Influence of fluctuations on spin systems with spatially isotropic competing interactions

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We consider spin systems with competing interactions isotropic with respect to the axes of a cubic lattice. A renormalization-group analysis carried to second order in $\epsilon = d - 4$ demonstrates that the critical behavior for the para-modulated phase transition is $2m$-component-like with $m = 3$ for a uniaxially modulated phase and $m = 4$ for a cubically modulated phase, in the same universality class as the m-component spin model with uniaxial competing interactions. Each case may describe different helical magnetic rare-earth alloys. Possible phase diagrams are proposed encompassing such compounds.

I. INTRODUCTION

Models with competing interactions that result in modulated phases have received much attention. Many magnetic materials undergo a phase transition from a paramagnetic phase to a phase with a modulated superstructure.[1] This superstructure can result from a competition between ferro and antiferromagnetic interactions. To study this problem, Garel and Pfeuty[2] used a model where m-component spins are coupled by nearest-neighbor ferromagnetic interactions and by next-nearest-neighbor antiferromagnetic interactions along a fixed direction. They found that the critical exponents belong to the same universality class as the O($2m$) Heisenberg model, where the $m = 1$ case is the usual ANNNI model which describes a para-sinusoidal transition having XY-model critical behavior, while $m = 2$ describes a para-planar helical transition with O(4) behavior.[3] Provided cubic anisotropy could be neglected, they concluded that the $m \geq 2$ cases have first-order transitions.

Although this analysis provides a explanation of the presence of modulated phases, the model has some defects. First, since its interactions are anisotropic, it does not have the possibility of modulated order in more than one direction. Indeed, in the analogous structurally modulated phases observed in binary alloys[4] we would normally expect the underlying couplings between the structural units to be isotropic. In order to repair this defect, Upton and Yeomans[5] introduced an Ising model with isotropic competing interactions. Within mean-field theory, it has a rich phase diagram with a paramagnetic phase, a ferromagnetic phase, and many modulated phases.[5] They found that the para-ferromagnetic Ising-like phase transition line meets a para-modulated continuous transition locus at a Lifshitz point.[5]

In this paper we allow for fluctuations by renormalization-group theory and show that the para-modulated phase transition in the Upton-Yeomans model is in the universality class of the $2m$-component spin model with cubic anisotropy between m components where $2m = 6$ for uniaxially-modulated phases while $2m = 8$ for cubically modulated phases. The $2m$-component model was studied previously by Mukamel and Krinsky.[6] They calculated fixed points and critical exponents to second order in the $d = 4 - \epsilon$ expansion. Using their results, we can predict the critical behavior for the paramodulated phase transitions. Recently Dawson, Walker, and Berera[7] studied the uniaxially modulated case by Monte Carlo simulation and suggested that the transition would be first order. For applications to structural systems, our analysis agrees with their results since the physical parameters prove not to be in the domain of attraction of the appropriate fixed point. In addition, we demonstrate similar behavior for cubical modulation. For applications to magnetic systems, however, we find that there is a possibility of a continuous transition.

Another defect in the Garel and Pfeuty model is the absence of cubic anisotropy between the m spin components. For applications to rare-earth compounds, for example, we should certainly allow for such a contribution since it is known that these materials are typically cubic ferromagnets.[8] It turns out that the inclusion of these terms changes the critical behavior. First, we show that the para-modulated phase transition present in the m-component spin model with uniaxial competing interactions and easy axes along m directions is in the same universality class as para-planar, para-uniaxial, and para-cubic modulated phase transitions occurring in the Ising model with isotropic competing interactions for $m = 2$, 3, and 4, respectively. Then, using the Mukamel and Krinsky[6] results, we present critical exponents for each case. It transpires that, owing to the cubic terms, the $m = 2,3,4$ systems can exhibit continuous transitions *if* the physical parameters are in the domain of attraction of the stable fixed point.

There has recently been a growing interest in helimagnets. It is well known[9] that the rare-earth metal erbium exhibits a uniaxial modulated structure which can be described by the XY model ($2m = 2$). Other materials such as Tb[10,11] Dy,[12,13] and Ho,[14] display a sinusoidal spin modulation which can be described by a four-component model ($2m = 4$). The nature of the para-modulated phase transition is not well known yet, since some experiments indicate continuous transitions,[10,12,14] while others indicate weak first-order transitions[11,13] for both Tb and Dy. Furthermore, the exponents obtained[14] for Ho are different from those[10,12] for Tb and Dy. We find that the theoretical exponents agree with the experimental ones obtained for Ho metal. We also suggest that the physical parameters related to Tb and Dy lie *outside* the domain of the stable fixed point, so yielding first-order transitions consistent with early experiments.[11,13]

Unfortunately there are no known physical examples of the $m=3,4$ cases. We propose that both cases may be realized by binary rare-earth alloys with orthogonal spin planes and uniaxial competing interactions when the wave vectors associated to each metal exhibit the same value. We present possible phase diagrams for different regions of the space of model parameters.

