial indecomposable permutations of order at most q. We consider the following three vectors

$$\mathbf{t}^q(\sigma) = (t(\tau_1, \sigma), \ldots, t(\tau_r, \sigma)),$$
$$\mathbf{t}^q_{\mathrm{mon}}(\sigma) = (t_{\mathrm{mon}}(\tau_1, \sigma), \ldots, t_{\mathrm{mon}}(\tau_r, \sigma)), \quad \text{and}$$
$$\mathbf{t}^q_{\mathrm{hom}}(\sigma) = (t_{\mathrm{hom}}(\tau_1, \sigma), \ldots, t_{\mathrm{hom}}(\tau_r, \sigma)).$$

Our aim is to understand possible densities of subpermutations in large permutations. This leads to the following definitions, which reflect the possible asymptotic densities of the indecomposable permutations of order at most q in permutations:

$$T^q = \{\mathbf{v} \in \mathbb{R}^r \mid \exists (\sigma_n)_{n=1}^{\infty} \text{ such that } \mathbf{t}^q(\sigma_n) \to \mathbf{v} \text{ and } |\sigma_n| \to \infty\},$$
$$T^q_{\text{mon}} = \{\mathbf{v} \in \mathbb{R}^r \mid \exists (\sigma_n)_{n=1}^{\infty} \text{ such that } \mathbf{t}^q_{\text{mon}}(\sigma_n) \to \mathbf{v} \text{ and } |\sigma_n| \to \infty\}, \quad \text{and}$$
$$T^q_{\text{hom}} = \{\mathbf{v} \in \mathbb{R}^r \mid \exists (\sigma_n)_{n=1}^{\infty} \text{ such that } \mathbf{t}^q_{\text{hom}}(\sigma_n) \to \mathbf{v} \text{ and } |\sigma_n| \to \infty\}.$$

In Section 2.2, we will see that T^q and T^q_{mon} have another, simpler definition in language of permutons. Now we give three observations on how the sets T^q, T^q_{mon} and T^q_{hom} relate to each other.

Observation 3. *The sets T^q_{mon} and T^q_{hom} are equal for every $q \in \mathbb{N}$.*

Proof. Observe that for every fixed integer k,

$$\Lambda_{\text{hom}}(\tau, \sigma) - \Lambda_{\text{mon}}(\tau, \sigma) \leq \binom{k}{2} n^{k-1} = O(n^{k-1}),$$

for every σ of order n and τ of order k.

Hence, for every permutation τ and every real $\varepsilon > 0$ there exists n_0 such that $|t_{\text{mon}}(\tau, \sigma) - t_{\text{hom}}(\tau, \sigma)| < \varepsilon$ for every permutation σ with $|\sigma| > n_0$. The statement now follows. □

In view of Observation 3, we will discuss only T^q_{mon} in the rest of the paper.

Observation 4. *For every $q \in \mathbb{N}$, the set T^q_{mon} is closed.*

Proof. Consider a convergent sequence $(\mathbf{w}_n)_{n \in \mathbb{N}} \subseteq T^q_{\text{mon}}$ and let $\mathbf{w} = \lim_{n \to \infty} \mathbf{w}_n$. For each n, choose σ_n such that $\|\mathbf{t}^q_{\text{mon}}(\sigma_n) - \mathbf{w}_n\| \leq 1/n$. Observe that $\mathbf{t}^q_{\text{mon}}(\sigma_n)$ converges to $\mathbf{w}$. □

Observation 5. *The set T^q is a non-singular linear transformation of T^q_{mon} for every $q \in \mathbb{N}$.*

Proof. Note that $\Lambda_{\text{mon}}(\pi, \sigma) = \sum_{\pi' \in \mathcal{P}} \Lambda(\pi', \sigma)$, where $\mathcal{P}$ is a set of permutations π' of the same order as π such that the identity mapping is a monomorphism from π to π'. Consequently, $t_{\text{mon}}(\pi, \sigma) = \sum_{\pi' \in \mathcal{P}} t(\pi', \sigma)$. This gives that T^q_{mon} is a linear transformation of T^q. Observe that if we order $\tau_1, \ldots, \tau_r$ by the number of inversions, the coefficient matrix of the induced linear mapping is upper triangular with diagonal entries equal to 1. We conclude that the linear transformation of T^q is non-singular. □

