AN INTERNATIONAL COMPARISON OF MATHEMATICAL TEXTBOOKS

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In this paper we report partial results of a study that compares Brazilian mathematics textbooks with textbooks from seven other countries (France, Germany, Italy, Japan, Portugal, Singapore and Spain) trying to assess whether they are suitably designed to support students on attaining a full range of mathematical abilities and to construct a view of mathematics as a science based on deduction. The analysis is based on the attributes of textbooks ranked by Howson (2013, p.653-654), including: mathematical coherence; clarity and accuracy of explanations; clarity on the presentation of kernels. Additionally, we consider one particularly important aspect: Do the textbooks allow the student the opportunity of experiencing abstract mathematical thinking? The partial results are presented through excerpts from textbooks. Those excerpts are examples of what may lead to misconceptions, misinterpretations, contradictions, or give the student an unclear idea of mathematical deduction.

Keywords: textbook comparison, mathematical accuracy, mathematical coherence, mathematical deduction

INTRODUCTION

As Howson (2013, p. 657) comments, mathematical textbooks “have played and will continue to play a vital role in mathematics education objectives, and not merely examination success.” In Brazil it is not different. As Borba (2013) points out, mentioning a government document, “textbooks are the main resources used by Brazilian teachers for planning classroom activities, so they have a very strong influence on what happens at school, and may affect students’ mathematical performance and understanding.”

Therefore, attention should be paid, for instance, to the accuracy and coherence of textbooks, as a criterion to assess whether they are suitably designed to support students on attaining a full range of mathematical abilities and constructing a view of mathematics as a science. And even more attention should be paid to the textbooks that are distributed by the government to the majority of the students in a country.

Since 1997 the Brazilian Ministry of Education has been providing textbook analysis and approval by specialists in different subjects (mathematics is one of them[1]) which are published in guides (Guia Nacional do Livro Didático). Each state-run school must choose

[1] From now on in this paper, for brevity, we just use ‘textbook’ to refer to ‘mathematics textbook’

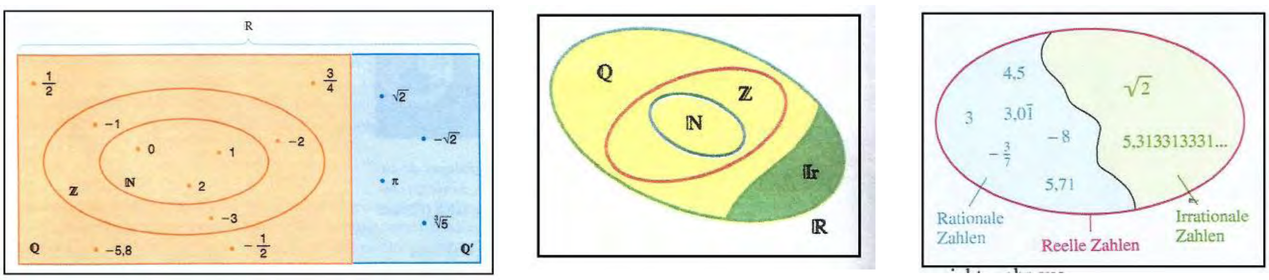
three titles among the approved ones. Then, for the next year, it receives from the government one of those three titles and each student receives a copy.[2]

Despite the analysis provided by the Brazilian government, one can still find aspects that can be criticized in the distributed textbooks mainly with respect to the attributes mentioned in the abstract. The question "Does any of those aspects to be criticized also appear in foreign textbooks?" was considered by the author and partial results comparing textbooks from Brazil with other countries are reported in this paper. All the textbooks under analysis are for 11 to 14-year-old-students, except the one from Portugal and Italy, which are for 4th grade students and for 15-year-old students, respectively.