Besides para-modulated transitions, models with competing interactions also display a para-ferromagnetic transition when the antiferromagnetic interaction dominates. Mean-field analysis of both n-component spin models with uniaxial competing interactions and Ising models with isotropic competing interactions indicates[3,5] that the para-, ferro-, and modulated phases meet at a Lifshitz point. Now renormalization-group analysis up to second order in $\epsilon=4.5-d$ expansion, performed for the n-component spin model with anisotropic competing interactions, predicts a continuous transition at the Lifshitz point if no cubic terms are considered.[15]

Since we are interested in applications to helimagnets, where the cubic terms can be relevant, we have included such contributions. We find that for $n \gtrsim 4$ and parameters $u > v$ (see below) the Lifshitz point behavior is controlled by the cubic fixed point. We have computed new exponents possibly relevant to the rare-earth compounds. For $u < v$, since the stable fixed point is not achievable, the three phases now meet at an ordinary triple point.

The study of the Lifshitz point for the isotropic case is more complex. First, since the para-ferromagnetic transition is continuous while the para-modulated transition is first order, one might anticipate that the supposed "Liftshitz point" is actually a critical endpoint. But because this result is based on the assumption that the wave vector $\mathbf{q}_c$ associated with the modulation is not too small we cannot be sure that this result is still valid when $\mathbf{q}_c \simeq 0$. In order to understand this region we employ renormalization-group theory up to second order in an $\bar{\epsilon}=8-d$ expansion. We find that the usual stable fixed point approaches the unstable Gaussian point as one lowers d from 8. We propose that higher-order contributions might make this fixed point enter the unstable region for $\bar{\epsilon}\simeq 4$ indicating a first-order transition. For $d \simeq 8$ dimensions, where a continuous para-modulated phase transition would be expected, a normal Lifshitz point should be observed. Leaving this point aside one has, as usual,[16] two critical lines described by

$$t_F(p)=(T_c^F-T_L)/T_L \sim A_F p^{1/\phi_p} , \tag{1.1}$$

$$t_M(p)=(T_c^M-T_L)/T_L \sim A_M |p|^{1/\phi_p} , \tag{1.2}$$

where $t=(T-T_L)/T_L$. These lines are defined by the zeros of $r+pq^2+q^4$ when $p>0$ (in this case $q=0$) and $p<0$ (in this case $q=q_c$ and $q_c \to 0$), respectively. Also, T_L given by $r=p=0$, T_c^F given by $p>0$, $r=0$, and T_c^M given by $p<0$, $r=r_c$ ($q=q_c$) are the critical temperatures of the Lifshitz point and, the ferromagnetic and modulated phases, respectively. The crossover exponent is given by

$$\phi_p=\tfrac{1}{2}+\tfrac{1}{24}\bar{\epsilon}+O(\bar{\epsilon}^2) . \tag{1.3}$$

For the anisotropic model, Mukamel and Luban[16] showed that A_F/A_M is a universal quantity. Assuming spherical symmetry within momentum space, they also suggested that for the isotropic case no long-range order should exist since A_M cannot be defined. However, the isotropic model proposed by Upton and Yeomans[5] does not have spherical symmetry owing to the presence of two kinds of antiferromagnetic interactions. In this case, it is possible to define A_M but we show that A_F/A_M is then *not* a universal ratio.

II. THE BEHAVIOR OF MODULATED PHASES

The Ising model with isotropic competing interactions introduced by Upton and Yeomans[5] may be described as follows. Ising spins s_i reside on the sites of a cubic lattice and interact through nearest-neighbor interactions of strength J, next-nearest-neighbor couplings along the cubic axes of strength $-\kappa_1 J$, and next-nearest-neighbor interactions across the face diagonals, $-\kappa_2 J$. On going to a continuous-spin representation by adding a weighting term for each spin, we obtain the effective Hamiltonian

$$H=-\tfrac{1}{2}\sum_{\mathbf{q}} s(\mathbf{q})u_2(\mathbf{q})s(-\mathbf{q}) - u_4 \sum_{\{\mathbf{q}_i\}} s(\mathbf{q}_1)s(\mathbf{q}_2)s(\mathbf{q}_3)s(\mathbf{q}_4)\delta\left[\sum_i \mathbf{q}_i\right] , \tag{2.1}$$

where, as usual, we have

$$u_2(\mathbf{q})=k_B T - J(\mathbf{q}) , \tag{2.2}$$

and for this model,

$$\tfrac{1}{2}J(\mathbf{q})=J[(\cos q_x+\cos q_y+\cos q_z) -\kappa_1(\cos 2q_x+\cos 2q_y+\cos 2q_z) -2\kappa_2(\cos q_x\cos q_y+\cos q_y\cos q_z +\cos q_x\cos q_z)] . \tag{2.3}$$

