On the consistency of the three-dimensional noncommutative supersymmetric Yang–Mills theory

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Abstract

We study the one-loop quantum corrections to the $U(N)$ noncommutative supersymmetric Yang–Mills theory in three space-time dimensions (NCSYM$_3$). We show that the cancellation of the dangerous UV/IR infrared divergences only takes place in the fundamental representation of the gauge group. Furthermore, in the one-loop approximation, the would be subleading UV and UV/IR infrared divergences are shown to vanish.

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Noncommutative supersymmetric models have a prominent place among the physically interesting field theories. Because supersymmetry favours the cancellation of dangerous divergences, they are the best candidates in a program to define consistent noncommutative field theories [1–5]. As part of a sequence of investigations devoted to this question [6–9], in this Letter we use the covariant superfield approach to study the noncommutative supersymmetric Yang–Mills model in $2+1$ spacetime dimensions. Based upon our previous experience with noncommutative supersymmetric QED$_3$ [7], we expect the absence of all one-loop divergences. More precisely, we shall show that the cancellation of the harmful (linear) UV/IR infrared divergences is achieved for the $U(N)$ group but only in the fundamental representation. We also verify the absence of the UV and UV/IR infrared logarithmic divergences for the entire theory in the one-loop approximation.

A general analysis of the divergence structure unveils that we must only care about the two-point vertex function of the spinor gauge superpotential. Indeed, although by power counting the three- and four-point vertex functions of this field are logarithmically divergent, they are in fact finite due to symmetric integration.

The action of the NCSYM$_3$ theory is [10]

$$S = \frac{1}{2g^2} \operatorname{Tr} \int d^5 z \, W^\alpha * W_\alpha, \tag{1}$$

where

$$W_\beta = \frac{1}{2} D^\alpha D_\beta \Gamma_\alpha - \frac{i}{2} [\Gamma^\alpha, D_\alpha \Gamma_\beta] - \frac{1}{6} [\Gamma^\alpha, \{\Gamma_\alpha, \Gamma_\beta\}], \tag{2}$$

is the superfield strength constructed from the spinor superpotential Γ_α which is a Lie algebra valued superfield, $\Gamma_\alpha(z) = \Gamma_\alpha^a(z) T^a$, $a = 1, \dots, N^2$. Hereafter, it is assumed that all commutators and anticommutators are Moyal ones. The above action is invariant under the $U(N)$ infinitesimal gauge transformation

$$\delta \Gamma_\alpha = D_\alpha K - i [\Gamma_\alpha, K], \tag{3}$$

where $K(z) = K^a(z) T^a$ is a supergauge parameter.

A generic covariant gauge (ξ) is fixed by adding to Eq. (1) the term

$$S_{\text{GF}} = -\frac{1}{4\xi g^2} \operatorname{Tr} \int d^5 z \, (D^\alpha \Gamma_\alpha) D^2 (D^\beta \Gamma_\beta). \tag{4}$$

One is also to include the Faddeev–Popov action

$$S_{\text{FP}} = \frac{1}{2g^2} \operatorname{Tr} \int d^5 z \, (c' D^\alpha D_\alpha c + i c' * D^\alpha [\Gamma_\alpha, c]), \tag{5}$$

where the ghost fields c and c' are also Lie algebra valued superfields.

Altogether, the resulting quadratic part of the action reads

$$S_2 = \frac{1}{2g^2} \operatorname{Tr} \int d^5 z \left[\frac{1}{2} \left(1 + \frac{1}{\xi}\right) \Gamma^\alpha \square \Gamma_\alpha - \frac{1}{2} \left(1 - \frac{1}{\xi}\right) \Gamma^\alpha i \partial_{\alpha\beta} D^2 \Gamma^\beta + c' D^\alpha D_\alpha c \right], \tag{6}$$

