THREE DIMENSIONAL FINITE ELEMENT CODE FOR FINITE STRAIN WITH ISOTROPIC SOFTENING

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Abstract. *This paper aims to validate a finite element code with isotropic softening against experimental data for three different loading cases (uniaxial tension, equibiaxial tension and pure shear). The softening effect is introduced here to model the Mullins effect, which softens the material only when the current elongation is smaller than the maximum elongation. In previous works of the authors, softening parameters were tested only with analytical expressions for the three loading cases aforementioned. In the present work, a finite element code is tested for a thetrahedral element that is subjected to stress controlled boundary conditions, instead of the usual strain boundary conditions. Mullins effect is modeled by pseudo-hyperelasticity, which introduces a softening parameter in a traditional hyperelastic model and it can be applied to any hyperelastic constitutive model. Incompressibility is assumed. The results were in good agreement with the experimental data for the three study cases. At the end a more complex loading case is analyzed with the finite element code.*

Keywords: *hyperelasticity, softening, constitutive models, Mullins effect, finite element method.*

1 INTRODUCTION

Hyperelastic material have been modeled with finite element softwares for the past few decades (Dorfmann and Ogden, 2003), however, usually only the elastic response of these materials is considered. This paper aims to model dissipative effects as the Mullins effect for isotropic material considering isotropic softening. This effect is is characterized by a stiffness reduction for stretches smaller than those from the loading history.

Several attempts to simulate Mullins effect have been proposed over the years, some of them include anisotropy induced by the Mullins effect, however, this paper considers the material response isotropic before and after the loading has been applied to the material.

The material was modeled using Hoss-Marczak hyperelastic constitutive model that presented a good response for the multiaxial experimental tests performed by the authors in previous papers (Wrubleski and Marczak, 2014). This constitutive model was first presented by Hoss et al. (2011) together with a correlation to estimate the error for nonlinear functions.

The material modeled in this paper is a Silicone rubber that is commonly used in the aeronautic industry. The experimental data used in the present paper was performed by Machado et al. (2010). This experimental data was chosen due to its variety of experimental data in the literature, as Ogden (1984) mentioned, this three experimental data are sufficient to characterize a three dimensional hyperelastic material.

This paper is presents the mathematical formulation for the constitutive model and the analytic responses for uniaxial tension test, equibiaxial tension test and pure shear test in Section 2. The mathematical formulation for the finite element implementation and the boundary conditions applied the finite element model are presented in section 3. Section 4 presents the results and the conclusions are presented in Section 5.

2 CONSTITUTIVE MODELING

This section presents the constitutive models mathematical formulation. The finite element analysis was performed using Bower's (2009) finite element code. The hyperelastic constitutive model and softening capabilities were implemented in this code.

The following sections presents the hyperelastic and pseudo-hyperelastic formulations that were used in this paper to model the hyperelastic and the dissipative effects present in the studied material.

2.1 Hyperelasticity

Before presenting the specific equations for the constitutive models mentioned above, we are going to present the basic equations needed to deduce the constitutive relation from the strain energy function. First, it is commonly postulated that a strain energy function represents the stored energy in the material. This strain energy function, W_0, depends on the deformation gradient **F**. So the constitutive relation can be obtained as (Holzapfel, 2000):

$$\boldsymbol{\sigma}_0 = J^{-1}\frac{\partial W_0}{\partial \mathbf{F}}\mathbf{F}^T, \tag{1}$$

where σ_0 is the Cauchy stress and J is the Jacobian ($J = \texttt{det}(\mathbf{F})$ and $\texttt{det}$ indicates the determinant operator).