The special feature of the model is that when $1-4\kappa_1-4\kappa_2<0$, the Fourier transform $J(\mathbf{q})$ has maxima at $\mathbf{q}=\mathbf{q}_c$, where $\mathbf{q}_c$ points along various special directions. For $\kappa_1<\frac{1}{2}\kappa_2$, $\mathbf{q}_c$ is oriented along the x, y, or z axes; this leads to a uniaxially modulated phase. Otherwise, when $\kappa_1>\frac{1}{2}\kappa_2$, $\mathbf{q}_c$ points along any diagonal direction. This leads to a cubically modulated region. Explicitly one has

$$\mathbf{q}_c \equiv (\pm q_c, 0, 0) \text{ (uniaxially modulated)}$$
$$\equiv (0, \pm q_c, 0)$$
$$\equiv (0, 0, \pm q_c) , \tag{2.4}$$

$$\mathbf{q}_c \equiv (\pm q_c, \pm q_c, \pm q_c) \text{ (cubic modulated)}$$
$$\equiv (\pm q_c, \mp q_c, \mp q_c)$$
$$\equiv (\mp q_c, \mp q_c, \pm q_c)$$
$$\equiv (\mp q_c, \pm q_c, \mp q_c) . \tag{2.5}$$

Evidently, for the modulated phases, the fluctuations with $\mathbf{q}$ close to $\mathbf{q}_c$ will play an important role. To take them into account, we introduce a new Brillouin zone given by $|q_i| \le \frac{1}{2} q_c$ with $i = x,y,z$. Then, as usual,[2,3] we fold the original zone into the new one by (i) dividing the sums over $\mathbf{q}$ into pieces running from

$$\tfrac{1}{2}(l_i - 1) q_c \le |q_i| \le \tfrac{1}{2} l_i q_c ,$$

with $i = x,y,z$ and $l_i = 1, 2, \cdots$; (ii) shifting all these sums to run from $|q_i| \le \frac{1}{2} q_c$; (iii) defining new spin variables by

$$s(\mathbf{q} \pm \mathbf{q}_c) = \frac{1}{\sqrt{2}} [\sigma_1^{(i)}(\mathbf{q}) \pm i \sigma_2^{(i)}(\mathbf{q})] \quad i = 1, 2, \ldots, m , \tag{2.6}$$

where m is the number of possible directions of $\mathbf{q}_c$ and we have integrated out all noncritical modes.

After these steps, the initial Hamiltonian (2.1) maps into a new one corresponding to a $2m$-component model, namely,

$$H = -\tfrac{1}{2} \sum_{\mathbf{q}}{}' \sum_{i=1}^{m} u_2'(\mathbf{q}) [\sigma_1^{(i)}(\mathbf{q}) \sigma_1^{(i)}(-\mathbf{q}) + \sigma_2^{(i)}(\mathbf{q}) \sigma_2^{(i)}(-\mathbf{q})] - u \sum_{\{\mathbf{q}_i\}}{}' \sum_{i=1}^{m} (\sigma_1^{(i)^2} + \sigma_2^{(i)^2})^2 - v \sum_{\{\mathbf{q}_i\}}{}' \sum_{j \ne 1}^{m} (\sigma_1^{(i)^2} + \sigma_2^{(i)^2})(\sigma_1^{(j)^2} + \sigma_2^{(j)^2}) , \tag{2.7}$$

where $u = 3u_4$ and $v = 6u$, while the sum $\sum'$ runs over the reduced Brillouin zone $|\mathbf{q}| \le \frac{1}{2} q_c$ and one has

$$u_2'(\mathbf{q}=0) = 1 - \frac{2J}{k_B T} [3\kappa_1 + \tfrac{3}{8}(\kappa_1 + \kappa_2)^{-1}] \tag{2.8}$$

for $\kappa_1 > \frac{1}{2}\kappa_2$ (modulation along any diagonal direction with $m = 4$) and

$$u_2'(\mathbf{q}=0) = 1 - \frac{2J}{k_B T} \left[2 - \kappa_1 - 2\kappa_2 + \frac{1}{8} \frac{(1 - 4\kappa_2)^2}{\kappa_1{}^2} \right] \tag{2.9}$$

for $\kappa_1 < \frac{1}{2}\kappa_2$ (modulation along any cubic axis direction with $m = 3$).

For the ANNNI model ($m = 1$), it was shown that the disordered-modulated phase transition is XY-model-like.[3] Let us consider the following model: Ising spins with nearest-neighbor ferromagnetic couplings J, next-nearest-neighbor antiferromagnetic interactions along the cubic axes in two fixed directions (x and y) of strength $-\kappa_1 J$, and next-nearest-neighbor antiferromagnetic interactions across the face diagonals in the plane (001) of strength $-\kappa_2 J$. This model will display uniaxial modulations along the two directions

$$(\pm q_c, 0, 0) \text{ and } (0, \pm q_c, 0) \tag{2.10}$$

if $\kappa_1 < \frac{1}{2}\kappa_2$, or along the four in-plane directions

$$(\pm q_c, \pm q_c, 0) \text{ and } (\mp q_c, \pm q_c, 0) \tag{2.11}$$

if $\kappa_1 > \frac{1}{2}\kappa_2$. Using the same process as above, we can show that both kinds of disorder-modulated phase transition can be described by the Hamiltonian (2.7) with $m = 2$. This result can be understood from the fact that each m-component modulated phase is characterized by an amplitude and a phase; consequently one needs $2m$ scalar components for a proper description of the order.