leading to the free propagators

$$\langle \Gamma^{\alpha a}(z_1) \Gamma^{\beta b}(z_2) \rangle = i g^2 \delta^{ab} \left[C^{\alpha\beta} \frac{1}{\square} (\xi + 1) - \frac{1}{\square^2} (\xi - 1) i \partial^{\alpha\beta} D^2 \right] \delta^5(z_1 - z_2) \tag{7}$$

and

$$\langle c'^a(z_1) c^b(z_2) \rangle = -2 i g^2 \delta^{ab} \frac{D^2}{\square} \delta^5(z_1 - z_2), \tag{8}$$

where $C^{\alpha\beta} = -C_{\alpha\beta}$ is the second-rank antisymmetric tensor with normalization $C^{12} = i$. Furthermore, we take

$$\operatorname{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \tag{9}$$

corresponding to the fundamental representation of the group generators. As known, at the classical level the use of this representation is mandatory to guarantee the closure of the gauge algebra [11–13].

The interacting part of the action is

$$\begin{aligned}
S_{\text{int}} = \frac{1}{g^2} \operatorname{Tr} \int d^5 z \Bigg\{ & -\frac{i}{4} D^\gamma D^\alpha \Gamma_\gamma * [\Gamma^\beta, D_\beta \Gamma_\alpha] - \frac{1}{12} D^\gamma D^\alpha \Gamma_\gamma * [\Gamma^\beta, \{\Gamma_\beta, \Gamma_\alpha\}] \\
& -\frac{1}{8} [\Gamma^\gamma, D_\gamma \Gamma^\alpha] * [\Gamma^\beta, D_\beta \Gamma_\alpha] + \frac{i}{12} [\Gamma^\gamma, D_\gamma \Gamma^\alpha] * [\Gamma^\beta, \{\Gamma_\beta, \Gamma_\alpha\}] \\
& + \frac{1}{72} [\Gamma^\gamma, \{\Gamma_\gamma, \Gamma^\alpha\}] * [\Gamma^\beta, \{\Gamma_\beta, \Gamma_\alpha\}] + \frac{i}{2} c' * D^\alpha [\Gamma_\alpha, c] \Bigg\}. \tag{10}
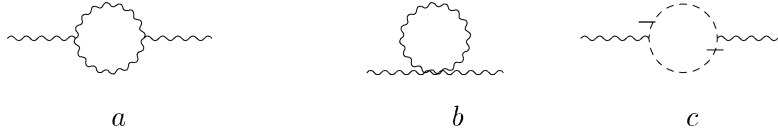\end{aligned}$$

Fig. 1. Diagrams contributing to the two-point function of the gauge superfield: gauge sector.

One then may convince oneself that the superficial degree of divergence for this model is

$$\omega = 2 - \frac{1}{2}V_c - 2V_0 - \frac{3}{2}V_1 - V_2 - \frac{1}{2}V_3 - \frac{1}{2}N_D, \tag{11}$$

where V_i, $i = 0, \ldots, 3$, is the number of pure gauge vertices involving i spinor supercovariant derivatives, V_c is the number of ghost vertices and N_D is the number of spinor derivatives moved to the external lines as consequence of the D-algebra transformations. By invoking the topological relation linking the numbers of loops, vertices and internal lines, one can convince oneself that all diagrams beyond the two-loop order are superficially finite. Hence, the theory under analysis is super-renormalizable. As we shall see, the same remains true after the introduction of matter superfields.