2.2. Permutation limits

In this subsection, we survey the theory of permutation limits, which was introduced in [12,15] (a similar representation was used in [21]). We follow the terminology used in [17]. An infinite sequence $(\sigma_i)_{i \in \mathbb{N}}$ of permutations with $|\sigma_i| \to \infty$ is *convergent* if $t(\tau, \sigma_i)$ converges for every permutation $\tau \in \mathfrak{S}$. Observe that every sequence of permutations has a convergent subsequence. A convergent sequence can be associated with an analytic limit object, a *permuton*. A *permuton* is a probability measure Φ on the σ-algebra of Borel sets of the unit square $[0, 1]^2$ such that Φ has *uniform marginals*, i.e., $\Phi([\alpha, \beta] \times [0, 1]) = \Phi([0, 1] \times [\alpha, \beta]) = \beta - \alpha$ for every $0 \leq \alpha \leq \beta \leq 1$. We denote the set of all permutons by $\mathfrak{P}$.

Given a permuton Φ, a *Φ-random permutation of order n* is a permutation $\sigma_{\Phi,n}$ obtained in the following way. Sample n points $(x_1, y_1), \ldots, (x_n, y_n)$ in $[0, 1]^2$ at random with the distribution given by Φ. Note that the values of x_i are pairwise distinct with probability one and the same holds for the values of y_i. Let $i_1, \ldots, i_n \in [n]$ be such that $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$. Then the permutation $\sigma_{\Phi,n}$ is the unique bijective mapping from $[n]$ to $[n]$ satisfying that $\sigma_{\Phi,n}(j) < \sigma_{\Phi,n}(j')$ if and only if $y_{i_j} < y_{i_{j'}}$ for every $j, j' \in [n]$. Informally speaking, the values x_i determine the ordering of the points and the relative order of the values y_i determines the relative order of the elements of the permutation.
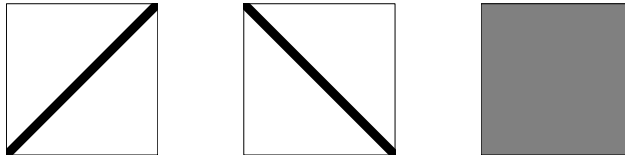
Fig. 1. The limits of sequences $\left(\pi_i^1\right)_{i\in\mathbb{N}}$, $\left(\pi_i^2\right)_{i\in\mathbb{N}}$, $\left(\pi_i^3\right)_{i\in\mathbb{N}}$.

If Φ is a permuton and σ is a permutation of order n, then $t(\sigma, \Phi)$ is the probability that a Φ-random permutation of order n is σ. We say that a permuton Φ is a *limit* of a convergent sequence of permutations $(\sigma_i)_{i\in\mathbb{N}}$ if

$$\lim_{i\to\infty} t(\tau, \sigma_i) = t(\tau, \Phi)$$

for every $\tau \in \mathfrak{S}$. Every convergent sequence of permutations has a limit and the permuton representing the limit of a convergent sequence of permutations is unique.

We now give some examples of the notions we have just defined (the corresponding permutons are depicted in Fig. 1). Let us consider a sequence $\left(\pi_i^1\right)_{i\in\mathbb{N}}$ such that π_i^1 is the identity permutation of order i, i.e., $\pi_i^1(k) = k$ for $k \in [i]$. This sequence is convergent and its limit is the permuton I with support $\{(x, x), x \in [0, 1]\}$ and measure uniformly distributed on its support. Similarly, the limit of a sequence $\left(\pi_i^2\right)_{i\in\mathbb{N}}$, where π_i^2 is the permutation of order i defined as $\pi_i^2(k) = i + 1 - k$ for $k \in [i]$, is the permuton with support $\{(x, 1 - x), x \in [0, 1]\}$ and measure uniformly distributed on its support.

Next, consider a sequence $(\pi_i^3)_{i\in\mathbb{N}}$ such that π_i^3 is a uniformly random permutation of order i. This sequence is convergent with probability one and its limit is the uniform probability measure on $[0, 1]^2$ with probability one.