Mukamel and Krinsky,[6] studied the $2m$-component spin model using renormalization-group techniques carried to second order in $\epsilon = 4 - d$. They obtained the fixed points

$$u^* = \epsilon / 40 K_4, \quad v^* = 0 , \tag{2.12}$$

$$u^* = \epsilon / 8(m+4) K_4, \quad v^* = u^* , \tag{2.13}$$

$$u^* = (m-1)\epsilon / 8(5m-4) K_4, \quad v^* = u^* / (m-1) , \tag{2.14}$$

where $K_4 = (8\pi^2)^{-1}$ and $u = u_1$ and $v = \frac{1}{2} u_3$ in the notation of Mukamel and Krinsky.[6] The first (XY like) fixed point is stable only if $v = 0$ (ANNNI model), the second (symmetric) fixed point is stable for $2m < 4$, while the last (asymmetric) fixed point is stable for $2m > 4$ (uniaxial and cubical modulation). The $2m = 4$ case has a stable asymmetric fixed point given by[6]

$$u^* = \frac{\epsilon}{48 K_d} + \frac{7\epsilon^2}{384 K_4}, \quad v^* = \frac{\epsilon}{48 K_d} - \frac{\epsilon^2}{384 K_4} , \tag{2.15}$$

with $K_d = 2^{-(d-1)} \pi^{-d/2} [\Gamma(d/2)]^{-1}$.

Now the stable fixed point are accessible only if initial physical parameters in (2.7) satisfy $u > v$. Then the continuous transition for $m = 1$, 2, 3, and 4 exhibits the critical exponents listed in Table I. However, recall that in (2.7) we have, by computation, $u = v/2$ initially.

Note that the Hamiltonian (2.7) is in the same universality class as that which describes m-component magnetic helical structures when modulation is imposed in one fixed direction (the m-component spin model with aniso-

TABLE I. Critical exponents for the $2m$-component model for $2m = 2$, 4, 6, and 8 and $d = 3$. The values are estimated from second-order $\epsilon = d - 4$ expansions (Ref. 6). Experimental data for holmium, corresponding to $2m = 4$, yield $\beta = 0.39 \pm 0.04$ (Ref. 14).

Number of components	Exponents β	ν
$2m = 2$	0.365	0.655
$2m = 4$	0.39	0.698
$2m = 6$	0.38	0.69
$2m = 8$	0.352	0.686

tropic competing interactions) and when a cubic anisotropy term of strength v is introduced among the m components. In contrast to our special isotropic case, v can now assume any initial value relative to u. This model has been studied previously[2] for $m=1,2,3$ and $u=v$. For $m=1$, since v is absent, the usual XY-like behavior is expected. This case has many physical realizations.[1] One is the modulated phase present[9] in erbium. The $m=2$ case is more complex. It can be realized in two ways either as two-component spins displaying modulation in one fixed direction perpendicular to the spin plane[2,3] or as Ising spins with competing interactions in two fixed directions. [See (2.10) and (2.11).] In the first case, a continuous transition is to be expected when $u>v$; otherwise, if $u<v$ the renormalization-group flow diagram shows that the stable fixed point (2.15) is not accessible and the transition should be first order. In the second case, our mapping leads to $2u=v$ [see Eq. (2.7)], the stable fixed point (2.15) cannot be reached, and the transition in this case should also be first order. Note, otherwise, that the critical behavior predicted by the fixed point is the same as predicted by the unstable (but attainable point) (2.13); this indicates a weak first-order transition. Both models exhibit helical phases and can be used to explain the weak first-order transition seen experimentally in the rare-earth metals[11,13] Tb and Dy and the continuous transition[14] present in Ho.

Unfortunately, there are no known physical examples for $m=3$. However, such a model might possibly be realized experimentally in binary rare-earth alloys such as Er, characterized by Ising spins with modulation in one fixed direction represented by a two-component order parameter,[9,2] and Ho or Dy characterized by XY spins lying in one plane and modulation perpendicular to it represented by a four-component order. The alloy might be represented by a six-component order-parameter ($2m=6$) with an anisotropy related to the difference between the wave vectors. In this way, besides a disordered phase, this system should have a modulation along the Er spin direction when the corresponding wave vector dominates, or orthogonal to the Ho spin plane, otherwise. When the overall anisotropy vanishes, the $m=3$ model should be realized provided one can neglect the quenched randomness in the alloy. However, it must be recognized that taking full account of randomness complicates the picture.[17]

Note that we are considering the m-vector model with uniaxial competing interactions, which, in contrast to the Ising (or $m=1$) model with spatially isotropic competing interactions, v has no fixed relation to u. In the absence of cubic terms, one has $u=v$ and the transition for $2m>4$ would be first order. When we allow $u\neq v$, a stable fixed point appears provided the physical parameters satisfy $u>v$. In this case the transition is continuous with exponents given in Table I. The cubic anisotropy terms, even in the case of a first-order transition, are relevant since they give rise to tricritical behavior: see Figs. 1 and 2 below for details. (Note that similar phase diagrams have been found previously for related but distinct models: see, e.g., Domany *et al.* and Blanckschtein and Aharony.[18])

The $m=4$ case is even more difficult to realize experimentally. A possibility is an alloy of two rare-earth metals such as Dy and Ho. Since each element alone exhibits modulation with spins lying in one plane and a wave vector perpendicular to it (the transition being described by a four-component order parameter), the alloy might be reasonably represented by an eight-component order parameter Hamiltonian ($2m=8$) provided the planes related to Dy and Ho are orthogonal. In this case, Dy and Ho spins will interact only through fourth-order terms.