It is important to say that the incompressiblility is considered in this study, so that $\texttt{det}(\mathbf{F}) = 1$, what is equivalent to say that $\lambda_1\lambda_2\lambda_3 = 1$, and $J = 1$, where the λ_i are the main stretches. In this case we assume that the elongation $\lambda_3 = (\lambda_1\lambda_2)^{-1}$, so that the stress can be rewritten as:

$$\boldsymbol{\sigma}_0 = \frac{\partial W_0}{\partial \mathbf{F}}\mathbf{F}^T - p\mathbf{I}, \tag{2}$$

where p is the hydrostatic pressure. That allow us to write Eq. (2) in terms of the main elongations as:

$$\sigma_{0i} = \lambda_i \frac{\partial W_0}{\partial \lambda_i} - p = \lambda_i \frac{\partial W_0}{\partial \lambda_i} - \lambda_3 \frac{\partial W_0}{\partial \lambda_3}, \;\; i = 1, 2, \tag{3}$$

where the index i indicates the main directions and the stress in the direction $i = 3$ has to be known and is equal to the hydrostatic pressure. In the three load cases the stress in the direction 3 is null, so the hydrostatic pressure is equal to the stress in the direction three to turn it zero, being only necessary to replace λ_3 by its respective λ as will be seen as follow.

HMI

The HMI is based on the study of the terms of different hyperelastic models (Hoss et al., 2011). Its strain energy function is written in terms of the strain invariants and has the form:

$$W_0 = \frac{\alpha}{\beta}\left(1 - e^{-\beta(I_1-3)}\right) + \frac{\mu}{2b}\left(\left(1 + \frac{b(I_1-3)}{n}\right)^n - 1\right) + C_2 \, ln\left(\frac{1}{3}I_2\right), \tag{4}$$

where α, β, b, n, μ and C_2 are the material constants and I_1 and I_2 are the first and second strain invariants, respectively.

When assuming incompressibility the constitutive relation for uniaxial tension test can be writtes as:

$$\sigma_0 = 2\left(\lambda - \frac{1}{\lambda^2}\right)\left(\alpha\, e^{-\beta(I_1-3)} + \frac{\mu}{2}\left(1 + \frac{b(I_1-3)}{n}\right)^{n-1} + \frac{1}{\lambda}\frac{C_2}{I_2}\right), \tag{5}$$

where again σ_0 is the stress in the main direction 1 without the inclusion of the softening effect. The term depending on λ is related to the derivative of the invariants to λ.

The constitutive relation for equibiaxial load is given by:

$$\sigma_0 = 2\left(\lambda - \frac{1}{\lambda^5}\right)\left(\alpha\, e^{-\beta(I_1-3)} + \frac{\mu}{2}\left(1 + \frac{b(I_1-3)}{n}\right)^{n-1} + \lambda^2\frac{C_2}{I_2}\right), \tag{6}$$

this relation is valid for both directions 1 and 2. Finally, for pure shear:

$$\sigma_0 = 2\left(\lambda - \frac{1}{\lambda^3}\right)\left(\alpha\, e^{-\beta(I_1-3)} + \frac{\mu}{2}\left(1 + \frac{b(I_1-3)}{n}\right)^{n-1} + \frac{C_2}{I_2}\right). \tag{7}$$

For pure shear only the stress in direction 1 is shown, stress in direction 2 can be evaluated but for the purposes of this paper only the stress in direction 1 is relevant.

2.2 Pseudo-hyperelasticity

For the non conservative behavior we assume a strain energy function different from the one shown in the previous section. We assume that this pseudo-energy function has the following form (Ogden Roxburgh, 1999):

$$W(\lambda_1, \lambda_2, \eta) = \eta W_0(\lambda_1, \lambda_2) + \phi(\eta), \tag{8}$$

where η is the softening parameter and $\phi(\eta)$ is the dissipation potential. This formulation does not dissipate energy, instead, it stores energy in ϕ that does not generate stress, so the following equation must hold:

$$\frac{\partial W(\mathbf{F}, \eta)}{\partial \eta} = 0. \tag{9}$$

Equation (9) leads to:

$$\frac{\phi(\eta)}{\eta} = -W_0(\lambda_1, \lambda_2), \tag{10}$$

which is a important equation that will be used in the following subsection.