Similarly to the subpermutation density, we can define the monomorphism density of a permutation τ in a permuton Φ as the probability that the identity mapping to a random Φ-permutation is a monomorphism of τ. Since we view permutons as representing large permutations, if we defined homomorphism densities in a natural way, they would coincide with monomorphism densities. So, we restrict our study to subpermutation densities and monomorphism densities in permutons. By analogy to the finite case, we define the vectors

$$\mathbf{t}^q(\Phi) = (t(\tau_1, \Phi), \ldots, t(\tau_r, \Phi)) \text{ and}$$
$$\mathbf{t}^q_{\mathrm{mon}}(\Phi) = (t_{\mathrm{mon}}(\tau_1, \Phi), \ldots, t_{\mathrm{mon}}(\tau_r, \Phi)),$$

where $q \in \mathbb{N}$ and $\{\tau_1, \ldots, \tau_r\}$ is the set of all non-trivial indecomposable permutations of order at most q.

If Φ is a permuton and σ_i is a Φ-random permutation of order i, then the sequence $(\sigma_i)_{i\in\mathbb{N}}$ is convergent with probability one and Φ is its limit. In particular, this means that for every finite set of permutations $\mathcal{P}$ and every $\varepsilon > 0$, there exists a permutation φ such that $|t(\pi, \Phi) - t(\pi, \varphi)| < \varepsilon$ for every $\pi \in \mathcal{P}$. This yields an alternative description of T^q as the set $\{\mathbf{t}^q(\Phi) \mid \Phi \in \mathfrak{P}\}$. Similarly, $T^q_{\mathrm{mon}} = \{\mathbf{t}^q_{\mathrm{mon}}(\Phi) \mid \Phi \in \mathfrak{P}\}$.

2.3. Permuton constructions

In this section we introduce constructions of *step-up* permutons and a *direct sum* of permutons, which we use in Section 3, and we derive formulae for densities of indecomposable subpermutations in the constructed permutons.

The step-up permutons are permutons with simple structure corresponding to a weighted permutation. They are defined as follows. Let σ be a permutation of order n and let $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}_+^n$ be such that $\sum_{i\in[n]} v_i \leq 1$, where $\mathbb{R}_+$ is the set of positive reals. The *step-up permuton* of σ and $\mathbf{v}$ is the permuton $\Phi_\sigma^{\mathbf{v}}$ such that the support of the measure $\Phi_\sigma^{\mathbf{v}}$ is formed by the segments between the points $(\sum_{j<i} v_j, \sum_{\sigma(j)<\sigma(i)} v_j)$ and $(\sum_{j\leq i} v_j, \sum_{\sigma(j)\leq\sigma(i)} v_j)$ for $i \in [n]$ and the segment between the
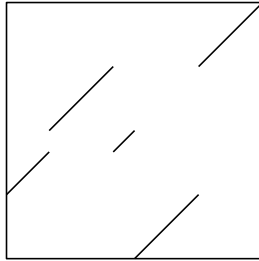
Fig. 2. The permuton $\Phi_\sigma^{\mathbf{v}}$ for $\sigma = (2, 4, 3, 1)$ and $\mathbf{v} = (1/6, 1/4, 1/12, 1/4)$.
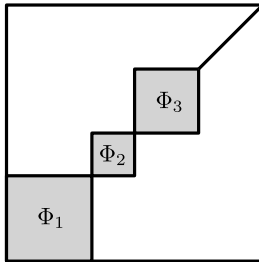

Fig. 3. The permuton $(1/3, \Phi_1) \oplus (1/6, \Phi_2) \oplus (1/4, \Phi_3)$.

points $(\sum_{j=1}^{n} v_j, \sum_{j=1}^{n} v_j)$ and $(1, 1)$. Note that this uniquely determines the permuton $\Phi_\sigma^{\mathbf{v}}$ because it must have uniform marginals. See Fig. 2 for an example.

We now define the direct sum of permutons with weights. For $k \in \mathbb{N}$, a sequence of permutons $(\Phi_i)_{i\in[k]}$ and $(p_i)_{i\in[k]} \in \mathbb{R}_+^k$ such that $\sum_{i\in[k]} p_i \leq 1$, the *direct sum of permutons* Φ_i with weights p_i is denoted by $\Phi = \bigoplus_{i\in[k]}(p_i, \Phi_i)$ and is defined as follows:

$$\Phi(S) = \sum_{i=1}^{k+1} p_i \Phi_i(\theta_i(S \cap C_i))$$

for every Borel set S, where $\Phi_{k+1} = I$ (the first permuton in Fig. 1), $p_{k+1} = 1 - \sum_{i=1}^{k} p_i$,

$$C_i = \left[\sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j \right]^2 \tag{3}$$

and θ_i is a map from C_i to $[0, 1]^2$ defined as

$$\theta_i((x, y)) = \left(\frac{x - \sum_{j=1}^{i-1} p_j}{p_i}, \frac{y - \sum_{j=1}^{i-1} p_j}{p_i} \right)$$

for every $i \in [k + 1]$. See Fig. 3 for an example.