One might anticipate, however, as Mukamel and Grinstein (as quoted in Ref. 17) concluded, that terms arising from the randomness in the alloy will make the transition first order when a continuous transition would, otherwise, be realized. Note that our considerations, presented below, suggest that the transition should be first order even without allowance for the randomness.

Since we are considering rare-earth alloys, we should also introduce a cubic anisotropy term that generates easy axes. Consequently the critical behavior of such a system should be represented by an m-component spin ($m=2,3,4$) system, with Hamiltonian

$$H=-\tfrac{1}{2}\int\left[\sum_{i=1}^{l}A_1(\mathbf{q})\phi_i^2-\sum_{i=l+1}^{m}A_2(\mathbf{q})\phi_i^2\right]-u\int\int\int\int\left[\sum_{i=1}^{l}\phi_i^4+\sum_{i=l+1}^{m}\phi_i^4\right]-v\int\int\int\int\sum_{i\neq j}^{m}\phi_i^2\phi_j^2\ , \tag{2.16}$$

where we use the notation

$$\int\equiv\int_0^{\pi/a}\frac{d^dq}{(2\pi)^d} \tag{2.17}$$

and

$$A_1(\mathbf{q})=1-\frac{2}{k_BT}J_1(\cos q_x+\cos q_y+\cos q_z-\kappa_1\cos 2q_z)\ , \tag{2.18}$$

$$A_2(\mathbf{q})=1-\frac{2}{k_BT}J_2(\cos q_x+\cos q_y+\cos q_z-\kappa_2\cos 2q_z)\ , \tag{2.19}$$

in which (J_1,κ_1J_1) and (J_2,κ_2J_2) represent nearest- and next-nearest-neighbor interactions between $(\phi_i-\phi_j)$ with $i,j=1\ldots,\ l$ spin components and $(\phi_i-\phi_j)$ with $i,j=l+1\ldots,\ m$ spin components, respectively. The cases $l=2$ and $m=4$ and $l=1$ and $m=3$ should describe Dy and Ho and Er and Ho alloys, respectively.

In the critical region, fluctuations with $\phi_i(q_c)$ or $\phi_i(\bar{q}_c)$ dominate where

$$q_c=\cos^{-1}(\tfrac{1}{4}\kappa_1)\quad\text{and}\quad\bar{q}_c=\cos^{-1}(\tfrac{1}{4}\kappa_2)\ . \tag{2.20}$$

In order to take these into account we apply the Brillouin-zone folding process obtaining a $2m$-component Hamiltonian, namely,

$$H_{\text{eff}}=-\tfrac{1}{2}\int\left[\sum_{i=1}^{l}(r_1+q^2)(\phi_i^2+\bar{\phi}_i^2)+\sum_{i=l+1}^{m}(r_2+q^2)(\phi_i^2+\bar{\phi}_i^2)\right]-u\int\int\int\int\sum_{i=1}^{m}(\phi_i^2+\bar{\phi}_i^2)^2-v\int\int\int\int\sum_{i\neq j}^{m}(\phi_i^2+\bar{\phi}_i^2)(\phi_j^2+\bar{\phi}_j^2)\ , \tag{2.21}$$

where

$$r_1=r-\left[1-\frac{l}{m}\right]g,\quad r_2=r+\frac{l}{m}g\ , \tag{2.22}$$

$$r=1-(2J/k_BT)(2+\kappa-\tfrac{1}{8}\kappa^{-1})\ , \tag{2.23}$$

$$g=-(2J/k_BT)(1-\kappa^{-2})\delta\ , \tag{2.24}$$

where we have written $J_1=J_2$, $\kappa_1=\kappa-[1-(l/m)]\delta$, and $\kappa_2=\kappa+(l/m)\delta$. One may reasonably suppose that both the parameters κ and δ will depend on the temperature, on the pressure, and on the concentration of each component. Variations of these various quantities could thus lead to phase diagrams similar to Figs. 1 and 2. Note that this Hamiltonian is identical to (2.7) if $r_1=r_2$ but now the initial values of u and v are not related.

In order to allow for $g\neq 0$, we have extended Mukamel and Krinsky's[6] analysis and this yields the crossover exponent

$$\phi=1+\frac{m}{2(m+4)}\epsilon+O(\epsilon^2)\ , \tag{2.25}$$

when $2m<4$ for the symmetric fixed point (2.13), and

$$\bar{\phi}=1+\frac{m}{2(5m-4)}\epsilon+O(\epsilon^2)\ , \tag{2.26}$$

when $2m>4$ for the asymmetric fixed point (2.14). In the case of cubic anisotropy among all $2m$ components, one has $\bar{\phi}=1+\epsilon/6$.[18]

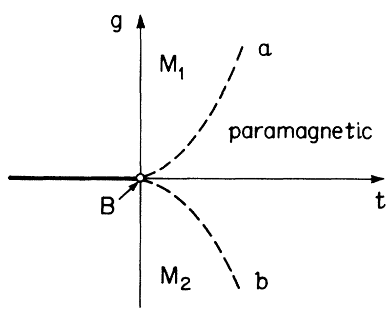

FIG. 1. Phase diagram for $u>v$. M_1 denotes the modulated phase associated with Ho; M_2 denotes the modulated phase associated with Er ($2m=6$) or Dy ($2m=8$). The critical lines (a) and (b) meet at the bicritical point B with crossover exponent $\bar{\phi}$ and amplitude ratio A^-/A^+. (See Table II for values.)