SOME EXCERPTS CONFIRMING SIMILARITIES

The diagrams illustrating the numerical sets

The apparently naive diagrams illustrating the numerical sets that were found in Brazilian textbooks (Figure 1, (*a*) and (*b*)) as well as in German ones (Figure 1(*c*)) may contribute to the misconception that the irrational numbers are exceptions in the set of real numbers by suggesting that the sets of rational numbers and of the irrational numbers are of the same "size" and/or that there are less irrational numbers than rational ones.[3] Additionally, the author could find no discussion about the irrationality of $2\sqrt{2}$ or $\sqrt{2}+\sqrt{3}$ in the section usually called *computing with radicals* and which can be found in many textbooks.



(*a*) (*b*) (*c*)

Figure 1: Diagrams illustrating numerical sets in (a) Paiva (2009), p.45; (b) Dante (2009, Vol. 8), p.30; (c) Griesel et all (2007), p.96

The definition of an irrational number

A conflict in thinking[4] may occur with some definitions of irrational numbers found in the textbooks. The assertion "an irrational number is a number the decimal expansion of which is

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[2] A more detailed description of this Program can be found in Borba (2013)

[3] By the author´s experience working with students in their first semester at the University, it is not rare that Brazilian students by the end of High School have difficulties in giving six examples of irrational numbers besides $\sqrt{2}, \sqrt{3}, \pi$ and sometimes e or ϕ.

[4] "A conflict (or inconsistency) in thinking occurs when there are two (or more) distinct ways of interpreting data that are not coherent. (…) It occurs in particular when experience in one context leads to incidental properties that do not carry over to other concepts. For instance, in counting whole numbers, after each number there is a 'next' number and there are no numbers between one number and the next. In fractions,

neither finite nor periodic" (Dante (2013, Volume 8), p.26) suggests that all the numbers have a decimal expansion, which is not true: this property does not carry over to the complex numbers. An analogous remark fits to the following excerpt from a Singaporean textbook (Hong et all (2012), p.44): "The decimal 0.5 can be expressed as ½ and thus is also rational. However, there are numbers which cannot be written as exact fractions. Such numbers are called irrational numbers."

Such a remark also emphasizes one of the principles of the mathematics as a science, namely, being based on accuracy (of definitions, in this case).

One can compare both definitions with the one found in a German textbook (Schmid (1996), p.14) in which the verb "can" prevents the imaginary numbers from being irrational: "Numbers that can be represented by non-terminating non-periodic decimal fractions are called irrational numbers."

The definition of π

One can notice in some Brazilian textbooks a lack of presentation of kernels while dealing with the number π. They inform that π is an irrational number approximately equal to 3.14, but then treat approximations with equalities, what gives rise to an incoherence: π is an irrational number and $\pi=3.14$. This fact can be found in Dante (2013, Vol.8, p. 28), but also in a Japanese textbook (Isoda (2012), p.42)[5] where the formula "Circumference = Diameter x 3.14" appears. We also point out that this substitution constitutes an obstacle in the learning of irrational numbers, if we intend that the student realizes that an irrational number can be arbitrarily approximated by rational numbers.

The definition of angle

The textbooks under analysis deal with the concept and the definition of angle in three different ways: *i)* using as definition "angle is the union of two rays with the same origin" as a Brazilian textbook (Figure 2(*a*); *ii)* using as definition "angle is a portion of the plane which is limited by two rays with the same origin" as a Brazilian (Ferrari (2000), p.47), an Italian (Cassina & Bondonno (2012), p.670) a Portuguese (Rodrigues et all (2009), p.85) and a Singaporean (Figure 2(*b*)) textbooks; *iii)* presenting only the idea of angle with illustrations that suggest the second definition (Brazilian and French textbook – Figures 3(a) and (b) resp.).