For a permutation τ of order k, we call an ordered partition $\mathcal{P} = (P_1, \ldots, P_\ell)$ of $[k]$ a *τ-compressive partition* if

- P_i is an interval for every $i \in [\ell]$,
- $a < b$ for every $a \in P_i$ and $b \in P_j$ with $i < j$, and
- for every $i \in [\ell]$, there exists an integer c_i, such that $\tau(a) = a + c_i$ for every $a \in P_i$. (In particular, $\tau(P_i)$ is an interval for every $i \in [\ell]$.)

We denote the set of all τ-compressive partitions by $\mathcal{R}(\tau)$. Note that for every permutation τ, there exists at least one τ-compressive partition: the partition into singletons.

For a permutation τ of order k and a τ-compressive partition $\mathcal{P} = (P_1, \ldots, P_\ell)$, let $\tau/\mathcal{P}$ be a subpermutation of τ of order ℓ induced by $\{a_1, \ldots, a_\ell\}$ where $a_i \in P_i$ for every $i \in [\ell]$. Note that $\tau/\mathcal{P}$ is unique, in particular, it is independent of the choice of the elements a_i.

In other words, the permutation $\tau/\mathcal{P}$ is a permutation that can be obtained from τ by shrinking each interval P_i and its image into single points, without changing the relative order of the elements of the permutation. For instance, $\mathcal{P} = (\{1, 2\}, \{3\}, \{4, 5\})$ is a $(4, 5, 1, 2, 3)$-compressive partition, with $(4, 5, 1, 2, 3)/\mathcal{P} = (3, 1, 2)$.

Observation 6. *Let τ be a non-trivial indecomposable permutation of order k, σ a permutation of order $n \geq k$ and let $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ be such that $\sum_{i \in [n]} p_i \leq 1$. It follows that*

$$t(\tau, \Phi_\sigma^{\mathbf{p}}) = k! \sum_{\mathcal{P} \in \mathcal{R}(\tau)} \sum_{\psi \in \mathrm{Occ}(\tau/\mathcal{P}, \sigma)} \prod_{i=1}^{|\mathcal{P}|} \frac{p_{\psi(i)}^{|P_i|}}{|P_i|!}.$$

Proof. Consider k distinct points in the support of $\Phi_\sigma^{\mathbf{p}}$ and label them (x_i, y_i), $i \in [k]$, in such a way that $x_i < x_j$ if $i < j$ for every $i, j \in [k]$. Let $i_1 < i_2 < \cdots < i_{k'}$ be the indices of the segments of the support (numbered from left to right) that contain at least one of the points and let $\mathcal{P} = \{P_1, \ldots, P_{k'}\}$ be a partition of $[k]$ such that $i \in P_j$ if the point (x_i, y_i) lies on the i_jth segment.

Assume that the points yield the permutation τ. Then, $\mathcal{P}$ is a τ-compressive partition and the subpermutation σ' of σ induced by $\{i_1, \ldots, i_{k'}\}$ is $\tau/\mathcal{P}$. (Note that since τ is irreducible, none of the points lies on the $(k + 1)$st segment of the support of $\Phi_\sigma^{\mathbf{p}}$.) The converse is also true; fix k' segments with indices $1 \leq i_1 < \cdots < i_{k'} \leq n$ and a τ-compressive partition $\mathcal{P} = \{P_1, \ldots, P_{k'}\}$ such that the subpermutation σ' of σ induced by $\{i_1, \ldots, i_{k'}\}$ is $\tau/\mathcal{P}$. Then any choice of points $(x_1, y_1), \ldots, (x_k, y_k)$ where each (x_t, y_t) lies on the segment i_j such that $t \in P_{i_j}$ yields the permutation τ.