2.3 Softening parameter

The authors have proposed in previous papers (Wrubleski and Marczak, 2014), a softening parameter which produces good agreement with experimental data when fitted against analytic expressions. However this paper aims to show that it can be also applied to a finite element implementation. The softening parameter has the form:

$$\eta = 1 - \frac{1}{r}\left[\tanh\left(\frac{W_m - W_0}{m}\right)\right]^q, \tag{11}$$

where r, m and q are the material constants. The constants in this case must hold the conditions $r \geq 1$, $m > 0$ and $q > 0$.

Isolating W_0 in the above equation, and using the relation from Eq. (10 leads to:

$$\phi'(\eta) = -W_0 = -m\,\tanh^{-1}\left\{[-r(\eta-1)]^{1/q}\right\} - W_m. \tag{12}$$

One must be write the above integration as a function of the hyperbolic function as:

$$\begin{aligned}\phi(\eta) = \frac{m}{q+1}\eta(r(\eta-1))^{1/q}\,{}_2F_1\left(1, \frac{q+1}{2}; \frac{q+3}{2}; (r(\eta-1))^{2/q}\right) - W_m\,(\eta-1) + \\ -m\,(\eta-1)\tanh^{-1}\left(r(\eta-1)\right)^{1/q}.\end{aligned} \tag{13}$$

where ${}_2F_1$ is the hyperbolic function (Abramowitz, 1972). This equation can also be written it in the for of a summation as follows:

$$\begin{aligned}\phi(\eta) = \frac{m}{q+1}\eta(r(\eta-1))^{1/q}\sum_{n=0}^{\infty}\frac{q+1}{q+2n+1}(r(\eta-1))^{\frac{2n}{q}} - W_m\,(\eta-1) + \\ -m\,(\eta-1)\tanh^{-1}\left(r(\eta-1)\right)^{1/q}.\end{aligned} \tag{14}$$

The authors tested the summation form of the potential and good results were obtained with only three terms in the summation ($n = 3$). This can be used instead of the hyperbolic function for less computational effort and decrease processing time.

3 FINITE ELEMENT MODELS

The present work analyzed three different loading cases, and to do so the finite element models used are described bellow.

The mathematical formulation for this type of problem is the same presented by Holzapfel (2000), where a decomposition of the strain energy function is done as follows:

$$W_0(\mathbf{F}) = W_{0iso}(\overline{\mathbf{F}}) + W_{0vol}(J), \tag{15}$$

where $W_{0iso}(\overline{\mathbf{F}})$ is the isochoric and $W_{0vol}(J)$ is the volumetric contribution of the strain energy function, $\overline{\mathbf{F}} = J^{-1/3}\mathbf{F}$ which represents the non volumetric part of the deformation gradient. That means only the isochoric part of the stain energy function is assumed to be responsible for the stress softening.

The isochoric strain energy function is taken to be the HMI model (where all the stresses are assumed isochoric) and the volumetric part is taken to be:

$$W_{0vol} = \frac{k}{4}\left(J^2 - 1 - 2\,\mathtt{log}J\right), \tag{16}$$

where k is the compressiblility modulus (assumed to be 10^6 to impose incompressible behavior to the material.

3.1 Uniaxial tensile test

For this loading case the faces of the cube are fixed in $u(0, y, z)$, $u(x, 0, z)$ and $u(x, y, 0)$, in the directions x, y and z respectively. A pressure is then applied to the surface $x = 1$ in the direction x positive, and its value varies to obtain the same values of the experimental data. The deformed shape is shown bellow.