Two rays with a common origin divide the plane into two sectors. Hence the assertion in *(i)* does not serve as definition of angle if one intends to *measure* this set in the way usually done in school, namely, measuring in fact the portion of the plane limited by those two rays. No student would be sure which portion of the plane in Figure 2(*a*) should he consider in measuring. One can notice in Figure 2(*b*) an attempt to indicate it in the picture. Nevertheless, in the next page of this Singaporean book, one can also find the phrase "An angle can be

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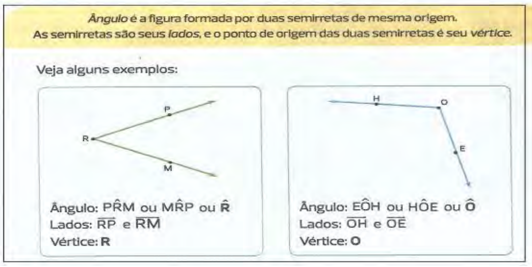
there is no 'next' number and between any two fractional numbers there is an infinite number of fractions" Tall (2014)

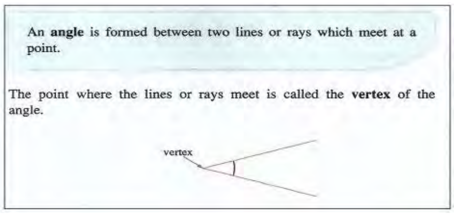
[5] This is a Japanese collection translated into Spanish.

named using the letters of the points on the rays, using the letter of the vertex in the middle", a notation that can be ambiguous if nothing more is said about the region.

In the Portuguese textbook (Rodrigues et al, 2009, p.85) one can find the definition indicated in *(ii)*, but also the following phrase: "Two rays with the same origin form an/one angle" which also makes this text ambiguous / incoherent[6].

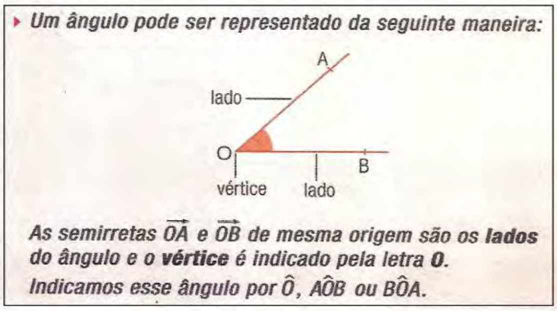


(*a*)

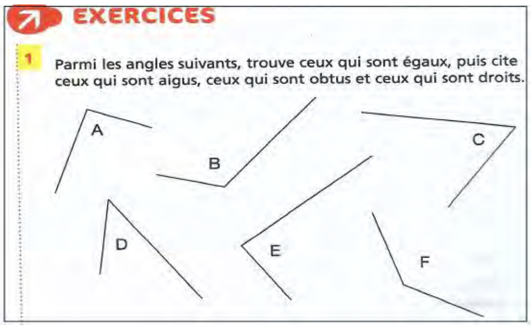


(*b*)

Figure 2: The definition of angle in (*a*) Dante (2013, Volume 7), p.168 and Figure 2(*b*) Hong et all (2012), p.189



(*a*)



(*b*)

Figure 3: The idea of angle in (*a*) Souza & Pataro (2012), p.206 and in (*b*) Peltier et all. (2009), p.58

Despite only presenting the idea of angle to the students, we remark that, in the teachers' guide of the French textbook (Peltier et al, 2009; Livre du Professeur, p.104) of the French textbook, the authors are very clear about the definition of angle: "The notion of angle is complex: it´s common that one identifies angle with its measure, forgetting that *angle is a geometric object.* It is a portion of the plane limited by two rays with the same origin, also called sometimes *angle sector*."

The opportunity of experiencing abstract mathematical thinking and arguing

Many properties of the numbers or of the operations with numbers are sometimes inadequately established, being based only on its validity in two or three instances. This strategy (inductive reasoning) does not give the student a clear idea of a fundamental principle of the mathematics as a science, namely, the deductive reasoning. This could be found in Brazilian (e.g. Souza, J. R. & Pataro, P.M. (2012), p.121) and Spanish textbooks

[6] "Duas semi-rectas com a mesma origem formam um ângulo". In Portuguese, the indefinite article "a" and the cardinal "one" are translated into the same word: *um.* Hence, depending on the interpretation of the "um" mentioned in the phrase, the text becomes either ambiguous or incoherent.