Note that k random points chosen based on the distribution $\Phi_\sigma^{\mathbf{p}}$ are distinct and lie in the support of $\Phi_\sigma^{\mathbf{p}}$ with probability one. The probability that they correspond to a given τ-compressive partition $\mathcal{P}$ and $\psi \in \mathrm{Occ}(\tau/\mathcal{P}, \sigma)$ is $k! \prod_{i=1}^{|\mathcal{P}|} \left(p_{\psi(i)}^{|P_i|}/|P_i|! \right)$. Since these events are disjoint for different pairs $(\mathcal{P}, \psi)$, the result follows. $\quad\square$

Observation 7. *Let τ be a non-trivial indecomposable permutation of order k and let m be a positive integer. Let $\Phi_1, \ldots, \Phi_m$ be permutons and let $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}_+^m$ be such that $\sum_{i \in [m]} x_i \leq 1$. The permuton $\Phi^{\mathbf{x}} = \bigoplus_{i \in [m]} (x_i, \Phi_i)$ satisfies*

$$t(\tau, \Phi^{\mathbf{x}}) = \sum_{i=1}^{m} x_i^k t(\tau, \Phi_i).$$

Proof. Observe that, if k random points chosen based on the distribution $\Phi^{\mathbf{x}}$ yield an indecomposable permutation, then all the points lie in the same C_i, for some $i \in [m]$ (where C_i is given by (3) in the definition of the direct sum of permutons). The probability that all the points are in C_i is x_i^k since $\Phi^{\mathbf{x}}(C_i) = x_i$. Conditioned on this event, the probability that the points yield τ is $t(\tau, \Phi_i)$. The result follows. $\quad\square$

Analogues of Observations 6 and 7 for densities of monomorphisms also hold.

2.4. Testing permutation parameters

A *permutation parameter* f is a function from $\mathfrak{S}$ to $\mathbb{R}$. A parameter f is *finitely forcible* if there exists a finite family of permutations $\mathcal{A}$ such that for every $\varepsilon > 0$ there exist an integer n_0 and a real $\delta > 0$ such that if σ and π are permutations of order at least n_0 satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$, then $|f(\sigma) - f(\pi)| < \varepsilon$. The set $\mathcal{A}$ is referred to as a *forcing family for f*.

A permutation parameter f is *finitely approximable* if for every $\varepsilon > 0$ there exist $\delta > 0$, an integer n_0 and a finite family of permutations $\mathcal{A}_\varepsilon$ such that if σ and π are permutations of order at least n_0 satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}_\varepsilon$, then $|f(\sigma) - f(\pi)| < \varepsilon$.

A permutation parameter f is *testable* if for every $\varepsilon > 0$ there exist an integer n_0 and $\tilde{f} : S_{n_0} \to \mathbb{R}$ such that for every permutation σ of order at least n_0, a randomly chosen subpermutation π of σ of size n_0 satisfies $|f(\sigma) - \tilde{f}(\pi)| < \varepsilon$ with probability at least $1 - \varepsilon$. The following was given in [14].

Lemma 8. *A bounded permutation parameter f is testable if and only if it is finitely approximable.*

3. Properties of the sets T^q and T^q_{mon}

In this section, we show that densities of non-trivial indecomposable permutations are mutually independent and, more generally, that T^q contains a ball. We start by considering the linear span of T^q.

Lemma 9. *For every $q \in \mathbb{N}$, $\mathrm{span}(T^q) = \mathbb{R}^r$, where r is the number of non-trivial indecomposable permutations of order at most q.*

Proof. Let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial indecomposable permutations of order at most q. For a contradiction, suppose that $\mathrm{span}(T^q)$ has dimension less than r, i.e., there exist reals $c_1, \ldots, c_r$, not all of which are zero, such that

$$\sum_{i=1}^{r} c_i v_i = 0$$

for every $(v_1, \ldots, v_r) \in \mathrm{span}(T^q)$. Therefore,

$$\sum_{i=1}^{r} c_i t(\tau_i, \Phi) = 0$$

for every permuton $\Phi \in \mathfrak{P}$.