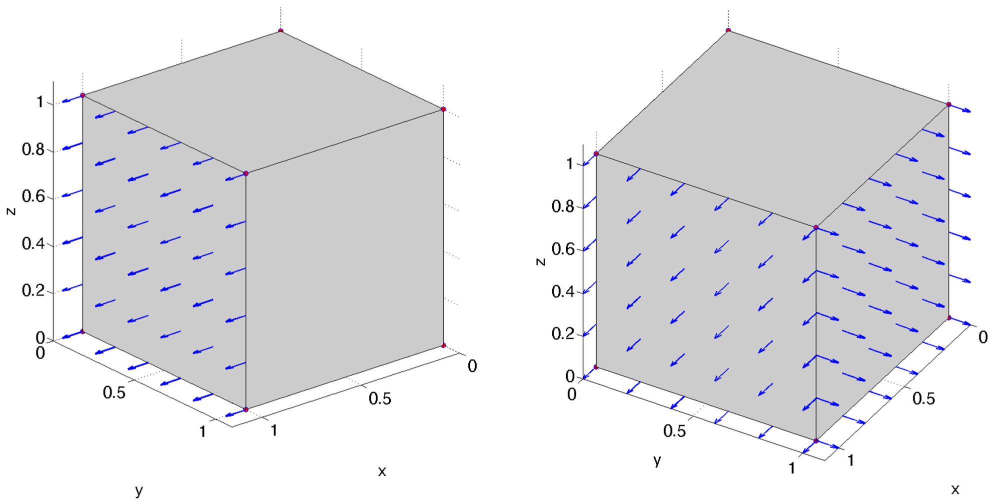


Figure 1: FEM model for: (a) uniaxial tension and pure shear, (b) equibiaxial tension.

3.2 Equibiaxial tensile test

The equibiaxial tension test also imposes the fixed boundary conditions in $u(0, y, z)$, $u(x, 0, z)$ and $u(x, y, 0)$, in the directions x, y and z respectively. Pressures are applied to the surfaces $x = 1$ and $y = 1$ in the directions $x$and y, respectively.

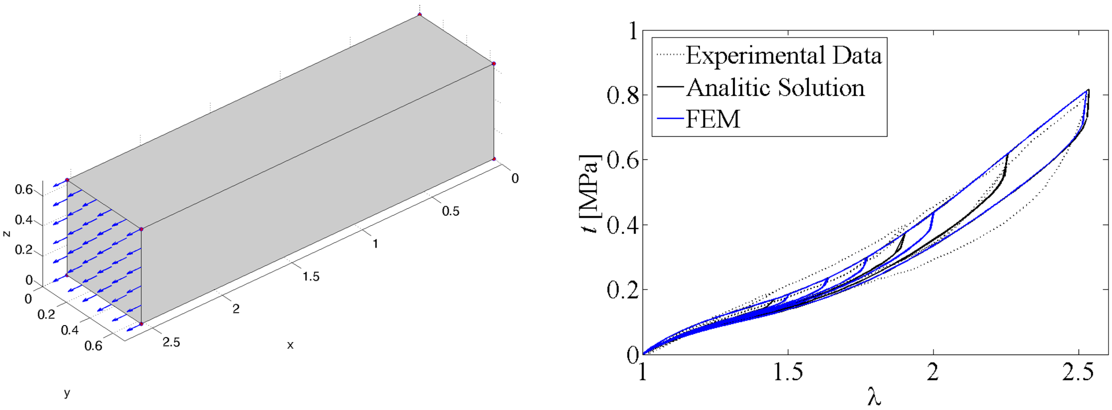
3.3 Pure shear test

As for the previous tests the boundary fixed conditions in $u(0, y, z)$, $u(x, 0, z)$ and $u(x, y, 0)$, in the directions x, y and z respectively. Also the surface $u(x, y, 1)$ is restricted in the direction z, in order to impose the pure shear loading case, and the pressure in applied in $x = 1$ in the direction x.