From Eq. (11) follows that the only linearly divergent graphs are those depicted in Fig. 1, where wavy and dashed lines represent gauge and ghost free propagators, respectively. They yield planar and nonplanar contributions. The linear UV divergences in the planar sectors are washed out by dimensional regularization, while the UV logarithmic ones vanish by symmetric integration. As for the nonplanar contributions, they are found to read

$$I_a = \xi \int \frac{d^3p}{(2\pi)^3}d^2\theta_1 \int \frac{d^3k}{(2\pi)^3} \frac{\cos(2k \wedge p)}{k^2} \Gamma^{b\beta}(-p, \theta_1)\Gamma_\beta^{b'}(p, \theta_1)A^{acb}A^{acb'} + \cdots, \tag{12a}$$

$$I_b = -\frac{1+\xi}{2} \int \frac{d^3p}{(2\pi)^3}d^2\theta_1 \int \frac{d^3k}{(2\pi)^3} \frac{\cos(2k \wedge p)}{k^2} \Gamma^{b\beta}(-p, \theta_1)\Gamma_\beta^{b'}(p, \theta_1)A^{abab'} + \cdots, \tag{12b}$$

$$I_c = \int \frac{d^3p}{(2\pi)^3}d^2\theta_1 \int \frac{d^3k}{(2\pi)^3} \frac{\cos(2k \wedge p)}{k^2} \Gamma^{b\beta}(-p, \theta_1)\Gamma_\beta^{b'}(p, \theta_1)A^{acb}A^{acb'} + \cdots, \tag{12c}$$

where

$$A^{a_1 a_2 \cdots a_n} = \text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}), \tag{13}$$

$k \wedge p \equiv \frac{1}{2}k_i \Theta^{ij} p_j$ and Θ^{ij} is the antisymmetric matrix characterizing the noncommutativity of the underlying spacetime.[2] The presence of harmful (linear) UV/IR infrared singularities in Eq. (12) should be noticed. They are accompanied by subleading (harmless) UV/IR infrared singularities as well as by finite terms, both of them indicated by dots. It is worth mentioning that these subleading UV/IR infrared singularities arise from integrals of the kind ($\tilde{p}^i \equiv \Theta^{ij} p_j$)

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^j e^{2ik \wedge p}}{k^4} = -\frac{i}{4\pi} \frac{\tilde{p}^j}{\sqrt{\tilde{p}^2}}, \tag{14}$$

which remains finite although depending on the direction as $\tilde{p} \to 0$.

From Eq. (12) we find that the linear UV/IR infrared divergences are cancelled if and only if

$$\text{Tr}(T^b T^a T^b T^c) = 2\,\text{Tr}(T^b T^d T^a)\,\text{Tr}(T^b T^d T^c), \tag{15}$$

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[2] We shall restrict to the case $\Theta^{0i} = 0$ to avoid unitarity problems.
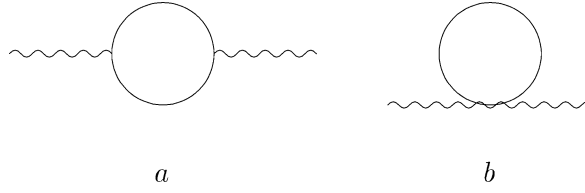
Fig. 2. Diagrams contributing to the two-point function of the gauge superfield: matter sector.

which is indeed verified in the fundamental representation of the gauge group. To see that this is the case, one may use the completeness relation for the $U(N)$ generators in the fundamental representation

$$\left(T^a\right)_{ij}\left(T^a\right)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk}. \tag{16}$$

When compared with the situation encountered in noncommutative supersymmetric QED$_3$ [7], one sees that Eq. (15) constitutes a new requirement arising from the non-Abelian character of the gauge group. Surprisingly, the same requirement secures the absence of quadratic UV/IR infrared divergences in the non-Abelian four-dimensional case, in spite of the strong differences in the structures of these theories. For a detailed discussion of the cancellation of divergences in the simple and extended supersymmetric noncommutative Yang–Mills theory in four spacetime dimensions we refer the reader to the paper in Ref. [9].