Consider the permutations τ_i such that $c_i \neq 0$. Among these pick a τ_k of maximum order. Observation 6 yields that the following holds for $s = |\tau_k|$ and every $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{R}^s_+$ such that $\sum_{i=1}^{s} x_i \leq 1$:

$$\sum_{i=1}^{r} c_i t(\tau_i, \Phi^{\mathbf{x}}_{\tau_k}) = \sum_{i=1}^{r} c_i |\tau_i|! \sum_{\mathcal{P} \in \mathcal{R}(\tau_i)} \sum_{\psi \in \mathrm{Occ}(\tau_i/\mathcal{P}, \tau_k)} \prod_{j=1}^{|\mathcal{P}|} \frac{x_{\psi(j)}^{|P_j|}}{|P_j|!} = p(x_1, \ldots, x_s),$$

where p is a polynomial. We now argue that p is a polynomial of degree s (and therefore it is a non-zero polynomial). Clearly, the polynomial p has degree at most s. Since $\mathrm{Occ}(\tau', \tau_k) = \emptyset$ for every τ' of order s such that $\tau' \neq \tau_k$, $c_k s! x_1 x_2 \ldots x_s$ is the only term of p containing the monomial $x_1 x_2 \ldots x_s$ with nonzero coefficient. Therefore, there exists $\mathbf{x}$ such that $\sum_{i=1}^{r} c_i t(\tau_i, \Phi^{\mathbf{x}}_{\tau_k}) \neq 0$, which is a contradiction. $\quad\square$

Now, we will prove the main result of this section, Theorem 1. It shows that the interior of T^q is non-empty. Observation 5 yields the same conclusion for T^q_{mon}.

Proof of Theorem 1. Let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial indecomposable permutations of order at most q and let $\Phi_1, \ldots, \Phi_r$ be permutons such that $\{\mathbf{t}^q(\Phi_i) \mid i = 1, \ldots, r\}$ spans $\mathbb{R}^r$. Consider the matrix $V = (v_{i,j})_{i,j=1}^{r}$, where $v_{i,j} = t(\tau_j, \Phi_i)$. Observe that the matrix V is non-singular.

Consider a vector $\mathbf{x} = (x_1, \ldots, x_r) \in (0, r^{-1})^r$ and let $\Phi^{\mathbf{x}} = \bigoplus_{i \in [r]} (x_i, \Phi_i)$. By Observation 7, we have

$$t(\tau_j, \Phi^{\mathbf{x}}) = \sum_{i=1}^{r} x_i^{|\tau_j|} t(\tau_j, \Phi_i) = \sum_{i=1}^{t} x_i^{|\tau_j|} v_{i,j}.$$

Let Ψ be a map from $\mathbb{R}^r$ to $\mathbb{R}^r$ such that

$$\Psi_j(\mathbf{x}) = \sum_{i=1}^{r} x_i^{|\tau_j|} v_{i,j} \quad \text{for all } j \in [r].$$

Since we have $\Psi(\mathbf{x}) = \mathbf{t}^q(\Phi^{\mathbf{x}})$, we get that

$$\Psi((0, r^{-1})^r) = \{\Psi(\mathbf{x}) \mid \mathbf{x} \in (0, r^{-1})^r\} \subseteq T^q.$$

The Jacobian $\text{Jac}(\Psi)(\mathbf{x})$ is a polynomial in $x_1, \ldots, x_r$. Since for $x_1 = \cdots = x_r = 1$ we have

$$\text{Jac}(\Psi) = \det(v_{i,j} \cdot |\tau_j|)_{i,j=1}^{r} = \left(\prod_{j=1}^{r} |\tau_j| \right) \det V \neq 0,$$

$\text{Jac}(\Psi)$ is a non-zero polynomial.

Hence, there exists $\mathbf{x} \in (0, r^{-1})^r$ for which $\text{Jac}(\Psi)(\mathbf{x}) \neq 0$. Consequently, T^q contains a ball around $\mathbf{w}$ for $\mathbf{w} = \Psi(\mathbf{x})$. □

Theorem 1 implies that for every finite family $\mathcal{A}$ of indecomposable permutations, there exist permutons Φ and Φ' and an indecomposable permutation τ such that $t(\pi, \Phi) = t(\pi, \Phi')$ for every $\pi \in \mathcal{A}$ and $t(\tau, \Phi) \neq t(\tau, \Phi')$. The following lemma shows that an analogous statement holds for any finite family of permutations, not only for indecomposable permutations.