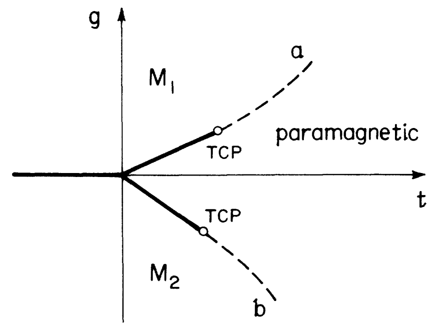

FIG. 2. Phase diagram for $u<v$. As in Fig. 1, M_1 denotes the modulated phase associated with Ho ($2l=4$), while M_2 labels the modulated phase appropriate for Er ($2m=6$) or Dy ($2m=8$). The critical lines (a) and (b) meet the first-order loci at the tricritical points $(g_t^{\pm},t_t^{\pm})$, labeled TCP.

Since u and v are not related initially, we can consider $v>u$. Then if $g<0$ the system orders along the Ho spin components with $O(2l)-(l=2)$ critical behavior through a continuous transition with exponents given in Table I. When $g<0$, the system orders in the Dy plane through *a* $O(2(m-l))-(m-l=2)$ continuous transition. At $g=0$, these two critical lines meet at a bicritical point, near which one has

$$g_c\approx\pm A^{\mp}|t|^{\bar{\phi}},\quad g\lessgtr 0\ , \tag{2.27}$$

where to first order in $\epsilon=4-d$ expansion one has[19]

$$A^-/A^+=[l/(m-l)]^{\bar{\phi}}\ , \tag{2.28}$$

where $\bar{\phi}$ is given by (2.26).

In order to understand the influence of the cubic term, we compare our results for $2m=6,8$ where we have considered cubic anisotropy within m components with known values for $2m=2,3,4$ calculated without a cubic term, since for $2m>4$ the symmetrical fixed point is the stable one. See Table II.

For $v>u$, since the stable fixed point (2.14) cannot be attained, the transition for $g=0$ is first order. However, for g sufficiently positive or negative we can still find two critical lines as illustrated in Fig. 2. These will meet the first-order loci at tricritical points specified by

$$g_t\approx\pm A_t^{\pm}|t_t|^{\phi}\ , \tag{2.29}$$

where ϕ is given by (2.25). In order to calculate $A_t^{\pm}$, we have to consider the $2l[2(m-l)]$-component Hamiltonian

$$\tilde{H}_{\text{eff}}==\tfrac{1}{2}\int(r_{\text{eff}}+q^2)\sum_i(\phi_i^2+\phi_i^{-2})-u_{\text{eff}}\int\int\int\int\sum_i(\phi_i^2+\phi_i^{-2})^2\ , \tag{2.30}$$

where $r_{\text{eff}}\simeq r_1$ for $g>0$ and $r_{\text{eff}}\simeq r_2$ for $g<0$, and

$$u_{\text{eff}}=u(l)-8K_4v^2(l)n(2\ln 2-1)\ , \tag{2.31}$$

in which $n=m-l$ for $g>0$ and $n=l$ for $g<0$.

This form of Hamiltonian is obtained from (2.21) by integrating out the noncritical fields $i\lessgtr l$ for $g\lessgtr 0$. Owing to

TABLE II. Universal amplitude ratios and crossover exponents ϕ ($u=v$) and $\bar{\phi}$ ($u\neq v$) for $2m=2$, 3, 4, 6, and 8 and $d=3$.

Number of components	A^-/A^+	ϕ	$\bar{\phi}$
2	1[a]	1.18±0.02[b]	
(Ising+Ising)		1.2[a]	
3	2.34±0.08[b]	1.25±0.02[b]	
(Ising+XY)	2.416[a]	1.27[a]	
4	1[c]	1.33[a]	1.167[c]
($XY+XY$)			
6	2.198[c]	1.214[a]	1.136[c]
[$XY+O(4)$]			
8	1[c]	1.25[a]	1.125[c]
[O(4)+O(4)]			

[a]The values were obtained from the ϵ expansion without cubic terms (Ref. 20).
[b]The values are derived from the high-temperature series (Ref. 19).
[c]The values were estimated from a first-order ϵ expansion including cubic anisotropy.

the cubic term, $u_{\rm eff}$ can become zero even if $u>0$. Using this condition together with $r_{\rm eff}=0$ and standard recursion-relation procedures, we obtain the universal amplitude ratio

$$A_t^-/A_t^+=[m/(m-l)]^{1-\epsilon m/2(m+4)} \; . \tag{2.32}$$

For $2m=6,8$ and $\epsilon=1$, we find $A_t^-/A_t^+\simeq 1.72, 1$, respectively.