4 RESULTS

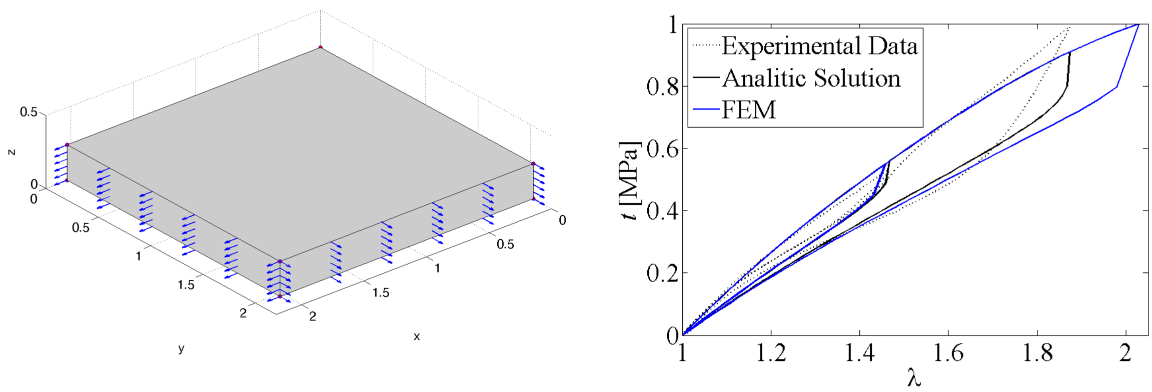
In this section the results for the three loading cases are shown. Figure 2(a) is the result for uniaxial tension case, Fig. 3(a) for equibiaxial tension case and Fig. 4(a) for pure shear. Figures 2(b), Fig. 3(b) and 4(b) show the experimental data, analytic and finite element results for t vs. λ.

Differently from the analytic solution which applies stretches and evaluate stresses, the finite element solution applies pressure as boundary conditions and measures stretches in the body. This is the reason why the results at maximum stretch are different from analytic and finite element solutions. They do, however, agree at intermediary stretches and thus they responses agree.



(a) b

Figure 2: Uniaxial tension results: (a) deformed finite element mesh, (b) tension vs. stretches.



(a) b

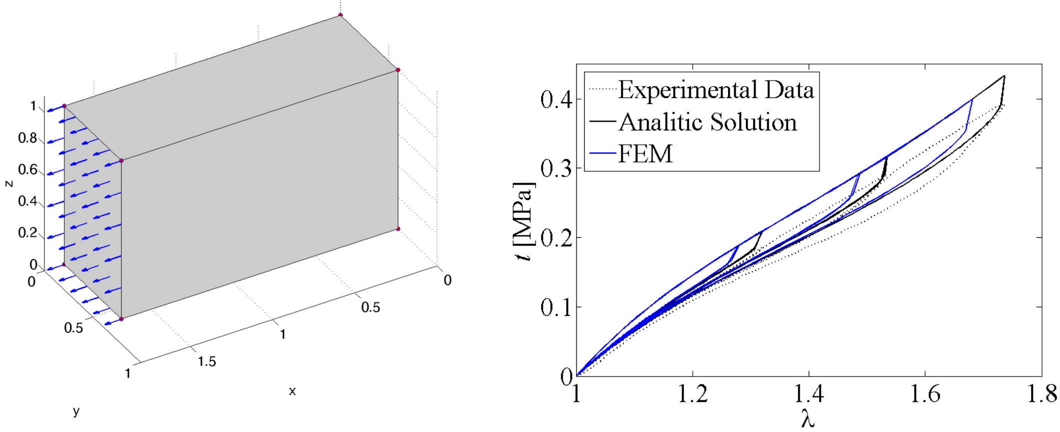
Figure 3: Equibiaxial tension results: (a) deformed finite element mesh, (b) tension vs. stretches.

5 CONCLUSIONS

The present work aimed the validation of a softening parameter so simulate Mullins effect with a finite element code against experimental data and analytic solution for three loading cases, uniaxial tension test, equibiaxial tension test and pure shear test. Previous papers from the authors only tested finite element code in uniaxial tension test with good results, however, it was not tested for other loading cases.

Through the present paper it is possible to assert that, for the material tested, the obtained results were in good agreement with the experimental data. It is known that Mullins effect induces anisotropy in the material as the amount of stress softening is different for each of the main stress directions. This effect is not taken into account and assumes the material is isotropic and remains isotropic after being loaded.

For future works the authors suggest to investigate more complex loading cases to be further investigated with this model.



(a) b

Figure 4: Pure shear results: (a) deformed finite element mesh, (b) tension vs. stretches.

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