Lemma 10. *For every finite set of permutations $\mathcal{A} = \{\tau_1, \ldots, \tau_k\}$, there exist a permutation τ and permutons Φ and Φ' such that $t(\tau_i, \Phi) = t(\tau_i, \Phi')$ for every $i \in [k]$ and $t(\tau, \Phi) \neq t(\tau, \Phi')$.*

Proof. Let $\mathcal{B} = \{\pi_1, \ldots, \pi_{k+1}\}$ be a family of indecomposable permutations each of order n with $n > |\tau_i|$ for every $i \in [k]$, such that for every $\pi_j \in \mathcal{B}$, there is no $\ell < n$ satisfying $\pi_j(\ell+1) = \pi_j(\ell)+1$. We call permutations with this property *thorough*. By (1) in Section 2.1 a random permutation of order n is indecomposable with probability tending to one as n tends to infinity. Moreover, by (2) in Section 2.1 such permutations are thorough with probability bounded away from zero, because every simple permutation is thorough. Therefore, a family $\mathcal{B}$ of $k + 1$ indecomposable thorough permutations exists for n sufficiently large.

Let $\Phi^{\mathbf{u}} = \bigoplus_{i \in [k+1]} (u_i, \Phi_{\pi_i}^{\mathbf{n}})$ for $\mathbf{u} = (u_1, \ldots, u_{k+1}) \in (0, \frac{1}{k+1}]^{k+1}$ where $\mathbf{n} = \underbrace{(1/n, \ldots, 1/n)}_{n\times}$.

Observe that for a thorough permutation π, the partition into singletons is the only π-compressive partition. Hence, by Observations 6 and 7, $t(\pi_i, \Phi^{\mathbf{u}}) = n!(u_i/n)^n$ for every $i \in [k+1]$. For every $j \in [k]$, the function $\mathbf{u} \mapsto t(\tau_j, \Phi^{\mathbf{u}})$ is continuous for every j. We consider the continuous map Γ from $(0, 1/(k+1)]^{k+1}$ to $\mathbb{R}^k$ such that

$$\Gamma(\mathbf{u}) = (t(\tau_1, \Phi^{\mathbf{u}}), \ldots, t(\tau_k, \Phi^{\mathbf{u}})).$$

Now, consider any k-dimensional sphere in $(0, 1/(k+1)]^{k+1}$. The Borsuk–Ulam Theorem [7] yields the existence of two distinct points on its surface that are mapped by Γ to the same point in $[0, 1]^k$. Hence, there exist distinct $\mathbf{v} = (v_1, \ldots, v_{k+1})$ and $\mathbf{v}' = (v_1', \ldots, v_{k+1}')$ such that $t(\tau_j, \Phi^{\mathbf{v}}) = t(\tau_j, \Phi^{\mathbf{v}'})$ for every $j \in [k]$. However, if, say $v_i \neq v_i'$, then $t(\pi_i, \Phi^{\mathbf{v}}) = n!(v_i/n)^n \neq n!(v_i'/n)^n = t(\pi_i, \Phi^{\mathbf{v}'})$. Therefore, we may take $\tau = \pi_i$, $\Phi = \Phi^{\mathbf{v}}$, and $\Phi' = \Phi^{\mathbf{v}'}$. □

4. Non-forcible approximable parameter

For this section, we fix a sequence $(\tau_i)_{i \in \mathbb{N}}$ of permutations of strictly increasing orders that satisfies the following: For every $k > 1$, there exist permutons Φ_k and Φ_k' such that $t(\sigma, \Phi_k) = t(\sigma, \Phi_k')$ for every permutation σ of order at most $|\tau_{k-1}|$, and $t(\tau_k, \Phi_k) > t(\tau_k, \Phi_k')$. Such a sequence $(\tau_i)_{i \in \mathbb{N}}$ exists by Lemma 10. We fix such Φ_k and Φ_k' for all $k \in \mathbb{N}$ for the rest of this section. Let $\gamma_k = t(\tau_k, \Phi_k) - t(\tau_k, \Phi_k')$ for every $k \in \mathbb{N}$.

Let $(\alpha_i)_{i\in\mathbb{N}}$ be a sequence of positive reals satisfying $\sum_{i\in\mathbb{N}}\alpha_i < 1/2$ and $\sum_{i>k}\alpha_i < \alpha_k\gamma_k/4$ for every k. The main result of this section is that the permutation parameter

$$f_\bullet(\sigma) = \sum_{i\in\mathbb{N}}\alpha_i t(\tau_i, \sigma)$$

is finitely approximable but not finitely forcible.