We shall next introduce matter minimally coupled to the gauge superfield. As consequence, the pure gauge action becomes augmented by the term

$$S_M = \operatorname{Tr}\int d^5z\left[\frac{1}{2}\left(D^\alpha\bar{\phi} + i\left[\bar{\phi}, \Gamma^\alpha\right]\right)*\left(D_\alpha\phi - i[\Gamma_\alpha, \phi]\right) + m\bar{\phi}\phi\right], \tag{17}$$

where $\phi(z) = \phi^a(z)T^a$. The superficial degree of divergence is now given by

$$\omega = 2 - \frac{1}{2}V_c - 2V_0 - \frac{3}{2}V_1 - V_2 - \frac{1}{2}V_3 - \frac{1}{2}E_\phi - \frac{1}{2}V_\phi^D - V_\phi^0 - \frac{1}{2}N_D, \tag{18}$$

where E_ϕ is the number of scalar legs, and V_ϕ^D (V_ϕ^0) is the number of vertices involving scalar superfields with one (none) spinor supercovariant derivatives. By power counting, the linearly divergent supergraphs involving matter vertices are those depicted in Fig. 2, where the continuous line represents the free matter field propagator

$$\left\langle\bar{\phi}^a(z_1)\phi^b(z_2)\right\rangle = -2i\delta^{ab}\frac{D^2 + m}{\square - m^2}\delta^5(z_1 - z_2). \tag{19}$$

After D-algebra transformations, the graph in Fig. 2a gives the contribution

$$\begin{aligned}
I_{2a} = -4\int\frac{d^3p}{(2\pi)^3}d^2\theta\int\frac{d^3k}{(2\pi)^3}&\frac{\cos(2k\wedge p)}{(k^2+m^2)[(k+p)^2+m^2]} \\
&\times\Bigg[-\left(k^2+m^2\right)C_{\alpha\beta}\Gamma^{\alpha b}(-p,\theta)\Gamma^{\beta b'}(p,\theta) + (k_{\alpha\beta} - mC_{\alpha\beta})\left[D^2\Gamma^\alpha(-p,\theta)\right]\Gamma^\beta(p,\theta) \\
&+\frac{1}{2}D^\gamma D^\alpha\Gamma_\alpha(k_{\gamma\beta} - mC_{\gamma\beta})\Gamma^\beta(p,\theta)\Bigg]\operatorname{Tr}\left(T^aT^cT^b\right)\operatorname{Tr}\left(T^aT^cT^{b'}\right) + \cdots, \tag{20}
\end{aligned}$$

whereas the graph in Fig. 2b yields

$$I_{2b} = -2\int\frac{d^3p}{(2\pi)^3}d^2\theta\int\frac{d^3k}{(2\pi)^3}\frac{\cos(2k\wedge p)}{(k+p)^2+m^2}C_{\alpha\beta}\Gamma^{\alpha b}(-p,\theta)\Gamma^{\beta b'}(p,\theta)\operatorname{Tr}\left(T^aT^bT^aT^{b'}\right) + \cdots. \tag{21}$$

We see again that the linear divergences are cancelled if and only if condition (15) is satisfied. As before, logarithmic divergences vanish due to symmetric momentum integration. Therefore, the completeness of the fundamental representation of the gauge group, expressed in Eq. (16), secures the one-loop finiteness of the NCSYM$_3$ also in the extended supersymmetric case. Furthermore, Eq. (18) implies in the absence of divergent corrections beyond two-loop order.

If, instead of adding Lie algebra valued matter superfields, we take matter to be a N component column vector, namely, in the fundamental representation, the corresponding part of the action reads

$$S_M = \int d^5 z \left[\frac{1}{2} \big(D^\alpha \bar{\phi} + i \bar{\phi} \Gamma^\alpha \big) * (D_\alpha \phi - i \Gamma_\alpha \phi) + m \bar{\phi} \phi \right]. \tag{22}$$

In this case, the supergraphs in Fig. 2 would be totally planar and hence finite in the framework of the dimensional regularization.

We conclude that NCSYM$_3$ is a consistent theory (without nonintegrable UV/IR infrared singularities) and a sound candidate for enlarging the class of finite noncommutative field theories proposed in [14].

Acknowledgements

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