For $u=v$, there is no stable fixed point when $2m\geq 4$. In this case the transition should be first order.

It should be stressed that even though qualitatively similar to the bicritical and tricritical behavior obtained when cubic anisotropy is allowed among $2m$ components,[18] our model lies, in fact, in a different universality class.

III. THE LIFSHITZ-REGION BEHAVIOR

Besides modulated and disordered phases, models with competing interaction models typically also exhibit a ferromagnetic phase. If the para-ferromagnetic and para-modulated phase transitions are continuous, these three phases meet at a Lifshitz point.[15] In this section we study this region for the different models considered above.

Consider first the n-vector model with a cubic term and uniaxially competing interactions. For $n\leq 4$, the para-ferromagnetic transition is continuous, as is well known, and so is the para-modulated transition if $u>v$, as we just saw. This leads to a Lifshitz point. In order to obtain the behavior in this region we must analyze the Hamiltonian (2.16) with $m=1=n$, $A_1(q)=A_2(q)$, and

$$A_1(\mathbf{q})\equiv r+pq_1^2+\sum_{\beta=2}^{d} q_\beta^2+q_1^4 \; , \tag{3.1}$$

that generalizes the Hornreich case[15] by allowing a cubic term. Since, in the vicinity of the Lifshitz point, the order-parameter fluctuations are dominated by the term q_1^4, the upper critical dimension must be $d_c=4.5$. In order to understand the influence of the cubic term, we have derived new recursion relations to second order in $\epsilon=4.5-d$. We obtain, besides the symmetric and Gaussian Lifshitz fixed points,[15] the Ising-Lifshitz fixed point

$$u^*=\epsilon/9\bar{K}, \quad v^*=0 \quad \text{with } \bar{K}=\Gamma(\tfrac{1}{4})/(2\pi)^{5/4} \tag{3.2}$$

and the cubic-Lifshitz fixed point

$$u^*=(n-1)\epsilon/9n\bar{K}, \quad v^*=\epsilon/3n\bar{K} \; . \tag{3.3}$$

Examination reveals that for $n<4$ the symmetric fixed point is the stable one; otherwise the cubic fixed point controls the behavior. In that case, we find the new Lifshitz exponents

$$\begin{aligned} \eta&=-\frac{1}{108}\frac{(n-1)}{n^2}(n+2)\epsilon^2+\cdots \; , \\ \nu&=\tfrac{1}{4}[1+\tfrac{1}{3}(n-1)\epsilon/n+\cdots] \; , \\ \beta_k&\equiv\nu/\phi_p=\frac{1}{2}+\frac{7(n+2)(n-1)}{216n^2}\epsilon^2+\cdots \; , \end{aligned} \tag{3.4}$$

that differ from usual cubic ones, in order ϵ by more than a shift in ϵ and a rescaling.[21,22] The negative sign of η is also characteristic of a Lifshitz point.

The Ising model with isotropic competing interactions displays quite different behavior. The para-ferromagnetic transition is continuous (Ising-like) but the paramagnetic to (cubic or uniaxial) modulated phase transitions are first order. Since this last result is due to fluctuations for $d<4$ dimensions and since the Brillouin-zone folding process does not work near $\mathbf{q}\simeq 0$, it is interesting to investigate the situation in more detail. To this end we expand (2.3) around $\mathbf{q}\simeq 0$ and redefine the spin variables appropriately which yields the Hamiltonian (2.16) with $m=l=1$ and

$$A_1(\mathbf{q})=r+pq^2+(q^2)^2+s\sum_{i=1}^{d} q_i^4 \; , \tag{3.5}$$

where

$$r=2[k_B\hat{T}-6(1-\kappa_1-2\kappa_2)]/\kappa_2,\quad \hat{T}=T/J\ ,$$
$$p=2(1-4\kappa_1-4\kappa_2)/\kappa_2\ , \tag{3.6}$$
$$s=2\kappa_1/\kappa_2-1\ .$$

Mean-field analysis of this Hamiltonian indicates that for $p<0$ the ferromagnetic phase is unstable against a modulated phase. For $s<0$, there is modulation along three axial directions (uniaxial modulated phases) but for $s>0$ along four diagonal directions (cubically modulated phases). In this approximation, the order-disorder transitions are continuous and there is a Lifshitz point.

In order to include fluctuations, Hornreich, Luban, and Shtrikman,[15] using the renormalization group to first order in $\bar{\epsilon}=8-d$, found $p^*=0$, $u^*=\bar{\epsilon}/36K_8$, and also the critical exponents ϕ_p given by (1.3) and $\nu=\frac{1}{4}+\bar{\epsilon}/48$. Note that owing to the isotropy in momentum space, one has $s=0$ in their analysis. In addition, Mukamel and Luban[16] showed that, in this case, the susceptibility $\chi(q)$ for $p<0$ apparently vanishes or cannot be defined, which they argue suggests that no long-range helical order exists. Here we specifically allow $s\neq0$ and have extended their analysis finding a stable fixed point with $s^*=0$. Neat this fixed point we obtain $(\Delta s)'=b^{\phi_s/\phi_p}\Delta s$ where b is the rescaling factor associated with momentum space with $\eta=-\epsilon^{-2}/180$ and $\phi_s/\phi_p=7\eta$. Since $\phi_s<0$ and $\phi_p>0$ one can say that s is an irrelevant variable. We will show, however, that it is actually a "dangerous irrelevant variable"[23] so that it cannot be neglected even asymptotically close to multicriticality.