Lemma 11. *The permutation parameter $f_\bullet$ is finitely approximable.*

Proof. Let $\varepsilon > 0$ be given. Since the sum $\sum_{i\in\mathbb{N}}\alpha_i$ converges, there exists k such that $\sum_{i>k}\alpha_i < \varepsilon/2$. Set $\mathcal{A} = \{\tau_1, \dots, \tau_k\}$ and $\delta = \varepsilon$. Consider two permutations σ and π that satisfy $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$. We obtain that

$$
\begin{aligned}
|f_\bullet(\sigma) - f_\bullet(\pi)| &= \left| \sum_{i\in\mathbb{N}}\alpha_i(t(\tau_i, \sigma) - t(\tau_i, \pi)) \right| \\
&\leq \sum_{i\in\mathbb{N}}\alpha_i |t(\tau_i, \sigma) - t(\tau_i, \pi)| \\
&< \sum_{i\leq k}\alpha_i\delta + \sum_{i>k}\alpha_i|t(\tau_i, \sigma) - t(\tau_i, \pi)| \\
&< \delta/2 + \sum_{i>k}\alpha_i \cdot 1 < \varepsilon.
\end{aligned}
$$

It follows that the parameter $f_\bullet$ is finitely approximable. □

In the following lemma, we show that $f_\bullet$ is not finitely forcible.

Lemma 12. *The permutation parameter $f_\bullet$ is not finitely forcible.*

Proof. Suppose that $f_\bullet$ is finitely forcible and that $\mathcal{A}$ is a forcing family for $f_\bullet$. Let τ_i, γ_i, Φ_i and Φ_i' be as in the definition of $f_\bullet$ and let k be such that maximum order of a permutation in $\mathcal{A}$ is at most $|\tau_{k-1}|$. We have $t(\rho, \Phi_k) = t(\rho, \Phi_k')$ for every $\rho \in \mathcal{A}$, $t(\tau_i, \Phi_k) = t(\tau_i, \Phi_k')$ for every $i < k$, and $t(\tau_k, \Phi_k) - t(\tau_k, \Phi_k') = \gamma_k$.

Let $\varepsilon = \alpha_k\gamma_k/4$. Let $\delta > 0$ be as in the definition of finite forcibility of $f_\bullet$. Without loss of generality we may assume that $\delta < \varepsilon$.

There exist a Φ_k-random permutation σ and a Φ_k'-random permutation σ' such that $|t(\rho, \sigma) - t(\rho, \sigma')| < \delta$ for every $\rho \in \mathcal{A}$, $|t(\tau_i, \sigma) - t(\tau_i, \sigma')| < \delta$ for every $i < k$ and $t(\tau_k, \sigma) - t(\tau_k, \sigma') > \gamma_k - \delta > 3\gamma_k/4$. Let us estimate the sum in the definition of $f_\bullet$ with the kth term missing.

$$
\begin{aligned}
\left| \sum_{i\in\mathbb{N}, i\neq k}\alpha_i(t(\tau_i, \sigma) - t(\tau_i, \sigma')) \right| \\
= \left| \sum_{i<k}\alpha_i\left(t(\tau_i, \sigma) - t(\tau_i, \sigma')\right) + \sum_{i>k}\alpha_i\left(t(\tau_i, \sigma) - t(\tau_i, \sigma')\right) \right| \\
< \sum_{i<k}\alpha_i\delta + \sum_{i>k}\alpha_i < \frac{\alpha_k\gamma_k}{8} + \frac{\alpha_k\gamma_k}{4} < \frac{\alpha_k\gamma_k}{2}.
\end{aligned}
$$

This leads to the following

$$
\begin{aligned}
|f_\bullet(\sigma) - f_\bullet(\sigma')| &= \left| \sum_{i\in\mathbb{N}}\alpha_i\left(t(\tau_i, \sigma) - t(\tau_i, \sigma')\right) \right| \\
&\geq \alpha_k\left(t(\tau_k, \sigma) - t(\tau_k, \sigma')\right) - \left| \sum_{i\in\mathbb{N}, i\neq k}\alpha_i\left(t(\tau_i, \sigma) - t(\tau_i, \sigma')\right) \right| \\
&> \frac{3}{4}\alpha_k\gamma_k - \frac{\alpha_k\gamma_k}{2} = \frac{\alpha_k\gamma_k}{4} = \varepsilon.
\end{aligned}
$$

This contradicts our assumption that $f_\bullet$ is finitely forcible. □

Lemmas 11 and 12 imply Theorem 2. Recall that, by Lemma 8 the testable bounded permutation parameters are precisely the finitely approximable ones.

Acknowledgements

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