(Colera & Gaztelu (2010), p.68): "Compare the two following expressions and remark that one obtains the same result.

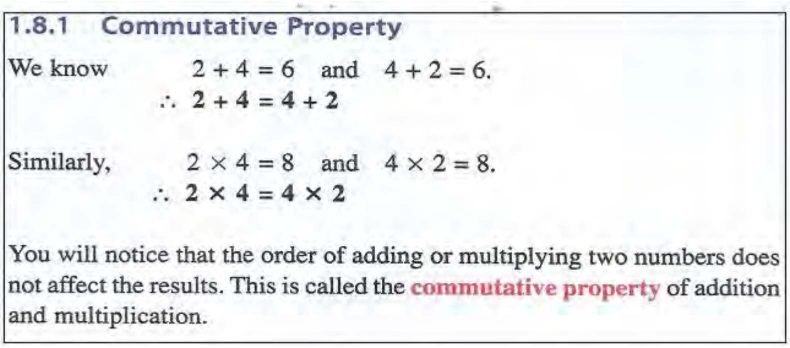
Example:

$$(2 \cdot 3)^3 = 6^3 = 6 \cdot 6 \cdot 6 = 216$$

$$2^3 \cdot 3^3 = (2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3) = 8 \cdot 27 = 216$$

The power of a product is equal to the product of the power of the factors."

Tall (2014) refers to *generic thinking* as "thinking of specific instances of a concept as representing the general idea itself." The excerpt from the Singaporean textbook (Figure 4) seems to be an attempt to make use of the generic thinking.



1.8.1 Commutative Property

We know $2 + 4 = 6$ and $4 + 2 = 6$.

$\therefore\ \mathbf{2 + 4 = 4 + 2}$

Similarly, $2 \times 4 = 8$ and $4 \times 2 = 8$.

$\therefore\ \mathbf{2 \times 4 = 4 \times 2}$

You will notice that the order of adding or multiplying two numbers does not affect the results. This is called the **commutative property** of addition and multiplication.

Figure 4: a slight allusion to generic thinking in Hong et all (2012), p.19

We do not disagree with that strategy, but remark that perhaps the students do not become aware of the necessity of the generality aspect of the given example without being previously exposed to assertions that are in some instances true and in some instances false. Such assertions were not found in the analysed textbooks.

Undoubtedly in a German textbook (Griesel et al (2007), p.66) the students are invited to the abstract mathematical thinking and it is offered to them an opportunity to be engaged in reasoning-and-proving. After one example, one can find the following reasoning:

We assert:

$$a^m . a^n = a^{m+n}$$

We justify:

$$a^m \cdot a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ factors} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ factors} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m+n\ factors}$$

We also remark that the above reasoning could be translated into colloquial language without loss of abstract mathematical thinking, i.e., it is not the symbolic language that we are emphasizing in the German textbook. There are many other instances in which we can exercise abstract mathematical thinking without making use of the symbolic language or of what Tall calls generic thinking.

CONCLUSIONS

All the topics mentioned in the previous section emerged from the analysis of Brazilian textbooks. Altogether, including others not mentioned here, they reveal, in the opinion of the author, a necessity of paying more attention to mathematics as a science at least in the textbooks distributed by the government in Brazil.

The fact that some of them (angle and irrational numbers, for instance) are also not adequately treated in other countries could indicate that this is effectively a difficult topic to be taught in school. In the opinion of the author, this is indeed the case for irrational numbers and angles, but for different reasons.

Finally, with respect to the foreign textbooks, it is obvious that the present analysis should not be taken as an overall evaluation of the textbooks here mentioned.

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