The singular part of the susceptibility, in the region around the Lifshitz point, can be written as

$$x_s\approx|p|^{-\gamma/\phi_p}X^{\mp}\left[\frac{t}{|p|^{1/\phi_p}},\frac{q}{|p|^{\nu/\phi_p}},\frac{s}{|p|^{\phi_s/\phi_p}}\right]\ , \tag{3.7}$$

where $t=(T-T_L)/T_L$ (at $p=0$), and the functions X^- and X^+ refer to $p<0$ and $p>0$, respectively. Now, the upper critical dimension for the para-ferromagnetic transition is only $d_c=4$ but we will calculate only $d=8-\bar{\epsilon}$ dimensions: thus the susceptibility for $p>0$ will diverge linearly on a locus $T_+(p)$ defined by $(X^+)^{-1}=0$, which we find leads explicitly to

$$t/p^{1/\phi_p}+A_+(y)=0\ , \tag{3.8}$$

where $t_+=[T_+(p)-T_L]/T_L$ and

$$A_+^{(y)}=\tfrac{1}{3}\bar{\epsilon}[\ln 2-1-\tfrac{1}{2}y(\ln 2+1)] \tag{3.9}$$

with $y\equiv s/|p|^{\phi_s/\phi_p}$. Likewise for $p<0$ one encounters the para-modulated phase transition and the susceptibility will diverge linearly on a locus $T_-(p,s)$ defined by $(x^-)^{-1}=0$ which, upon calculation, is given by

$$(t/p^{1/\phi_p})+\tfrac{1}{4}(s/p^{\phi_s/\phi_p})-\tfrac{1}{4}-A_-(y)=0\ , \tag{3.10}$$

where $t_-=[T_-(p,s)-T_L]/T_L$ and

$$A_-(y)=\tfrac{1}{6}\bar{\epsilon}+\tfrac{1}{24}\bar{\epsilon}\ln\tfrac{5}{4}-\tfrac{1}{16}y^{-1/2}\bar{\epsilon}(1-\tfrac{1}{2}y)I \tag{3.11}$$

with

$$I=\int\frac{d\Omega}{(1-\hat{f})^{1/2}},\quad \hat{f}\equiv\sum_{i=1}^{d}(q_i)^4\Big/\left[\sum_{i=1}^{d}|q_i|^2\right]^2\ , \tag{3.12}$$

and where Ω denotes a solid angle.

Note that in the spatially isotropic case, one has $s=0$ and the function X^- then vanishes identically indicating, following Mukamel and Luban,[16] that no long-range modulated phase persists. As we allow $s=0$, however, $T(p,s)$ is defined for $p=0$. In this sense, we argue that the contribution of y to A_-, even being apparently irrelevant, makes a para-modulated phase transition possible. We may anticipate also that the ratio A_+/A_- will be a universal function of y; however, since we may not actually set $y\neq0$, noting the divergence in $A_-(y)$, the ratio A_+/A_- will not take a universal value, for given values of s and p. This is in agreement with the general surmise that this sort of ratio need not be universal when many perturbations leading to different types of critical behavior are present.[24]

This analysis for $\bar{\epsilon}=8-d\ll1$ cannot reveal the weak first-order disorder-modulated phase transition arising from fluctuations which is expected below $d_c=4$ dimensions. In order to understand what happens, even qualitatively, when d is lowered from eight dimensions one has to look at higher orders in the $\bar{\epsilon}$ expansion. We have found that the leading higher-order term gives a negative contribution to the value of u at the stable fixed point, suggesting that a first-order transition may arise for larger $\bar{\epsilon}$. For this reason we believe that the Lifshitz point should, in fact, become a critical endpoint in dimensions $d<4$.

IV. SUMMARY

For applications to binary alloys and helimagnets, we have studied two models with competing interactions: Ising spins with spatially isotropic competing interactions and n-component spins with anisotropic competing interactions. We showed first that the paramagnetic-to-planar, or to uniaxial, or to cubic-modulated phase transition present in the first model lies in the same universality class as the para-modulated transition (for which the wave vector is unique) present in the second model when the number of spin components is $n=2$, 3, or 4, respectively. The Ising-spin model displays first-order transitions; the second model can have continuous transitions. This result may explain why some helimagnets exhibit first-order transitions while others do not. We have also proposed phase diagrams encompassing such compounds.

In order to understand how the first order or continuous transitions meet the para-ferromagnetic transition locus, we studied the Lifshitz region. For a continuous transition we calculated new Lifshitz point exponents for the anisotropic n-component spin model; for the isotropic Ising model it was suggested that the supposed Lifshitz point in $d<4$ dimensions should actually become a critical end point.

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